Locality enhances consensus in opinion dynamics

Abstract

Social dynamics is a multidisciplinary field that encompasses a wide range of fields. Opinion dynamics, one of the fields of social dynamics, is investigated to understand how individual opinions shape the whole society. Some of the common models of opinion dynamics include discrete and continuous models. For the discrete opinion dynamics, the majority rule model and voter model are introduced. For the continuous opinion dynamics, the Deffuant model and Hegselmann-Krause model are introduced. In this work, the Deffuant model was the primary focus.

Past literature in the Deffuant model assumed a two-dimensional grid (square lattice), a complete graph, a random graph and a scale-free graph. These graphs do not cover the effect of local connectivity. A graph between a square lattice and a complete graph is proposed to hypothesize how local interactions may influence the whole population. Therefore the spatial parameter of "globality" is introduced into the model.

It was found by numerical stochastic simulations that the degree of globality and confidence bound have an impact on the thresholds for reaching "almost consensus". In addition to the previous findings that larger confidence bound increases the probability of the population attaining a state of "almost consensus", it was found that smaller number of neighbors, or "low globality" in other words, increased the likelihood of attaining "almost consensus", regardless of the value of the confidence bound.

As a population capable of attaining "almost consensus" evolves toward this state, agglomerations of centrists were found to appear and grow in size until the whole population is covered. There may be minority agents with outlying opinions scattered across the two-dimensional space, but the overall trend is not affected.

It was hypothesized that conversion of agents to centrist agents occur at the interface of growing agglomeration of centrists. It was also hypothesized that small confidence bound and large globality prevents nucleation of centrist agglomerations, resulting in polarization of the population. It must be noted that polarization of opinion occurs without spatial polarization.

As the next step, the spatially defined phenomenon such as the growth of centrist agglomerations can be quantified and investigated in more detail. This can be achieved by employing image processing methods such as clustering algorithms and dimensionality reduction. These techniques may allow us to identify properties of nucleation phenomena, give us insights into what local agent clusters might qualify as a nucleus to agglomeration, and to help us gain a deeper understanding of the underlying dynamics of opinion formation.

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1. Introduction
1.1. Social dynamics
Social dynamics is a broad and multifaceted field that encompasses a wide range of topics and phenomena, including group formation and cohesion, leadership, communication, conflict and cooperation, power dynamics, and social influence.(Fortunato 2005; Castellano, Fortunato, and Loreto 2009)(Edmonds 2006)(Helbing 2010)(Stauffer 2009)

Researchers in this field use a variety of methods and approaches, including experimental studies, field observations, and computational simulations, to understand the ways in which individuals influence each other and how these interactions shape group behavior.(Figure 1) (Castellano, Fortunato, and Loreto 2009)(Castellano 2012)

One key aspect of social dynamics is the study of how group behavior can emerge from the interactions of individual group members. This is often referred to as "emergent behavior," and it occurs when the behavior of a group as a whole cannot be fully explained by the individual behavior of its members. This can occur in a variety of contexts, such as when people are participating in social media networks, engaging in political activism, or working together in a team.

Another important aspect of social dynamics is the study of how group behavior changes over time.(Castellano, Fortunato, and Loreto 2009) This can include examining how groups form and dissolve, how group norms and values evolve, and how group members adapt to changes in their social environment. By understanding the ways in which group behavior changes and develops over time, we can shed light on important questions about how social structures and institutions emerge and function.

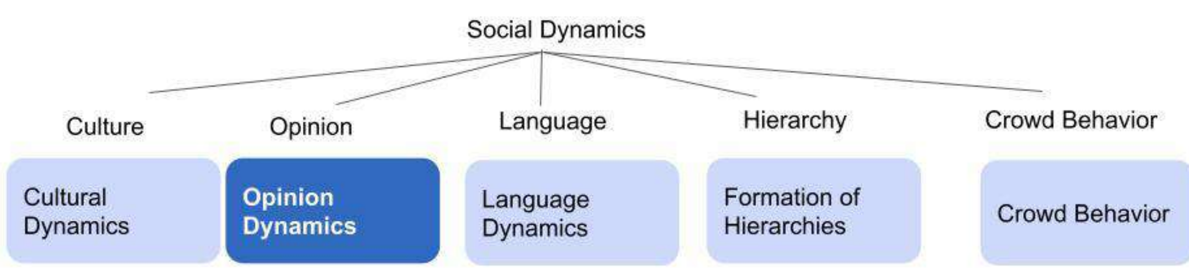


Figure 1 : Categories within social dynamics

1.2. Opinion dynamics
The discussions over a concerned agenda among the members of a society is a crucial process in maintaining a functioning community or society. Opinion dynamics is a field

that examines how the opinion “landscape” of a society is shaped through individual interactions of its members. Opinion dynamics was first conceptualized by Weidlich in 1971.(Weidlich 1971)

Opinion dynamics usually employ agent-based modeling methods, a methodology that describes social phenomena numerically and analytically. The opinions of each agent are quantified, representing the fact that each person in a given society has a specific viewpoint on a particular topic.(Kozitsin 2022) Then, the agents are allowed to interact with other agents according to a certain update rule.

Through interactions with others in society, individuals may update their opinions or maintain their current viewpoints. These interactions and opinion updates can be analyzed at both macroscopic and microscopic levels, to elucidate how microscopic interactions evolve into a macroscopic opinion "landscape".(Figure 2)

Some of the early examples of opinion dynamics include the use of the Ising model to simulate how opinions evolve and change within a population of individual agents. The spin-spin coupling in the Ising model represents the pairwise interaction between agents, for example, the influence that one individual's opinion has on another's. The magnetic field in the model represents external factors such as cultural majority or propaganda, which can exert a powerful influence on the opinions of the individuals within the population.(Serge Galam, (Feigenblat), and Shapir 1982)(Serge Galam and Moscovici 1991)(Castellano, Fortunato, and Loreto 2009)(Li et al. 2019)

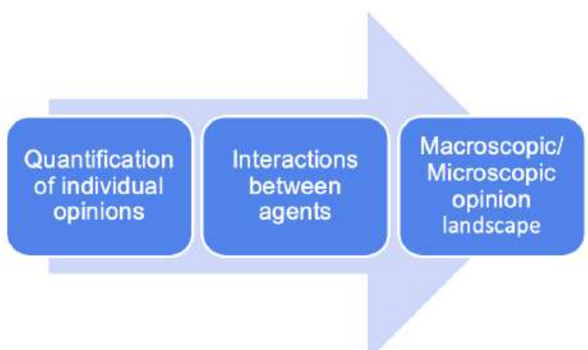


Figure 2: General scheme of modeling in opinion dynamics - opinion dynamics quantifies the opinions of each agent and models what happens to society as a collection of agents.

One application of this field is the use of simulations to study hypothetical elections. These simulations can help researchers investigate how different factors, such as campaign strategies or the influence of media, may impact the outcome of an election.(Bail et al. 2018) Effectively, we gain insights into how collective decision-making processes work.

Using these models several major real political events were successfully predicted including the victory of the French extreme right party in the 2000 first round of French presidential elections, the voting at fifty-fifty in several democratic countries (Germany, Italy, Mexico), and the victory of the no to the 2005 French referendum on the European constitution.(Serge Galam 2011)

Another application is the study of opinion polarization in social networks.(Loy, Raviola, and Tosin 2022)(Matakos, Terzi, and Tsaparas 2017) By creating simulations of how opinions may spread and evolve within a social network, we can gain a deeper understanding of how and why opinions may become more extreme or divisive over time. These simulations can provide valuable insights to identify potential interventions that may be effective in reducing polarization.(Prasetya and Murata 2020)

1.3. Models of opinion dynamics

Some of the common models of opinion dynamics include discrete and continuous models. In discrete opinion dynamics, opinions are either 0 or 1, such as in the case of political party affiliation (e.g., Republican or Democrat). Examples of discrete models include the majority rule model, the voter model, social impact theory, and the Sznajd model.(Castellano, Fortunato, and Loreto 2009) These models differ in how agents interact and update their opinion to form a macroscopic opinion landscape.(Figure 3)(Sîrbu et al. 2017) The Social Impact Theory and the Sznajd model are outside the scope of this work.

In continuous opinion dynamics, opinions can take any value from 0 to 1, such as in the case of tolerance for tax rates. Continuous opinion dynamics was first studied in 1977 by Chatterjee et al.(Chatterjee and Seneta 1977) Some of the common examples of continuous models include the Deffuant model and the HK model.(Deffuant et al. 2000)(Hegselmann and Krause 2002)(Lorenz 2007)

In addition, all models of opinion dynamics assume a certain network structure, which is simply a collection of connected objects. It is often depicted by mathematical equations or visual graphs.

A network structure depicted by a graph consists of N vertices that represent each agent, and the edges between the vertices represent connections that allow for connected agents to interact. Any opinion updates in opinion dynamics occur by the edges.(Castellano, Fortunato, and Loreto 2009)

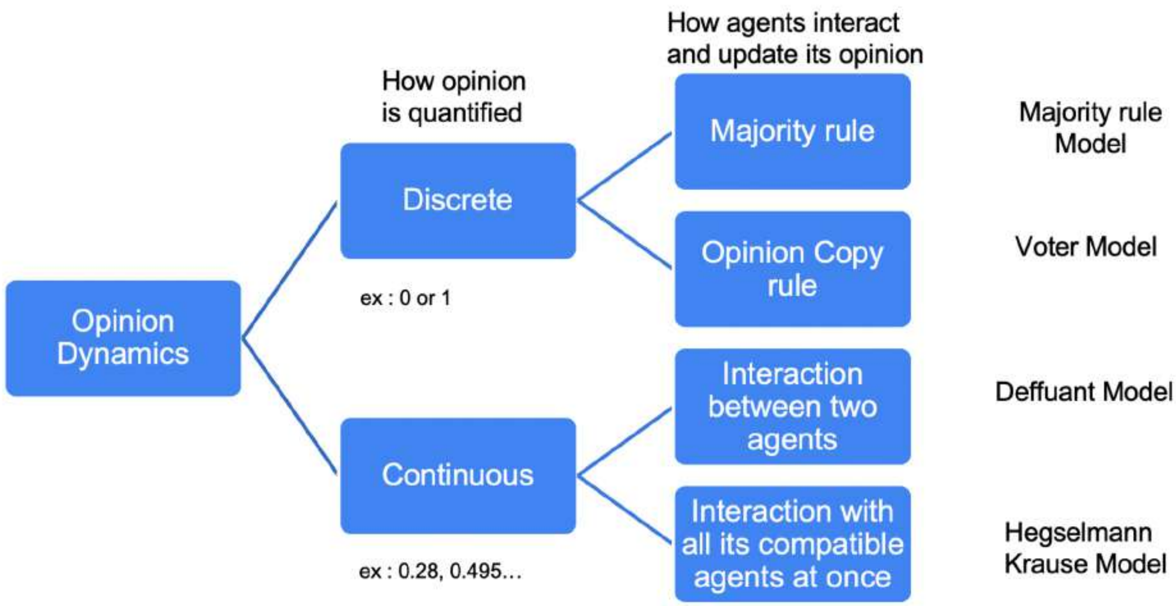


Figure 3 : Examples of opinion dynamics in literature

In a real society, out of all possible connections, only a limited number of connections actually exist. (i.e., not all 8 billion people in the world are friends with each other.) In opinion dynamics that models a real society, the topology of such limited connections determines the spread of opinion and the resulting opinion landscape.(Prettejohn and McDonnell 2011)(Stern and Livan 2021)

1.4. Scope of this research

In Chapter 1, the field of social dynamics, a broad and multifaceted field that encompasses a wide range of topics and phenomena, is introduced, with a particular focus on opinion dynamics. Opinion dynamics is a field that examines how the opinion "landscape" of a population is shaped through individual interactions of the population's members.

In chapter 2, some discrete opinion dynamics models, namely the majority rule model and the Voter model are introduced. In the majority rule model, the final configuration is solely dependent on the initial proportion of agents, as the model is based on the majority rule.

In chapter 3, some continuous opinion dynamics models, namely the Deffuant model and the HK model, are introduced, with a particular focus on the Deffuant model that can be based on either a complete graph or a two-dimensional grid. For each of these cases, we examine the opinion dynamics under different values of the confidence bound, an internal parameter of agents that determines the outcome of an agent's opinion update. Previous works examined the value of confidence bound at which complete consensus is obtained. This work also takes a closer look at these states.

In chapter 4, the Deffuant model in a two-dimensional grid is modified to account for local interactions that are not limited to the ones with the immediate neighbors. This range of interaction beyond the immediate neighbors is termed "globality". Along with other terms of the Deffuant model such as the confidence bound, the effect of these parameters to the opinion dynamics is explored.

Chapter 5 describes how the degree of globality and confidence bound impact the thresholds for reaching "almost consensus." One of the key findings is that small globality increases the likelihood of reaching "almost consensus", regardless of the value of confidence bound.

2. Discrete models of opinion dynamics

The two most known models in discrete opinion dynamics that assume binary opinion, in the majority rule model and the Voter model. Both models introduce similar properties, but they differ in their opinion update rules. While the majority rule is introduced in the majority rule model, the Voter model is based on the rule that agents update their opinion to that of one of their neighbors.(Redner 2019)

The majority rule model is based on the idea that individuals make decisions based on both their own opinions and the opinions of the group they are interacting with.(Serge Galam 2011) The Voter model is based on the idea that their opinions change to align with those of their neighbors such that each agent votes for one of its neighbors' opinions.(Fernández-Gracia et al. 2014)

2.1. Majority rule model

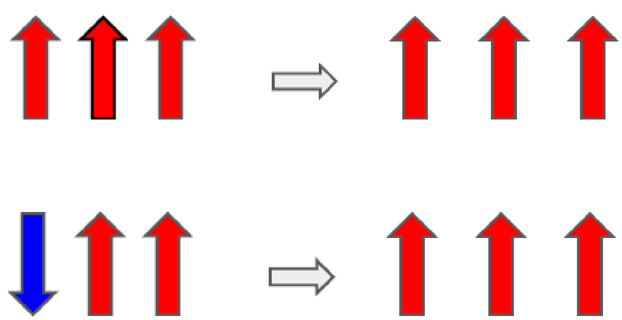


Figure 4: An illustration of how opinions are updated in a group by the majority rule

The model is based on a group-based majority rule. The majority rule is a means for making decisions in a group of people.(Figure 4) Individuals are influenced by the opinions of those around them, and they are more likely to adopt the opinion of the majority within their social network.(Krapivsky and Redner 2003)(S. Galam 2002)

2.1.1. Properties and definition

The majority rule model assumes a group of agents with a total size of N. Each agent within the group holds either opinion A or opinion B. The number of agents that support opinion A at a given time step is represented by $N_+(t)$, while the number of agents that support opinion B is represented by $N_-(t)$. It is stated that the total number of people supporting either opinion A or B at any given time must equal N, meaning that $N_+(t) + N_-(t) = N$.

The proportions of people supporting each opinion at a given time step are represented by $p_+(t)$ and $p_-(t)$, respectively. These proportions are calculated by dividing the number of people supporting each opinion by the total size of the group N. It is also stated that the sum of the proportions of people supporting each opinion must equal 1, meaning that $p_+(t) + p_-(t) = 1$.

The majority rule model is implemented on a complete graph, which means that every agent is connected to every other agent and can interact with them. The model can be applied to situations where individuals can interact with others regardless of physical distance, such as in the case of a large-scale online community.

In this model, a “floater” behavior is assumed in all agents. A floater is not fully committed to a particular opinion and is open to persuasion from others. It listens to the opinions of others and may change its initial opinion.

2.1.2. Opinion update procedure

In the majority rule model, all N agents have an opinion on a particular topic. Opinions are binary represented by $+1$ or -1. In a political context, this could represent support for the Republican Party or the Democratic Party. The proportion of agents having opinion $+1$ is represented as p_+, while the proportion of agents holding opinion -1 is represented as p_-.

In this model, the agents update their opinion through interactions with others. The process starts with selecting a group of r agents randomly from a population of size N. Then the majority rule is applied to update the individual opinions of the selected agents. We chose r as an odd number, typically r =3, so that the majority rule can decide a winner. Under the majority rule, agents adopt the majority opinion of the group they are with, regardless of their own initial opinion. Then finally, the agents are reshuffled back into the population for the next random selection. This process is iterated until a stable configuration is reached.

A stable configuration refers to a state in which the opinions of the agents in the system are no longer changing. This can occur when all agents settle on the same opinion, or the agents distribute themselves in multiple distinct groups with different opinions.

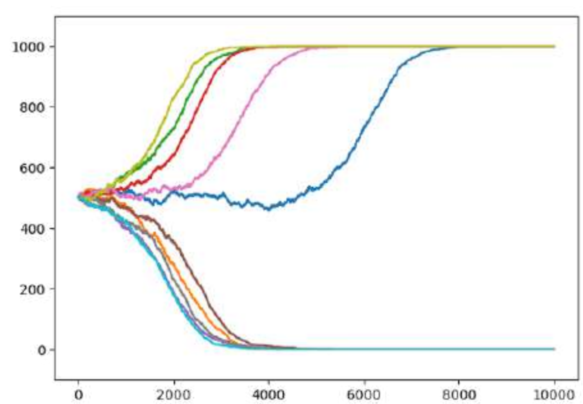
The majority rule model is considered global because it picks agents entirely randomly from the whole population. The model is simple, but it is capable of describing how different initial conditions lead to different outcomes such as consensus or polarization.

2.1.3. Empirical results

The convergence of opinion is determined solely by the initial state, as it is the only variable. Specifically, the population is more likely to converge towards an opinion with more than $\frac{N}{2}$ agents with the opinion, where N is the population size. This tendency is stronger when the initial number of agents is significantly bigger than $\frac{N}{2}$.

If the initial number is only marginally bigger than $\frac{N}{2}$, there is still a non-negligible probability of the population's opinion converging toward the side that started as the minority (but only marginally). The probability of such happening nears zero as the initial value departs from $\frac{N}{2}$.

The initial values also play a role in determining the speed at which the population reaches equilibrium. It has been observed that the farther the initial values are from N/2, the faster the convergence rate becomes. Understanding these relationships can be crucial in predicting and controlling the behavior of a given population.



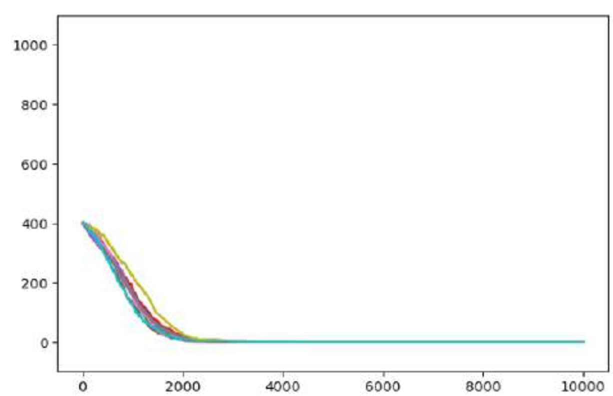


Figure 5: Number of agents with opinion A changing over time in the majority rule model at N=1000, with 500 initial opinion A supporters(left) and 400 initial opinion A supporters (right).

2.1.4. Majority rule model with contrarians

In a particular community, the majority of people hold a certain belief or follow a certain trend. However, some individuals always strongly disagree with the majority and actively oppose the majority's beliefs and actions. Agents with such a psychological feature are called a contrarian.(Serge Galam and Cheon 2019)

When contrarians are present within the community, they can have substantial effects on the dynamics.(Serge Galam and Cheon 2019) For example, it may become difficult for an opinion to spread and gain acceptance.

The majority rule model was modified to incorporate contrarians to investigate how their presence influences the spread and evolution of opinions and the overall opinion dynamics.(Serge Galam and Cheon 2019)

The proportion of contrarians within the group is denoted by the time-independent variable "a".(Serge Galam and Cheon 2019) In this model, the number of contrarians in a given group is assumed to be constant over time. All non-contrarian agents are assumed as floaters as defined in the original majority rule model.

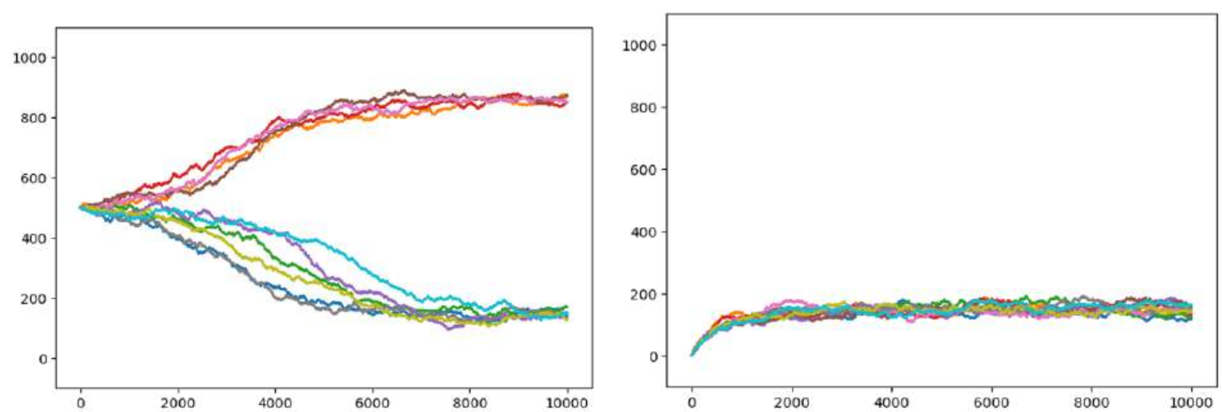


Figure 6: Number of agents with opinion A changing over time in the majority rule model in a population consisting of 900 floaters and 100 contrarians - in a case with 500 initial opinion A supporters(left) and in a case with only one initial supporter(right)

A contrarian shifts their opinion to oppose the local majority once it has been revealed.

The contrarian shift is independent of whether the majority opinion is A or B. Contrarian behavior is activated randomly with probability a, which represents the proportion of contrarian behavior within a social group. Contrarian behavior can be constant for some agents and temporary for others. However, at any given time, a proportion a of individuals exhibit contrarian behavior. The value a is a given fixed external parameter, independent of the dynamics, and satisfies $0 \leq a \leq 1$.(Serge Galam and Cheon 2019)

2.1.5. Spatial majority rule model

The spatial majority rule model is different from the majority rule model in a way that while agents are allocated in a complete graph in the majority rule model, the spatial majority rule model assumes agents to be distributed in a two-dimensional grid. The majority rule model is based on the global majority rule and assumes interactions between agents held at a global scale, but it does not model interactions occurring at a local scale. In the spatial majority rule model, interactions occur locally by letting agents interact only with its adjacent four neighbors.

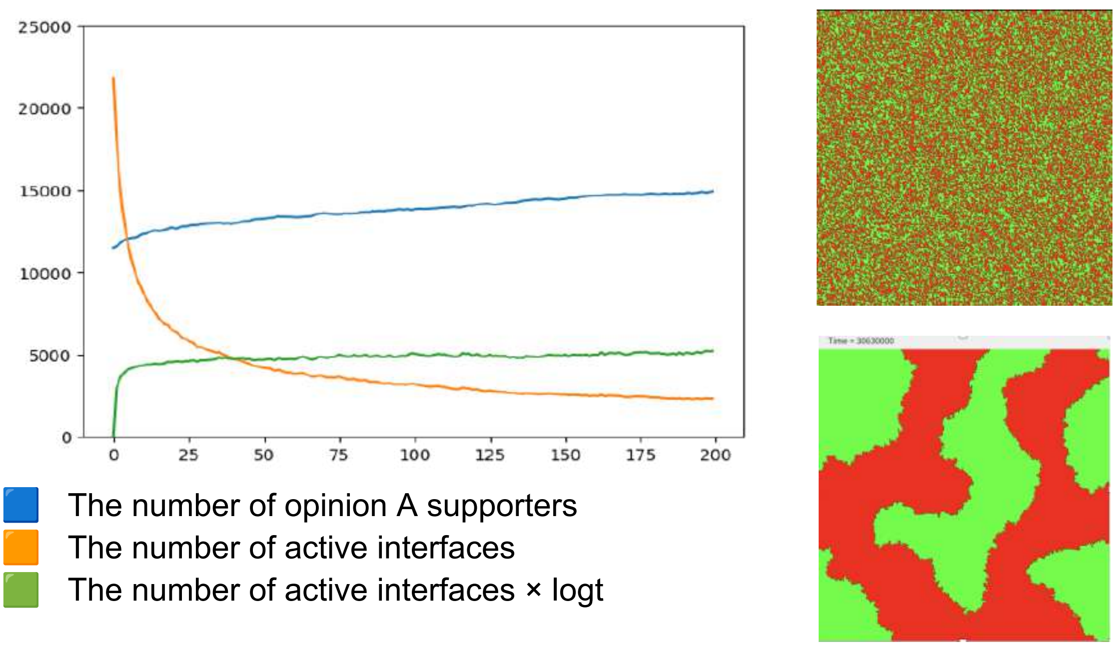


Figure 7: A spatial majority rule model simulation of a population of 200*200 agents at a cyclic boundary condition - the number of agents with each opinion and the number of active interfaces (left), and the spatial distribution of opinions before (top right) and after (bottom right) 30,630,000 opinion update procedures.

In this model, the update procedure starts with selecting one agent and other r agents out of its four neighbors randomly. After picking $(1 + r)$ agents, the majority rule is applied to update the agents' opinions. Since agents are spatially fixed, opinion changes occur only when agents with different opinions are adjacent to each other. Hence, it can be said that opinion updates occur only at such an "active interface".

Figure 7 shows the initial state and the result of running a simulation according to the spatial majority rule model. The number of supporters fluctuates slower than in the original majority rule model. This is possibly caused by coarse graining, the decrease in the amount of active interfaces where opinion updates actually occur. (Figure 7 left)

2.2. Voter model

The Voter model established in 1973 is a simple model based on the idea of social influence, where individuals are influenced by the opinions of those around them.(Clifford and Sudbury 1973)(Holley and Liggett 1975)

In the Voter model, each agent is assigned an opinion x that can be represented by a binary variable as in the majority rule model. Agents with opinions of either 0 or 1 are randomly assigned in the initial configuration. At each time step, an agent i is randomly selected and its opinion is updated to match that of one of its neighbors j, also chosen at random. $x_i = x_j$.

This process is repeated until a stable configuration is reached, at which point the opinions of the agents are no longer changing.

2.2.1. Spatial Voter model

In the spatial Voter model, agents are distributed in a two-dimensional grid, as opposed to the Voter model based on a complete graph. The opinion update rule is common between the two models. Every agent interacts with one of its four neighbors. This implies, interactions between agents are held locally.

The opinion update procedure starts with selecting one agent and one of its four neighbors randomly. After selection, the first chosen agent takes the opinion of the second chosen agent as in Figure 8.

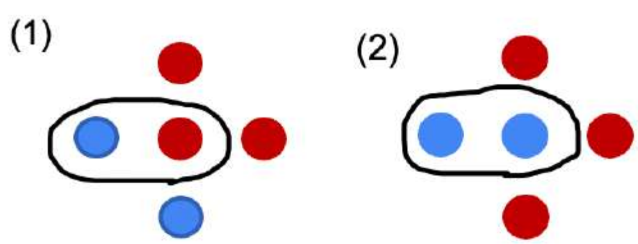


Figure 8: Step 1 and 2 of the opinion update rule of the Voter model - the first randomly selected agent takes the opinion of the second chosen agent.

Figure 9 shows the initial state and the result of running a simulation according to the Voter model. Compared to the majority rule model, while the number of active interfaces similarly decreases, the coarse graining appears less distinct in the spatial Voter model.

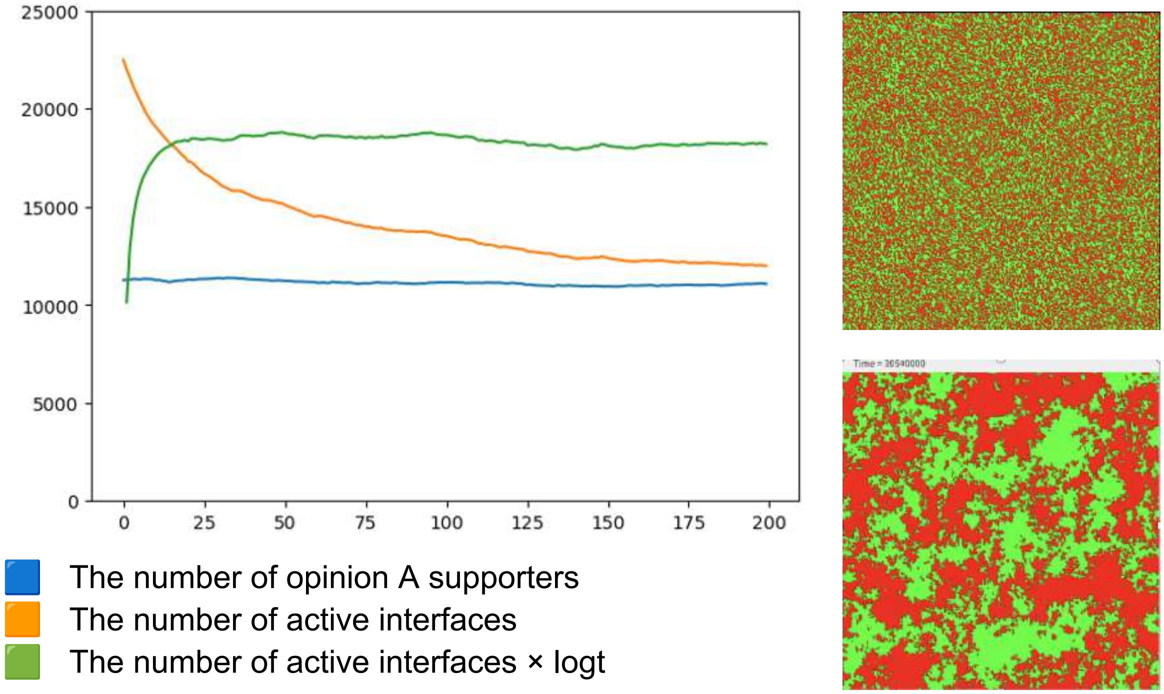


Figure 9: A spatial Voter model simulation of a population of 200*200 agents at a cyclic boundary condition - the number of agents with each opinion and the number of active interfaces (left), and the spatial distribution of opinions before (top right) and after (bottom right) 30,540,000 opinion update procedures.

3. Continuous models of opinion dynamics

The continuous opinion dynamics models including the Deffuant model and the HK model assume the opinions of individuals to take a value on a continuous opinion range rather than a binary.(Chatterjee and Seneta 1977) For example, an opinion range from 0 to 1 may be mapped between “conservative” and “liberal” to define a person’s “liberal-ness” or “conservative-ness” by a value like 0.23. An update to the opinion means a change in this value.

A density histogram can be drawn by taking the opinion value on the horizontal axis and the number of agents on the vertical axis. Such a histogram would feature a single cluster if the population has reached a consensus. It would be a representation of the majority holding a “centrist” opinion near 0.5 on the opinion range.

In continuous opinion dynamics, all individuals usually start with different opinions. The possible outcomes are often more complex than in discrete opinion dynamics. The final stable configuration may include one or more opinion agglomerations, which represent consensus, polarization, or fragmentation.(Castellano, Fortunato, and Loreto 2009)

3.1. Deffuant model

The Deffuant model is a continuous opinion dynamics model that was proposed by Deffuant, Neau, Amblard, and Weisbuch in 2000.(Deffuant et al. 2000) It has been used to study the formation and evolution of group opinions, as well as the emergence of social polarization, fragmentation, and consensus in populations.

3.1.1. Properties and definition

In this model, each agent holds an opinion on a particular topic, which is in the interval [0,1].(Castellano, Fortunato, and Loreto 2009) The initial opinions of the agents are randomly assigned values uniformly between 0 and 1. The agents are willing to compromise and can be influenced by the opinions of others, but only if they are not too far apart in opinion range.

Here, the model introduces the concept of confidence bound ε that represents the degree of tolerance of agents for letting opinion updates occur. We take $\varepsilon \in (0,1)$.

The agents in this model are assumed to interact with each other only if the absolute difference between their opinions is less than ε. Therefore, two people having opinions far different from each other are assumed to give up on making compromises.

To be more precise, let $s_i(t)$ and $s_j(t)$ denotes the opinion of a pair of interacting agents i and j at time *t*. Then the update rule is characterized as follows:

If $\left|s_i - s_j\right| \leq \varepsilon$,then:

$$s_i(t+1) = s_i(t) + \mu\left[s_j(t) - s_i(t)\right]$$

$$s_j(t+1) = s_j(t) + \mu\left[s_i(t) - s_j(t)\right]$$

When $\left|s_i - s_j\right| > \varepsilon$, opinion updates do not occur.

The convergence parameter, represented by μ, determines how much weight an agent gives to the opinions of others upon an opinion update. In other words, it determines the “stickiness” of “individuals’ opinions. The value of the convergence parameter lies in the interval [0, ½].(Castellano, Fortunato, and Loreto 2009)

When an opinion update occurs, the interacting agents update their opinions to take the middle ground according to the two opinions and the value of μ. When μ=0.5 and the opinions of two agents are within the confidence bound, they take the average of their opinions and adopt the new averaged opinion. Since they are only taking their average, the sum of all opinions remains constant throughout this process. The convergence parameter affects how fast stable configurations are reached and also determines the final outcome of the opinion dynamics.(Laguna, Abramson, and Zanette 2004)(Porfiri, Bollt, and Stilwell 2007)

The distribution of opinions can be illustrated by the histogram that takes the opinion range from 0 to 1 as the horizontal axis and the number or density of agents on the vertical axis. A localized mass in the histogram is termed a “cluster”.(Figure 10)

3.1.2. Network structure

Network structures in opinion dynamics models result in different patterns of opinion formation and convergence. There are two types of networks that have been implemented into the Deffuant model - a complete graph and a two-dimensional grid.

A complete graph is a network in which every pair of vertices (representing agents) is connected by an edge. This means that every agent is connected to every other agent in the network.

A two-dimensional grid is a network in which the vertices are arranged on a square lattice and the vertices are connected to their nearest neighbors on the lattice, forming a regular grid structure. A periodic boundary condition is often employed for such a grid.(Fortunato 2004)

3.1.3. Empirical results in a complete graph

In a complete graph, all agents are connected and “next to each other” by definition. Therefore, spatial distribution is not defined and only the histogram of opinions is concerned.

Initially, the agents in the population have randomly distributed opinions ranging from 0 to 1. The opinion update process is iterated until a stable configuration is reached. In this state, opinion distribution clusters appear to no longer influence each other. This is called “stable configuration”. Such a state is reached when the opinions of agents are all in a single cluster, or in clusters that are at least ε away from other clusters on the opinion space. At this state, the clusters appear independent from each other and static except for some microscopic internal fluctuations.

The evolution of opinion distribution starts with the agents near the edges updating their opinions toward the center. As more agents take centrist opinions, the histogram exhibits a propagation of higher concentration toward the center of the opinion space. This is especially evident when $\varepsilon > 0.5$. Under this condition, the centrist cluster appears from the first and becomes sharper over time. Finally, the histogram reaches the state of “complete consensus”, which is defined as all agents in the population being in a single cluster.(Figure 10)(Laguna, Abramson, and Zanette 2004)(Fortunato 2004)(Lorenz and Urbig 2007)

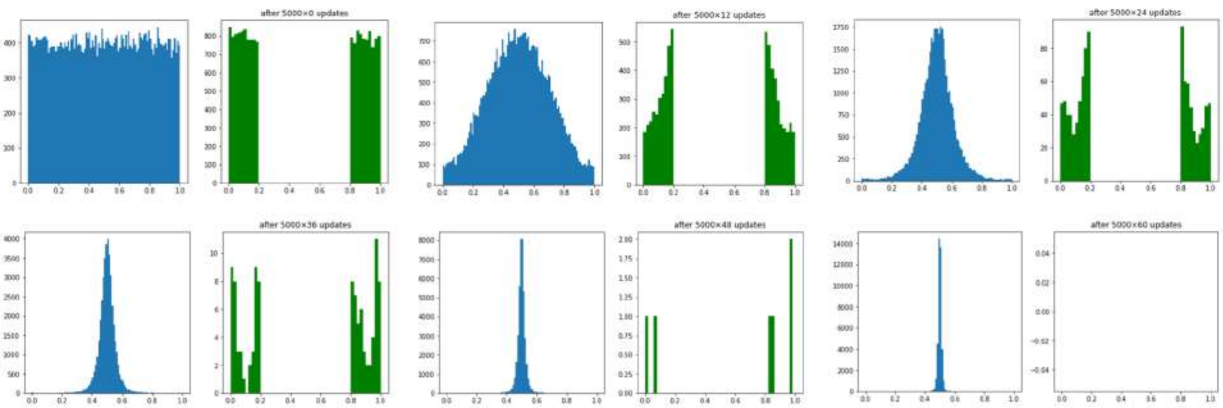


Figure 10: Deffuant model in a complete graph with 40000 agents at $\varepsilon = 0.5$, simulated for 60 iterations of 5000 updates when $\varepsilon > 0.5$ - blue represents the whole histogram and green indicates the distribution of agents in the opinion space 0-0.2 and 0.8-1.0. A single cluster emerges from the first and becomes larger and sharper as more agents join. Over time, a complete consensus is obtained. All agents held opinions within the range of 0.4 and 0.6 after 60 iterations of 5000 updates.(Fortunato 2004)

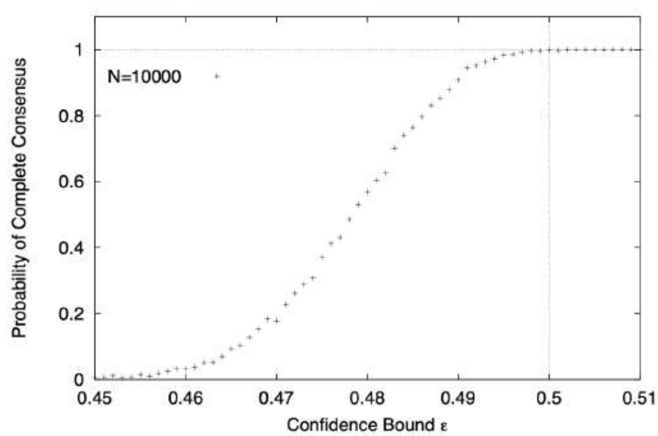


Figure 11 : The relations between the confidence bound and the probability of reaching a complete consensus in the Deffuant model in the complete graph (Fortunato 2005) - the probability tends to zero by ε=0.45.

As the value of ε becomes smaller than 0.5, the population starts failing to reach a complete consensus, meaning that not all agents join the final centrist cluster. Works by Fortunato shows that when ε shrinks to below 0.5, the possibility of failing to reach a complete consensus arises and by ε=0.45, the such possibility becomes practically nil.(Figure 11) Figure 12 shows an example of such a state, where there is a major centrist cluster but also some outlying agents, whose opinions are at least ε away from the centrist cluster.(Fortunato 2004; Laguna, Abramson, and Zanette 2004)

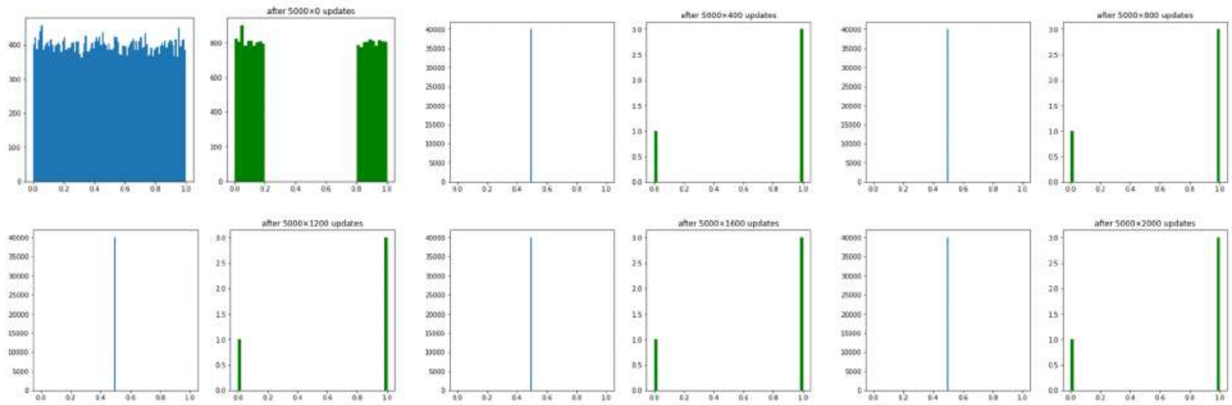


Figure 12 : Deffuant model in a complete graph with 40,000 agents at ε＝0.48, simulated for 2000 iterations of 5000 updates, leaving 4 outlying agents outside the centrist cluster - compared to the case of ε=0.5 (Figure 10), a complete consensus is not reached even after 2000 iterations of 5000 updates as opposed to 60 iterations of 5000 updates. (Figure 10)

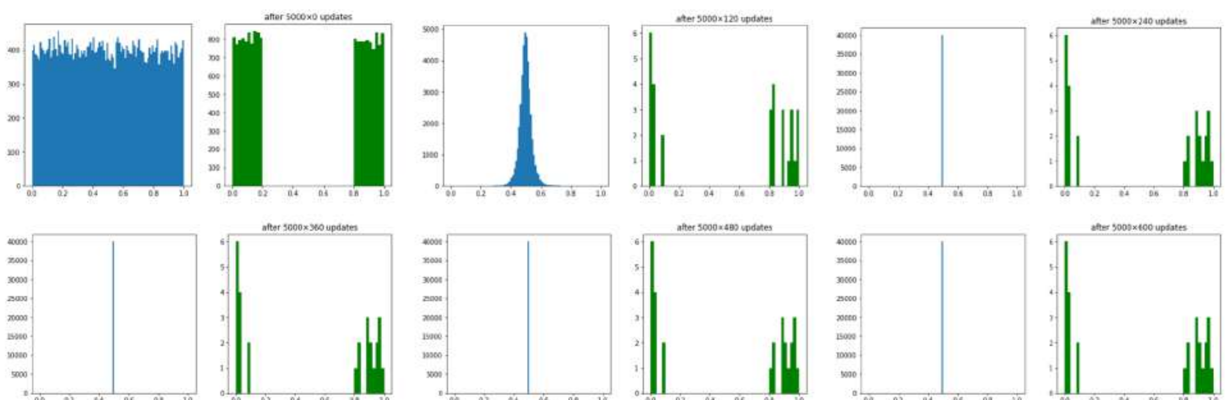


Figure 13: The development of the histogram of the opinions of agents in a complete graph of 40000 agents, over 600 iterations of 5000 updates with ε=0.28 by the Deffuant model - The twin clusters initially appear and then merge into a centrist cluster. In the final state, agents in the centrist cluster do not interact with outlying agents as their opinions are further than ε.

It is notable that as ε becomes smaller than 0.5, the histogram starts exhibiting a different dynamics. In this condition, the initial opinion updates by agents toward the center results in formation of a twin cluster in the opinion distribution histogram. Nevertheless, the clusters eventually merge into a single cluster. (Figure 13)

When $\varepsilon < 0.28$, the initial twin cluster feature becomes permanent and a single cluster distribution is not observed even after a prolonged simulation time.(Figure 14(b)) When the clusters are separated by distances above ε, the difference of opinions between agents in different clusters would always exceed ε. At this state, only agents within the same cluster can interact with each other.

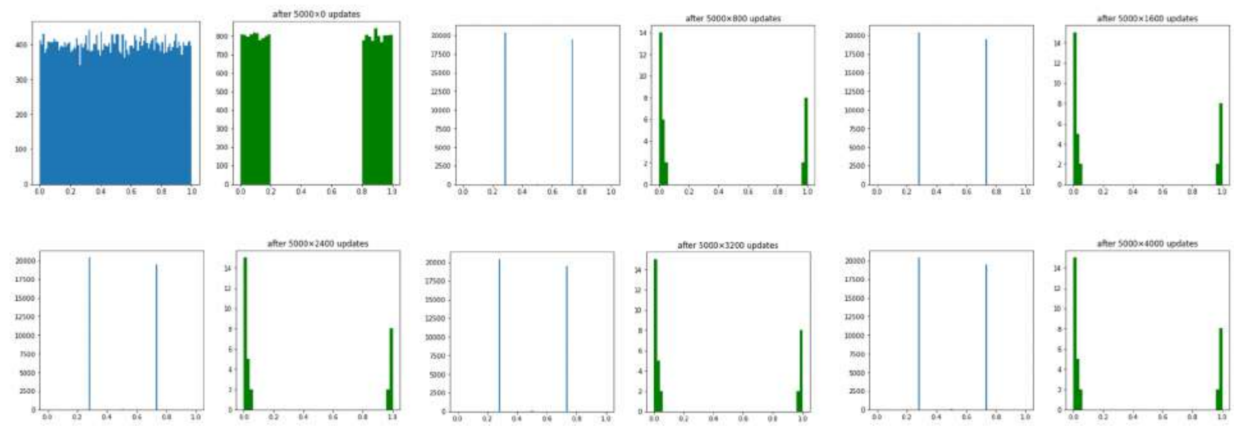


Figure 14: Deffuant model in a complete graph with 40,000 agents at ε=0.2, simulated for 4,000 iterations of 5,000 updates, shows that the initial twin clusters are a permanent feature.

As mentioned above, the average opinion of the agents within a pair remains unchanged before and after their interaction for any given values of the confidence bound and the convergence parameter. As a result, the global average opinion of the entire population is invariant of the Deffuant model dynamics, throughout the whole process of opinion updates to reach a stable configuration. This is expressed by the equation:

$$\sum_{i=1}^{N} x_i = \frac{N}{2}$$

While the sum of all opinions stays constant, local distribution clusters can occur. When the threshold value ε is greater than a critical value ε_c, it results in all agents converging to a shared opinion of $\frac{1}{2}$, meaning that there is complete consensus among them. (Fortunato, 2004; Lorenz and Urbig, 2007). This is true for a complete graph, regular lattice, random graph, and a scale-free network.(Fortunato, 2004; Lorenz and Urbig, 2007).

There are cases when multiple clusters occur. Monte Carlo simulations have shown that the number of agglomerations, n_c, in the final configuration can be approximated by the expression $\frac{1}{2\varepsilon}$.(Fortunato 2004) Because the opinions of each agglomeration need to be separated by the distance at least 2ε to its neighboring agglomerations. As a result, no other agglomerations can exist within an interval of 2ε around any given agglomeration.

The convergence parameter is known to affect the time it takes for an opinion landscape to change.(Laguna, Abramson, and Zanette 2004) When the value of the convergence parameter is small, agents resist updating their own opinions, resulting in a slower change of the opinion landscape. When it is high, individuals are more willing to update their opinions, leading to faster opinion dynamics.(Gargiulo and Huet 2010)

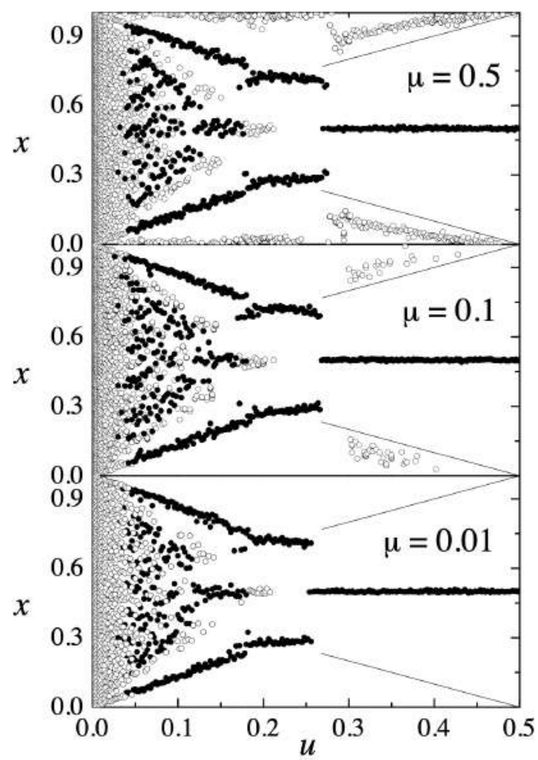


Figure 15: The positions of cluster peaks across different values of the confidence bound u, under the cases of 3 different values of the convergence parameter μ. Empty dots indicate minor clusters and full dots indicate major clusters with a population larger than 1000 agents, i.e. than 10% of the total population. Any agents within the pair of lines are absorbed into the major cluster at x=0.5 to leave an empty space.(Laguna, Abramson, and Zanette 2004)

3.1.4. Empirical results in a two-dimensional grid graph

Unlike in a complete graph, the physical location of agents is clearly defined in a two-dimensional grid graph. This makes both the histogram and spatial opinion distribution important properties of the two-dimensional Deffuant model.

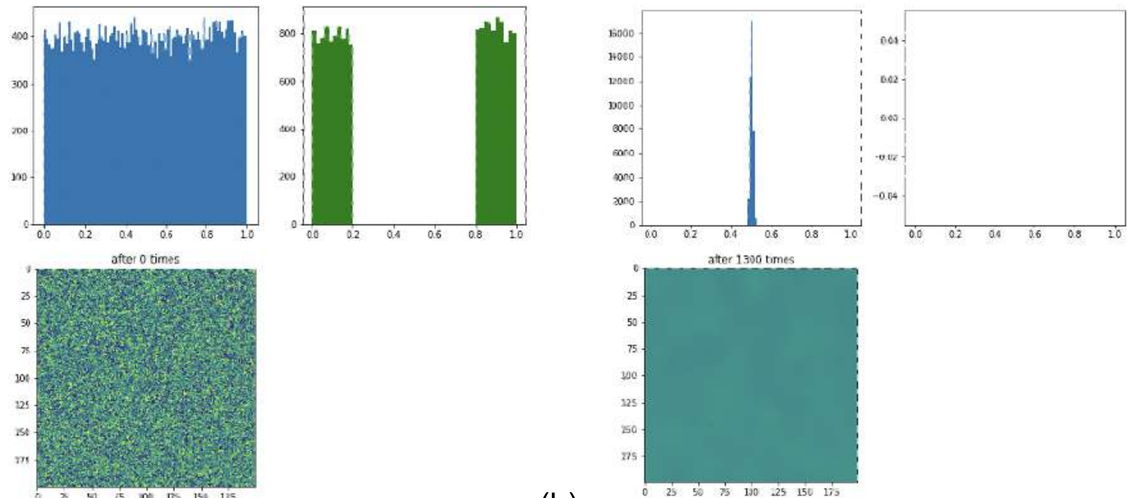
A spatial opinion distribution shows how opinion clusters emerge in a two-dimensional grid. This allows us to understand how geographic proximity can influence the formation of clusters of similar opinions, and how these clusters can contribute to attainment of consensus over time. We define the group of agents that have similar opinions and are physically close in the spatial opinion distribution as an agglomeration.

In the same manner as in the complete graph, the agents in the population have randomly distributed opinions ranging from 0 to 1. The opinion update process is iterated until a state, where opinion distribution clusters appear to no longer influence each other.

When $\varepsilon>0.51$, the evolution of opinion distribution starts with the agents near the edges updating their opinions toward the center. As more agents update their opinions, the histogram exhibits a propagation of density toward the center to make the centrist peak sharper. Finally, the histogram reaches the state of "complete consensus".(Figure 16) In such a state, the two-dimensional distribution appears nearly featureless without any blips, with agents in all locations holding similar opinions near 0.5.

When the value of ε becomes smaller than 0.51, the population starts failing to reach a complete consensus, meaning that not all agents join the final centrist cluster. The initial two clusters appearing immediately after the start of simulation become more evident as ε goes below 0.51. When ε shrinks to below 0.51, it becomes harder for a population to reach a complete consensus. In general, smaller populations are more likely to reach a complete consensus even with a smaller value of ε. Nonetheless, the chances of reaching a complete consensus drops to practically nil even for a smaller population like $N=2500$ when $\varepsilon=0.47$.(Figure 18)

Under the condition of $\varepsilon<0.51$, an overwhelming majority of agents join the centrist cluster as the simulation progresses. Yet this time, some outliers close to 0 or 1 on the opinion space are observed.(Figure 17) In the spatial distribution, this appears as scattered blips representing the outliers in the sea of agents with centrist opinions.(Figure 17)



(a) (b)

Figure 16: Deffuant model in a two-dimensional grid graph with 200*200 agents in a cyclic boundary condition at ε=0.51, before (a) and after (b) the simulation for 1300 iterations of 5000 updates - the histogram reaches a complete consensus with all agents holding opinions near 0.5. No blips are found in the two-dimensional distribution.

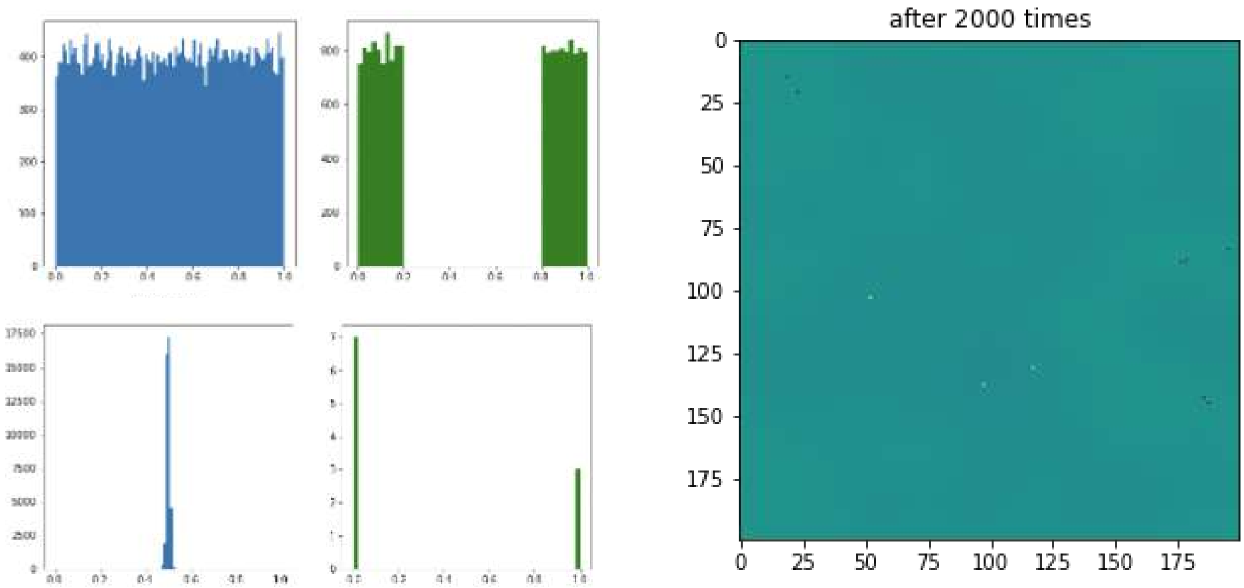


Figure 17: Opinion distribution with 200*200 agents with ε=0.50 in a two-dimensional grid at a cyclic boundary condition, before (top left) and after (bottom left) simulation for 2000 iterations of 5000 updates - some agents stay outside the centrist major cluster, appearing as blips in the spatial distribution (right).

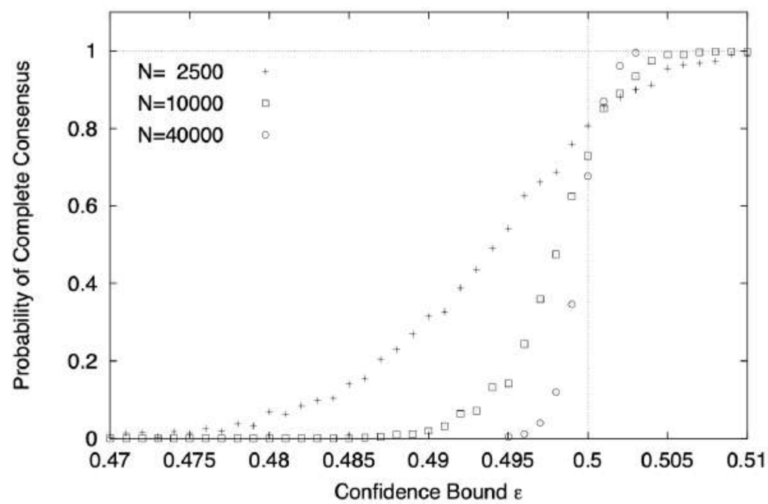


Figure 18: The relations between the confidence bound and the complete consensus in the Deffuant model in the two-dimensional grid.(Fortunato 2005)

The outlying agents that did not settle inside the centrist cluster locate themselves on the histogram ε away from the centrist cluster. This means when ε decreases, the distance between the centrist major cluster and the small satellite clusters also decreases. Additionally, as the centrist major cluster absorbs fewer agents with decreasing ε, more agents join the satellite clusters.(Figure 19 bottom) Similar property is also observed in the complete graph.(Figure 15)

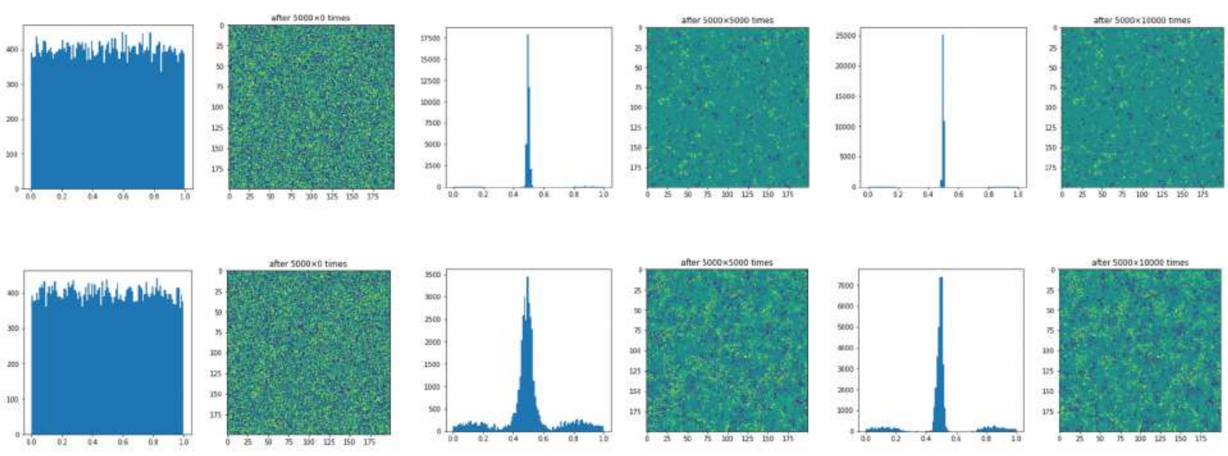


Figure 19: The histogram and spatial distribution from a population with ε=0.3 over 10,000 iterations of 5,000 updates, showing the emergence of scattered agents with non-centrist opinions (top) and those from a population with ε=0.23, showing more agents joining the scattered non-centrist agglomerations.(bottom)

On the spatial opinion distribution, two types of non-centrist agents are visible.(Figure

19) In general, the number of non-centrists are dependent on the value of ε, and at a given value of ε, the number of agents with opinions between 0 and 0.25 appear comparable to the number of agents with opinions between 0.75 and 1. They both appear randomly scattered in the spatial distribution.(Figure 19)

3.2. HK model

The HK model is a discrete opinion dynamics model that was proposed by Hegselmann and Krause in 2002. In this model, opinions take real values on an interval $[0, 1]$. The agent i, with opinion xi, interacts with neighboring agents whose opinions lie within the range $\left[x_i + \varepsilon,\ x_i - \varepsilon\right]$, where ε represents the degree of the confidence bound.(Hegselmann and Krause 2002)

The update rule differs from the Deffuant model in that agent i interact with others one by one, but all at once. The Deffuant model is suitable for describing the opinion dynamics of large populations where people meet in small groups, like pairs, whereas the HK rule is intended to describe formal meetings where there is an effective interaction involving a large population at the same time.(Hegselmann and Krause 2002)

In this model, the opinion of agent i at time t is denoted:

$$x_i(t + 1) = \frac{\sum_{j:\left|x_i(t) - x_j(t)\right| < \varepsilon} a_{ij} x_{ij}(t)}{\sum_{j:\left|x_i(t) - x_j(t)\right| < \varepsilon} a_{ij}}$$

where a_{ij} is the adjacency matrix of the graph. The HK model is fully determined by the confidence bound ε. Unlike the Deffuant model, the agent i takes the average opinion of its compatible neighbors. As the model calculates the averages of opinions upon every opinion update, it is computational expensive and simulation times tend to be longer.

The dynamics in the HK model are similar to those in the Deffuant model at the stationary state. As ε increases, the number of final clusters decreases until ε exceeds the threshold ε_c, at which point only one agglomeration emerges in the final configuration.

The agents in the HK model adopt the average opinion of their group, resulting in a final configuration that is symmetric with 0.5 as the center of symmetry.(Fortunato 2005)

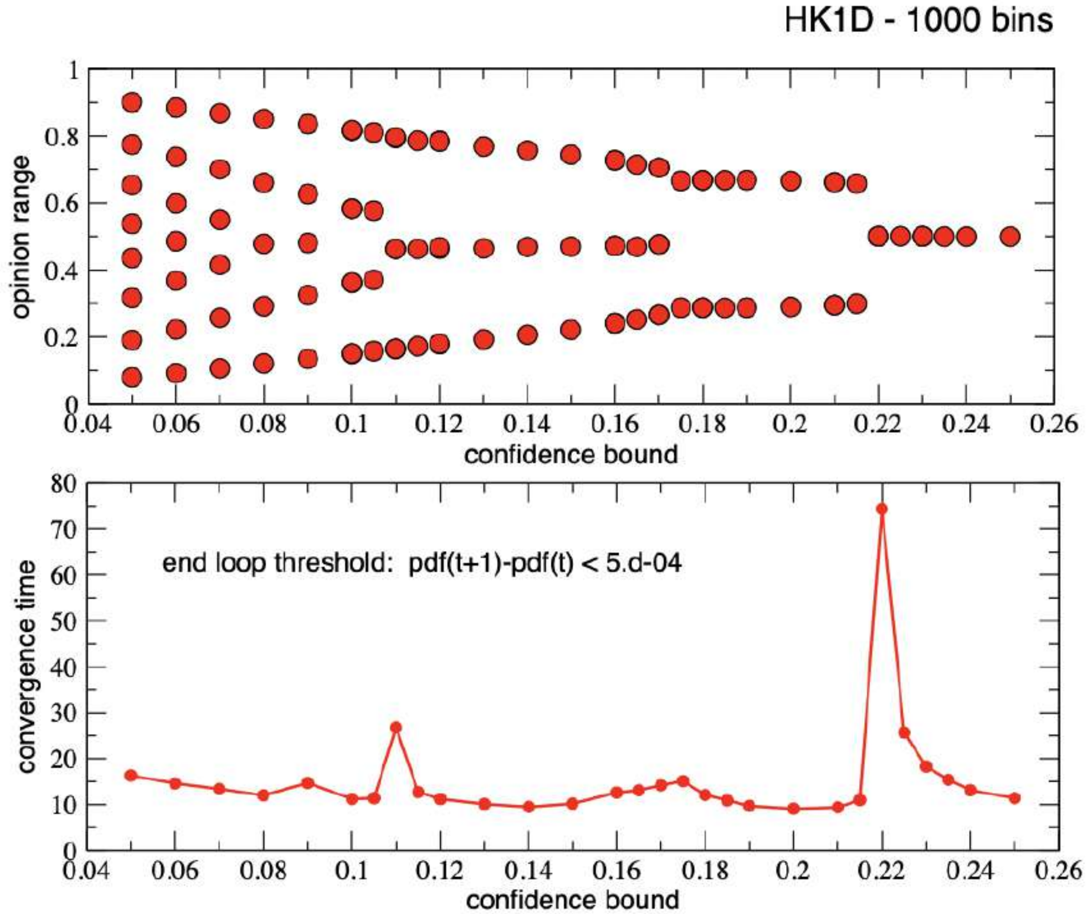


Figure 20: The positions of clusters over different confidence bound (top) and the time to reach a state in which clusters appear static(bottom)(Fortunato et al. 2005)

4. Spatial continuous Deffuant model

In addition, past literature assumed a two-dimensional grid (square lattice), a complete graph, a random graph a la Erdos and Renyi, and a scale-free graph a la Barabasi-Albert for the internal connectivity of the population.(Fortunato 2004) Realistically speaking, limiting the possibility of opinion updates to the neighboring 4 agents as assumed by the square lattice model, is hardly an accurate representation of modern society. Likewise, an agent listening to every single member of the society as assumed by a complete graph is equally unrealistic. Here we aim to assume a graph that falls somewhere in between these two extremes.

4.1. Introduction of “globality” - a spatial parameter

A spatial Deffuant model, which assumes a two-dimensional grid, but with a different opinion update model, is introduced to better represent the reality. In this model, an agent interacts with not only the immediate neighbors of a square lattice but also with the agents at a distance R which represents "globality".

The opinion update rule starts with randomly selecting an agent (j, k) from the population of N agents. Next, another agent within a specified two-dimensional vector $||v|| = max(|x|, |y|)$ is randomly selected.

The length of the two-dimensional vector is defined by the maximum of the absolute values of the vector, denoted as $||v|| = max(|x|, |y|)$. The length of this vector is assumed to satisfy $||v|| < R$, where R represents "globality".

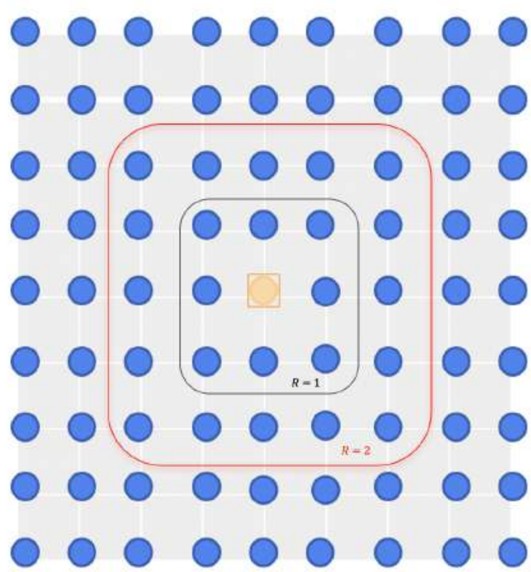


Figure 21: The definition of globality R on a two-dimensional lattice

Setting the distance vector allows us to model a local community to describe intimate relationships. For example, when $R=1$, an agent interacts with another agent existing within the difference vector of 1. This means an agent can interact with one of the 8 candidates within its vicinity.

A larger value of R allows an agent to interact with another agent further away in the two-dimensional grid. When R=2, the agent can interact with one of 24 agents selected within a two-grid radius in all directions, or in other words, one of 24 candidates existing within the difference vector of 2. To put it in a general term, an agent selects one of the $(2R + 1)^2 - 1$ candidates to interact. When $R \rightarrow \infty$, an agent can interact with any other agents in the population, equalling the state of a complete graph.

4.2. Redefinition of consensus

Most studies of the Deffuant model in the past defined “consensus” as 100% of the population holding the same opinion.(Fortunato 2004) Although, the distribution of opinions often converges to a state without significant changes long before the “consensus” is reached. Therefore, it is possible to assume the attainment of consensus before taking extremely long simulation times. In addition, the real society does not expect all members to have exactly the same opinion to declare a consensus.

Therefore in this study, we define “almost consensus” as 80% of the population belonging to the same cluster on the histogram. Still, it is often the case that reaching “almost consensus” requires long simulation times.

Figure 22 shows some of the examples of a population reaching “almost consensus”. In these cases, a centrist cluster in which more than 80% of all agents belong to, and the satellite clusters in which the remaining less than 20% of all agents belong to, are observed. The distance between these clusters are close to the value of the confidence bound.

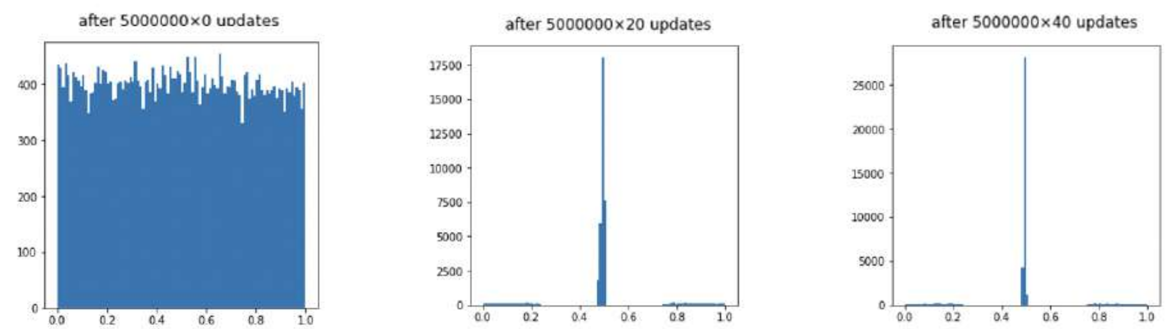


Figure 22: Deffuant model in a square lattice with ε=0.25, over 40 iterations of 5,000,000 updates.

Depending on the value of ε and R, the population may reach a stable configuration at a polarized state, in which the clusters do not move or widen over long simulation times. At such a state, we are likely to observe a twin roughly identical major clusters accompanied with some outlying agents near the edges of opinion space and in between the twin clusters. This means any polarized state would not qualify as “almost consensus”, since its condition of “more than 80% of all agents in the same cluster” would not be satisfied.

4.3. Opinion update procedure

In the spatial Deffuant model, all agents are assumed to be distributed on a two-dimensional grid under periodic boundary conditions, which effectively makes it a toroidal graph. On the histogram, initially, the agents have opinions randomly distributed within 0 and 1.

The opinion update rule starts with randomly selecting one agent (j, k) from the population of N agents. Next, another agent within a specified two-dimensional vector $||v|| = max(|x|, |y|)$ is randomly selected.

The length of this vector is assumed to satisfy $||v|| < R$, where R represents "globality". When the value of R, the globality is large, an agent can interact with another agent in a further distance in the two-dimensional grid.

Setting the distance vector allows us to model a local community to describe intimate relationships. For example, if the value of R is two, an agent can interact with another agent selected within a two-grid radius in all directions. After selecting these two agents, the Deffuant rule is applied as described.

The conditions and threshold of reaching "almost consensus" over a range of R and ε in this spatial continuous Deffuant model were investigated by running a number of simulations.

4.4. Empirical results

Work by Laguna et al on the Deffuant model has identified that when the confidence bound ε is larger than 0.3, the population always reaches an "almost consensus" state with a single centrist peak on the histogram. Likewise, when ε is smaller than 0.2, the population tends to exhibit a fragmented state.(Figure 10) Therefore, the scope of this work was set to the dynamics when the confidence bound ε is between 0.2 and 0.3.

For all values of confidence bound ε and globality R, it was found that both values affect the speed at which "almost consensus" is reached.

When ε is small, such as ε=0.2, it is harder for two agents with different opinions to agree. Then, it takes more iterations to exit a labile state in which fast or significant changes in the distribution occur. Figure 23 and Figure 24 show the opinion distributions when ε=0.2 and ε=0.25 with the same globality $R = 1$. The population reaches almost consensus after 300 iterations when ε=0.25, but the population does not yet reach such a state even after 300 iterations when ε=0.20.

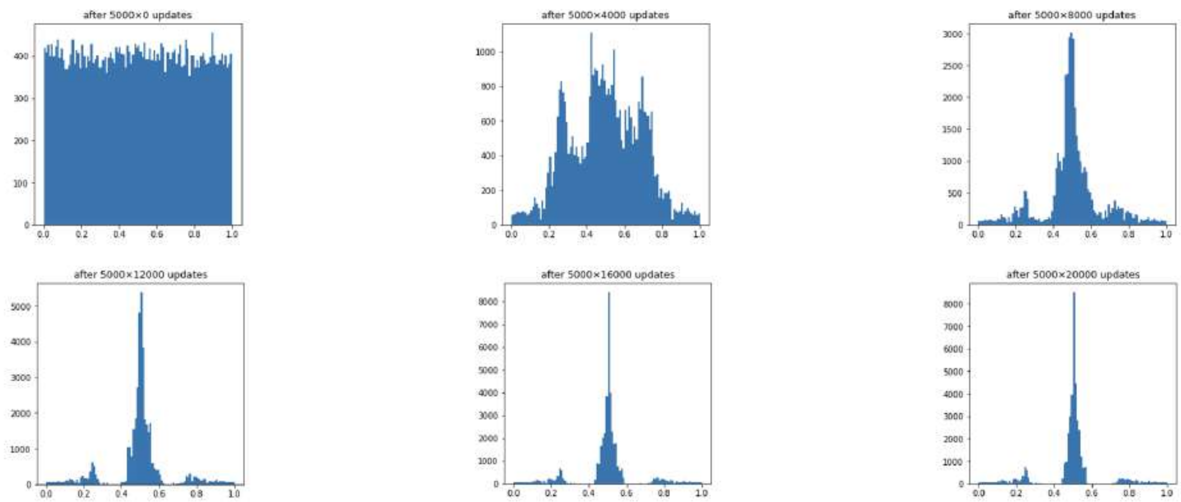


Figure 23: The histogram at $R = 1$, ε=0.2 over 20000 iterations of 5000 updates - the time to “almost consensus” is markedly different from the case when ε=0.25 shown in Figure 24.

Similarly, a small value of R makes it harder for agents to exit a labile state. This is because if the number of neighbors that surround an agent is small, fewer opportunities for opinion exchanges occur. Figure 24 and Figure 25 show the opinion distributions where $R = 1$ and $R = 6$ respectively when ε=0.25. After 700 iterations of 5000 updates, the population with R=6 has exited the labile state while the population with R=1 has not.

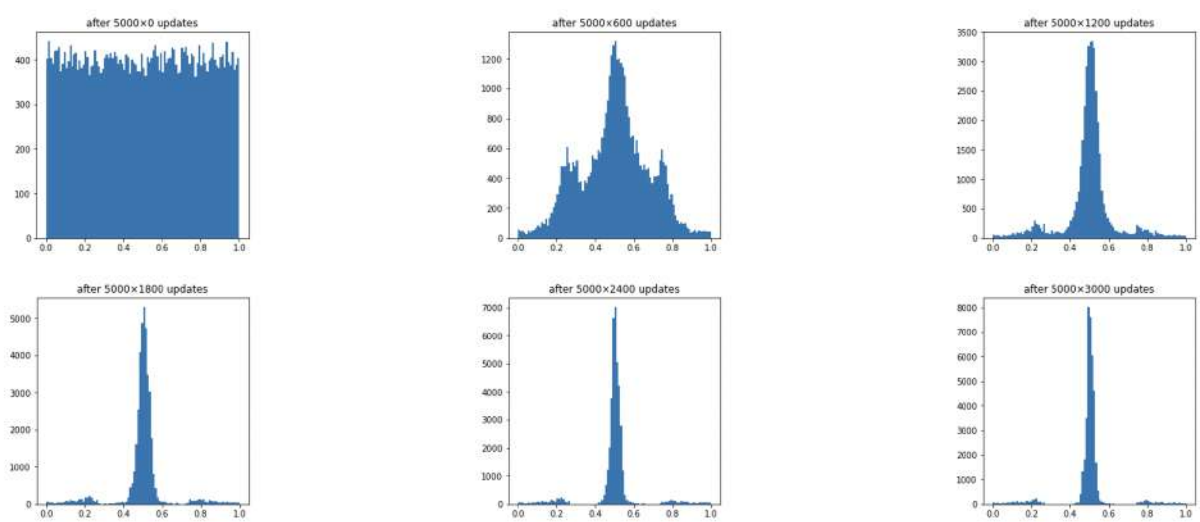


Figure 24: The histogram at $R = 1$, ε=0.25 over 3000 iterations of 5000 updates - larger confidence bound ε appear to reduce the time to “almost consensus”.

Overall, longer simulation times are necessary to reach stable configurations in the Deffuant models when both ε and R are small. In these cases, larger numbers of updates or iterations and therefore longer simulation times are needed to exit a labile state and observe an “almost consensus” state, polarization state, or any other opinion patterns.

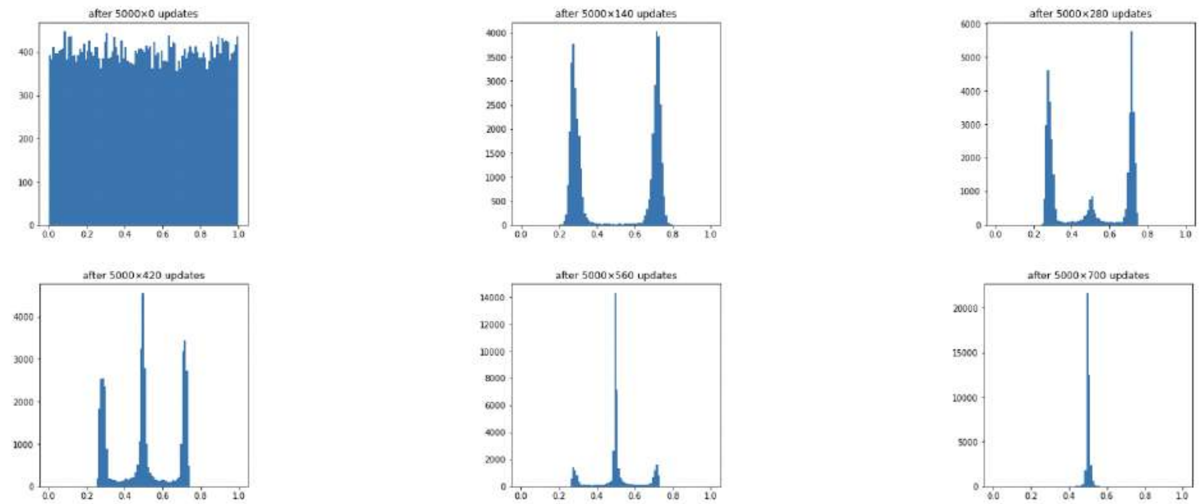


Figure 25: The histogram at $R = 6$, ε=0.25 over 700 iterations of 5000 updates - larger globality appears to reduce the time to “almost consensus”.

4.4.1. Thresholds for convergence

The probabilities of reaching “almost consensus” for different values of globality R and confidence bound ε were calculated from ten instances of simulation runs. The time required until reaching “almost consensus” was significantly longer for smaller values of ε and R as mentioned earlier. The number of iterations was increased tenfold for these cases.(Figure 26 top) The constructed map of probabilities indicated that the likelihood of reaching "almost consensus" is strongly dependent on the values of R and ε. (Figure 26 bottom)

The population is more likely to reach “almost consensus” when the confidence bound is high, regardless of the globality. On the other hand, if ε<0.27, the population may not always reach “almost consensus”. In this case, the possibility of reaching “almost consensus” depends on the value of R and at a given value of ε, the population is more likely to reach “almost consensus” when R is smaller. In addition, the “threshold” of R for reaching consensus is dependent on ε, and the threshold tends to occur at a smaller value of R when ε is smaller. The “steepness” of the threshold does not appear to vary significantly over ε.

	0.2	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.3
local	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1
3	0.8	1	1	1	1	1	1	1	1	1	1
4	0.3	0.5	0.9	1	1	1	1	1	1	1	1
5	0.1	0.2	0.3	0.6	1	1	1	1	1	1	1
6	0	0	0	0.3	0.3	1	1	1	1	1	1
7	0	0	0	0	0	0.6	1	1	1	1	1
8	0	0	0	0	0.2	0.2	0.9	1	1	1	1
9	0	0	0	0	0	0.2	0.9	1	1	1	1
10	0	0	0	0	0	0.1	0.6	1	1	1	1
global	0	0	0	0	0	0	0	0.5	1	1	1

	0.2	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.3
local	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1
3	0.9	1	1	1	1	1	1	1	1	1	1
4	0.3	0.5	0.9	1	1	1	1	1	1	1	1
5	0.1	0.2	0.3	0.6	1	1	1	1	1	1	1
6	0	0	0	0.3	0.3	1	1	1	1	1	1
7	0	0	0	0	0	0.6	1	1	1	1	1
8	0	0	0	0	0.2	0.2	0.9	1	1	1	1
9	0	0	0	0	0	0.2	0.9	1	1	1	1
10	0	0	0	0	0	0.1	0.6	1	1	1	1
15	0	0	0	0	0	0	0.1	0.9	1	1	1
20	0	0	0	0	0	0	0	0.8	1	1	1
30	0	0	0	0	0	0	0	0.9	1	1	1
40	0	0	0	0	0	0	0	0.4	1	1	1
global	0	0	0	0	0	0	0	0.5	1	1	1

Figure 26: The map of probabilities of reaching "almost consensus" across different values of R and ε - while most instances of simulation was done at 50 iterations of 500000 updates, the number of updates was increased by tenfold for cases painted in orange, when the values of R and ε are small. (top) A heatmap constructed on the map of probability of reaching "almost consensus", elucidating a certain threshold of R over different values of ε.(bottom)

4.4.2. Opinion distribution

In the spatial Deffuant model, the final opinion distribution relies on both the confidence bound ε and the globality R.

When ε<0.5, the population initially splits into two clusters regardless of the value of R. This includes the cases when the value of ε is between 0.2 and 0.3. In such cases, the histogram features initial formation of a twin cluster, followed by formation of a large centrist cluster around 0.5 accompanied by smaller satellite clusters.(Figure 25) The majority agents with opinions around 0.5 form the centrist cluster and the minority agents with opinions near 0 or 1 form the satellite clusters. This minority usually makes up less than 20% of the total population.(Figure 25)

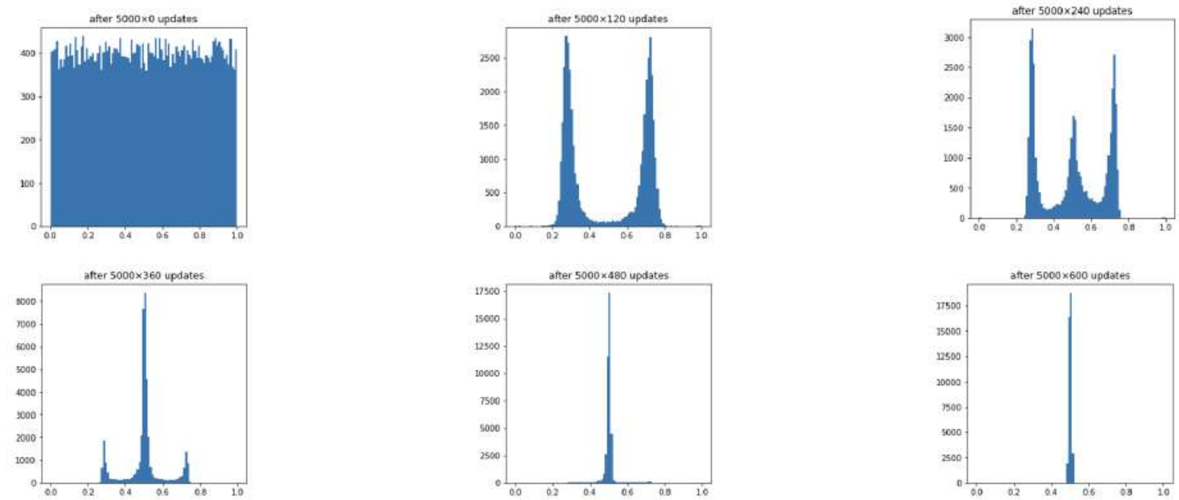


Figure 27: Evolution of clusters at ε=0.25, R=5 over 600 iterations of 5000 updates

Figure 28 looks at this phenomenon in more detail. The initial twin clusters are distinctly visible after 120 iterations. Then, a centrist cluster appears around the 180th iteration. Over time, the centrist cluster grows while the initial twin cluster shrinks. The twin clusters do not appear to move during the process. After around 500 iterations, the population reaches “almost consensus” and by 600th interaction, it further resembles the state of a complete consensus.

At a fixed value of ε, larger globality appears to entail a deeper initial trench between the initial twin clusters, as evident from Figure 28 and Figure 29. Around the 100th iteration of 5000 updates, the initial twin clusters are the most distinct. At this state, a large value of R results in fewer agents in the middle that would have otherwise bridged the two clusters, and a smaller value of R results in more agents to be found in the middle. Yet in both cases, the agents in the middle that bridge the twin clusters eventually form a third cluster, which ultimately absorbs the initial twin clusters to become the main and only cluster. This process appears more distinct in cases where the values of R and ε are close to the threshold illustrated in Figure 26.

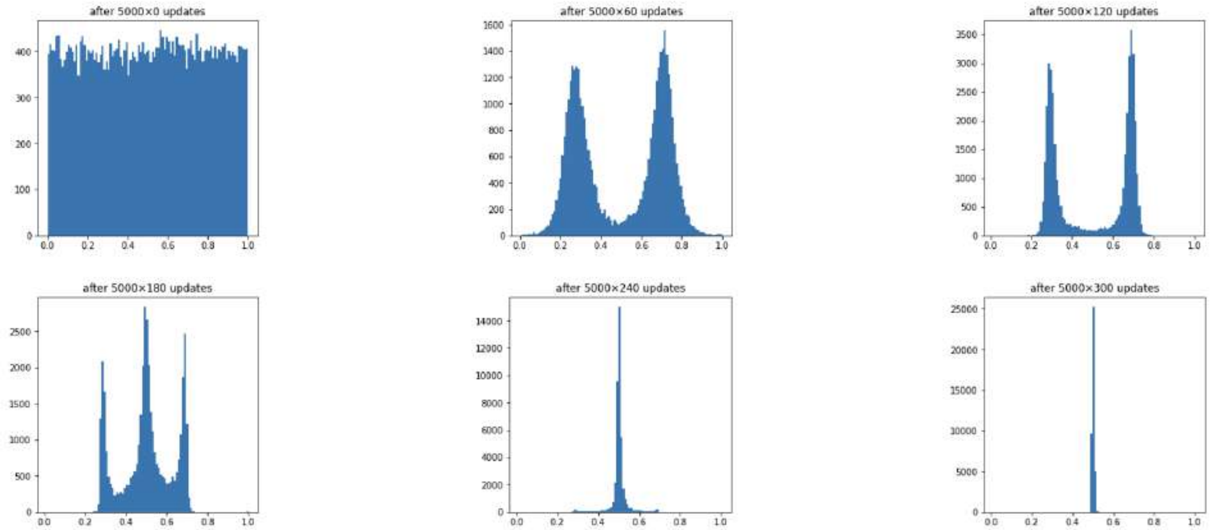


Figure 28: Evolution of clusters at ε=0.27, R=10 over 300 iterations of 5000 updates

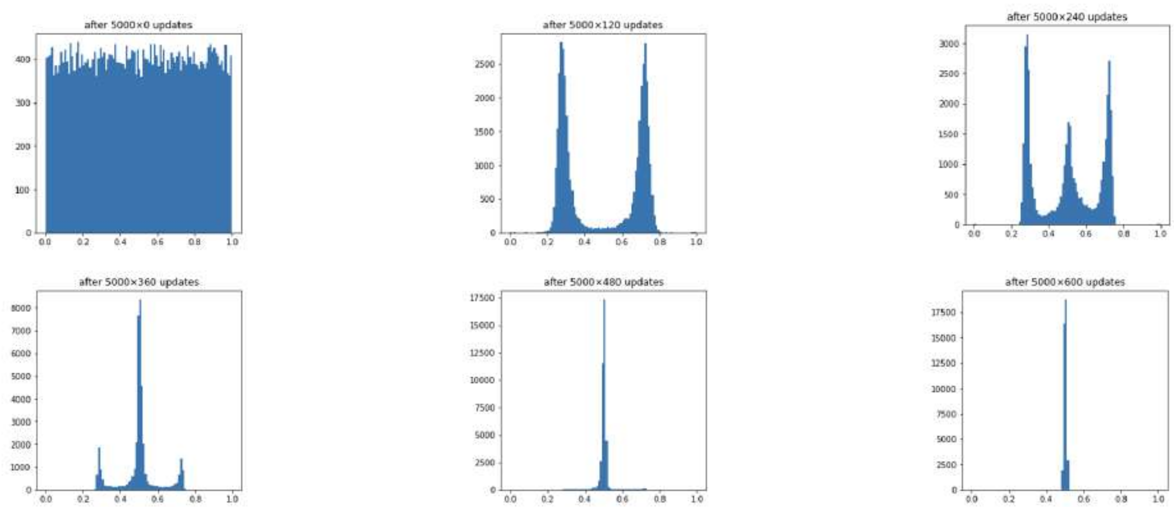


Figure 29: Evolution of clusters at ε=0.27, R=1 over 600 iterations of 5000 updates.

When the value of ε is low such as 0.2 but R is high, the population does not reach the state of “almost consensus”, but a state of multiple clusters such as polarization. These clusters position themselves at a distance approximately 2ε from each other. Most agents are found in these clusters, except a few that are close to the edge of 0 or 1, or in between the clusters.(Figure 30)

It is also observed that as long as the values of ε and R stays on the side of threshold, in which “almost consensus” does not occur (Figure 26), the dynamics are relatively unaffected by the values of ε and R.

Figure 30 and Figure 31 compare the dynamics at a fixed value of ε but different values of R, and Figure 31 and Figure 32 compare the dynamics at a fixed value of R but a different value of ε. In all of the above cases, the dynamics, more specifically the order of events, shape of histogram features, and the number of iterations required to reach a certain state, are not distinctively different.

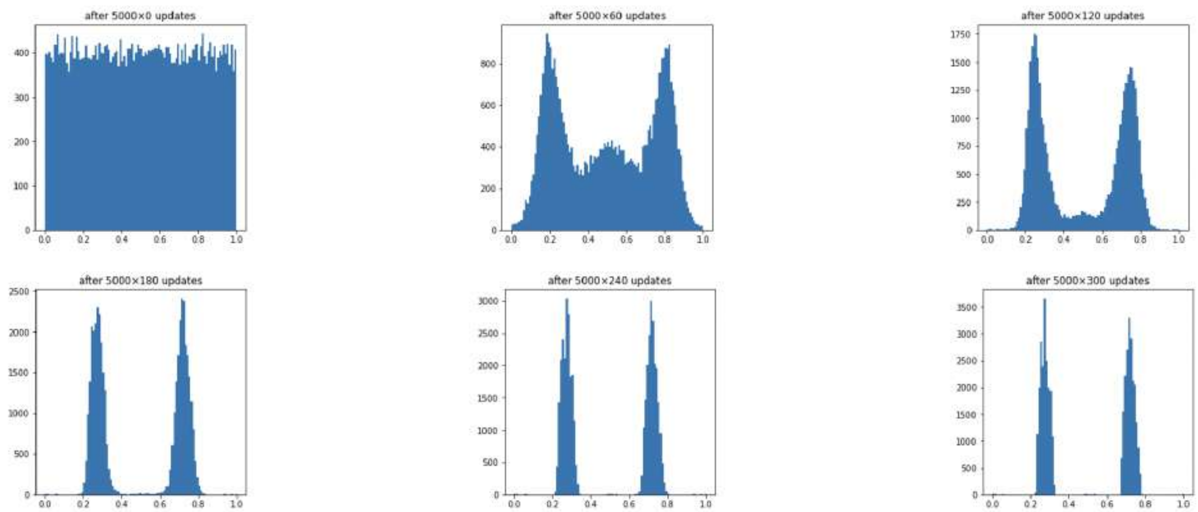


Figure 30: Evolution of clusters at ε=0.20, R=6 over 300 iterations of 5000 updates

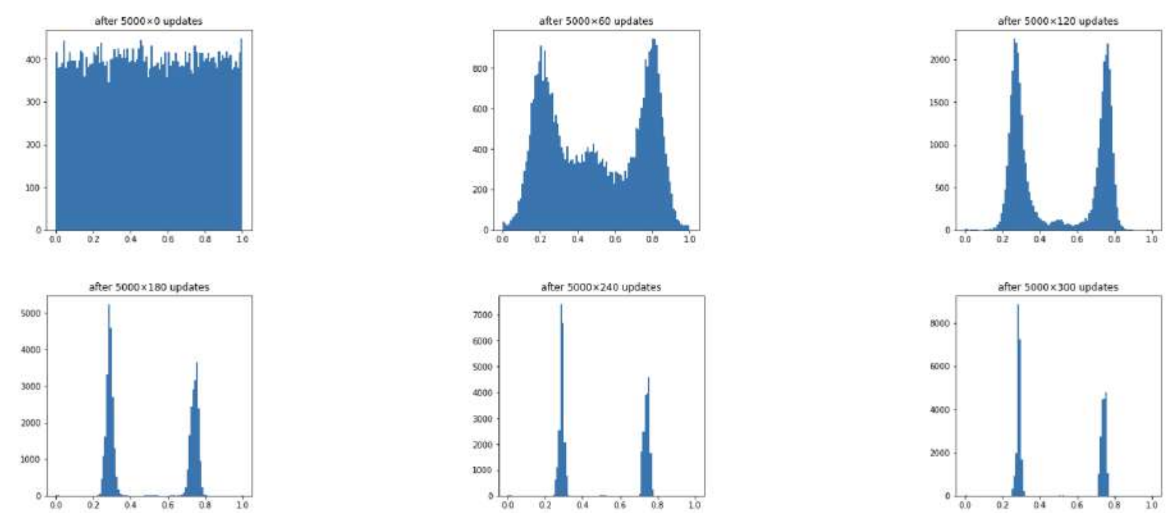


Figure 31: Evolution of clusters at ε=0.20, R=10 over 300 iterations of 5000 updates

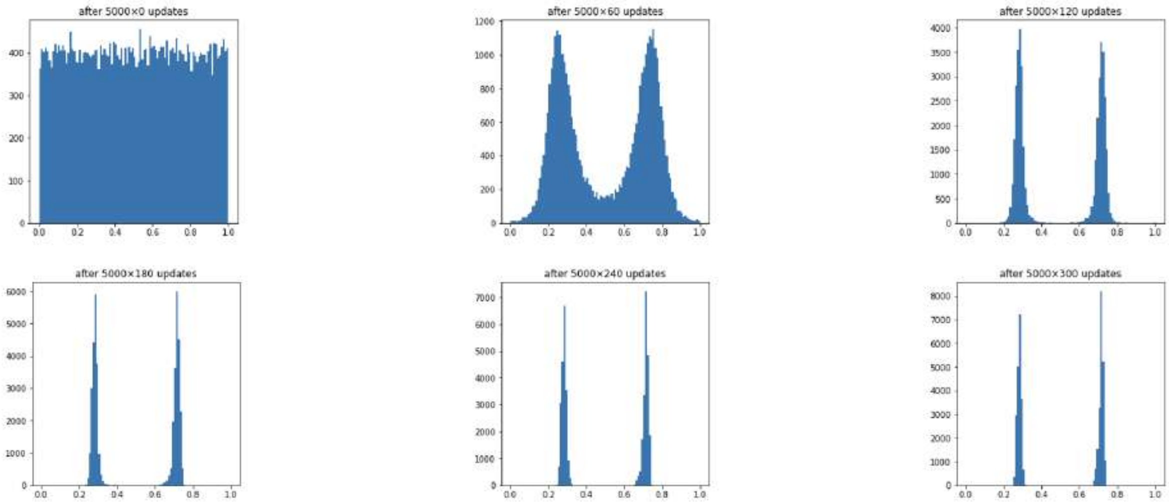


Figure 32: Evolution of clusters at ε=0.24, R=10 over 300 iterations of 5000 updates

4.4.3. Spatial Opinion distribution

The spatial component of the spatial Deffuant model reveals which agent plays what role in the opinion dynamics under what circumstances. For example, the spatial component reveals if opinion polarization is geometrically dependent or not. The histogram and map of spatial distribution at different confidence bound ε, globality R, and the number of iterations were compared to establish how the evolution of spatial distribution determines the outcome.

Figure 33 shows the evolution of histogram and its corresponding spatial distribution of opinions that are represented using a color gradient. The lighter colors represent opinions closer to 0 and the darker colors represent the opinions closer to 1. In the initial configuration, agents holding their opinion are randomly distributed in a two-dimensional space. In this state, different colors representing a diverse range of opinions are found in any locality.

For populations that are expected to reach “almost consensus” (the values of ε and R marked in red in Figure 27), the opinion formation process follows the same pattern regardless of the value of ε and R - initial twin cluster, formation of centrist peak, followed by attainment of “almost consensus” with centrist opinions.

Meanwhile on the spatial distribution, no distinct spatial pattern is observed while the twin clusters initially form on the histogram. In this state, the population’s opinions are polarized, but spatial clustering of opinions is not happening.

Then as the centrist cluster starts growing on histogram, small agglomerations of agents with centrist opinions emerge in the population. As the small agglomerations grow, they sometimes merge with nearby agglomerations, until they cover the entire population. This phenomenon may be compared to nucleation of crystals when liquid freezes. The same process appears to be followed regardless of the value of ε and R. Nonetheless, the details of this process, such as the number of agglomeration nuclei, may be different.

Two populations, one with ε=0.24, R=1 (Figure 34) and the other with ε=0.24, R=5 (Figure 35), are compared to the difference in the details of the process.

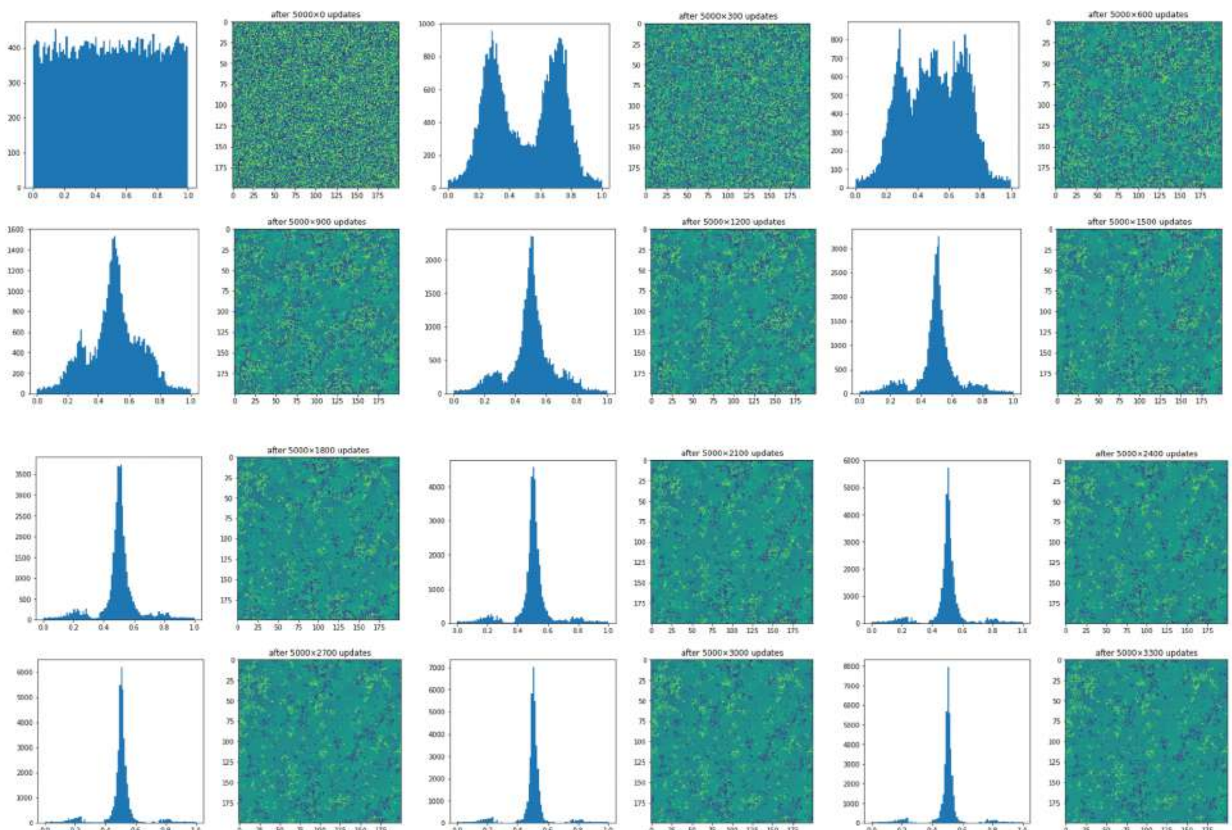


Figure 33: The histogram and spatial distribution after 3300 iterations of 5000 updates at ε=0.24, R=1 - the “nuclei” of agglomerations appear to be scattered across the map.

Around the 300th iteration, the spatial agglomerations are not distinct in either populations. Then as simulations continue, the number of centrist agents between the initial twin cluster increases on the histogram. At the same time, spatial agglomerations of agents with centrist opinions become clearly visible in the spatial opinion distribution. These agglomerations often lose their circular shape over their growth and merger with other agglomerations. (Figure 33) (Figure 34)

By 4000th to 5000th interactions and the population moves toward a consensus, the small agglomerations grow and merge with each other to form a larger agglomeration of centrists. When “almost consensus” is reached, the spatial opinion distribution is mostly occupied by the single majority agglomeration of centrists represented by similar colors.

There are some agents with non-centrist opinions represented with different colors scattered across the distribution, but as previously mentioned, the presence of such agents with outlying opinions is a typical feature of populations with ε<0.5. Their opinions are outside the confidence bound of the majority centrists, and are not influenced by the surrounding agglomeration.

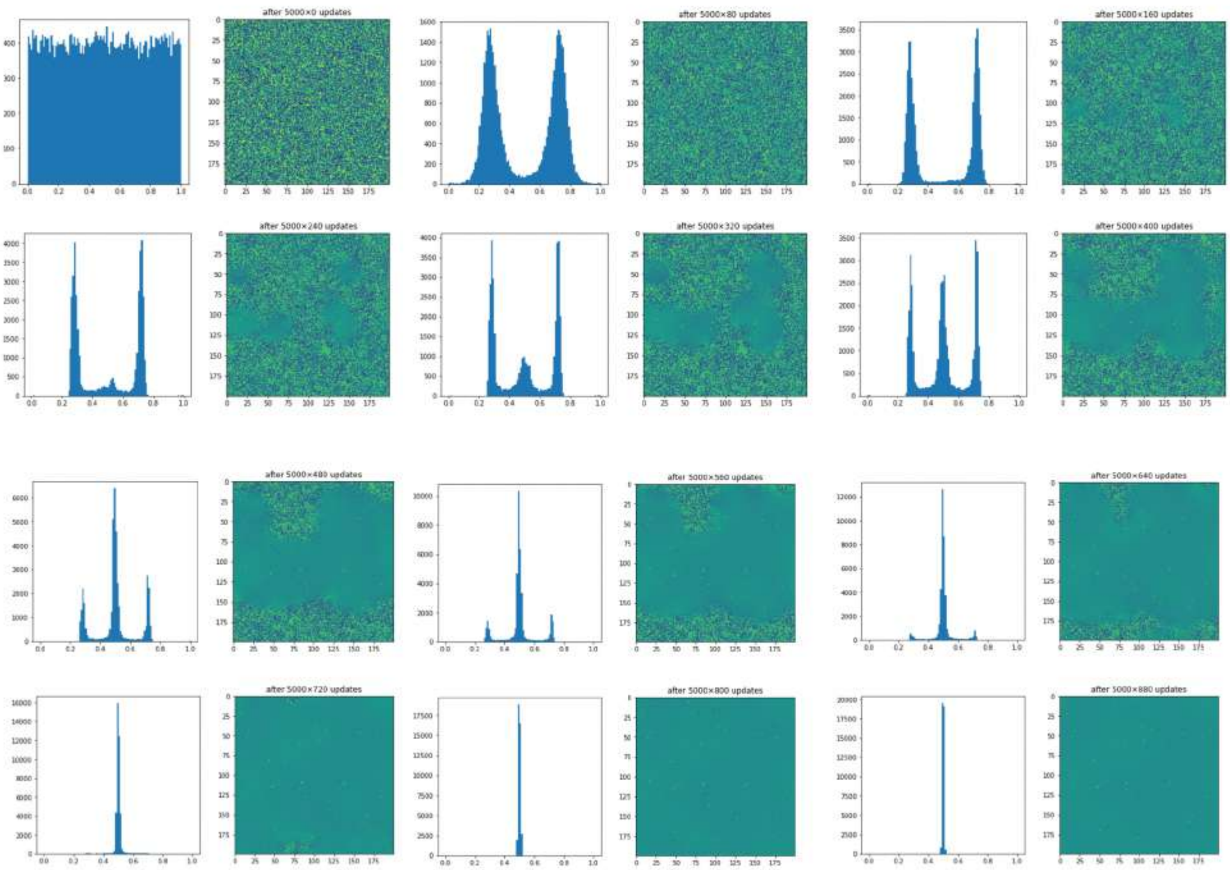


Figure 34: The histogram and spatial distribution after 880 iterations of 5000 updates at ε=0.24, R=5 - fewer “nuclei” of agglomerations were found compared to the case of ε=0.24, R=1(Figure 33)

When the cases of ε=0.24, R=1 and ε=0.24, R=5 (Figure 34 and Figure 35) are compared, we see more outliers when ε=0.24, R=1. This is a reproduction of the result visible from the comparison of Figure 25 and Figure 26. There, a smaller value of R results in more outliers to occur in the state of “almost consensus”.

We also find that the growth of agglomerations starts from fewer locations when ε=0.24, R=5. It may be explained that a larger value of R results in deeper trench between the initial twin clusters, and that leads to fewer chances of nucleation by centrists to occur. But even with fewer nuclei of centrists, the agglomeration of centrists grows until the attainment of “almost consensus”.

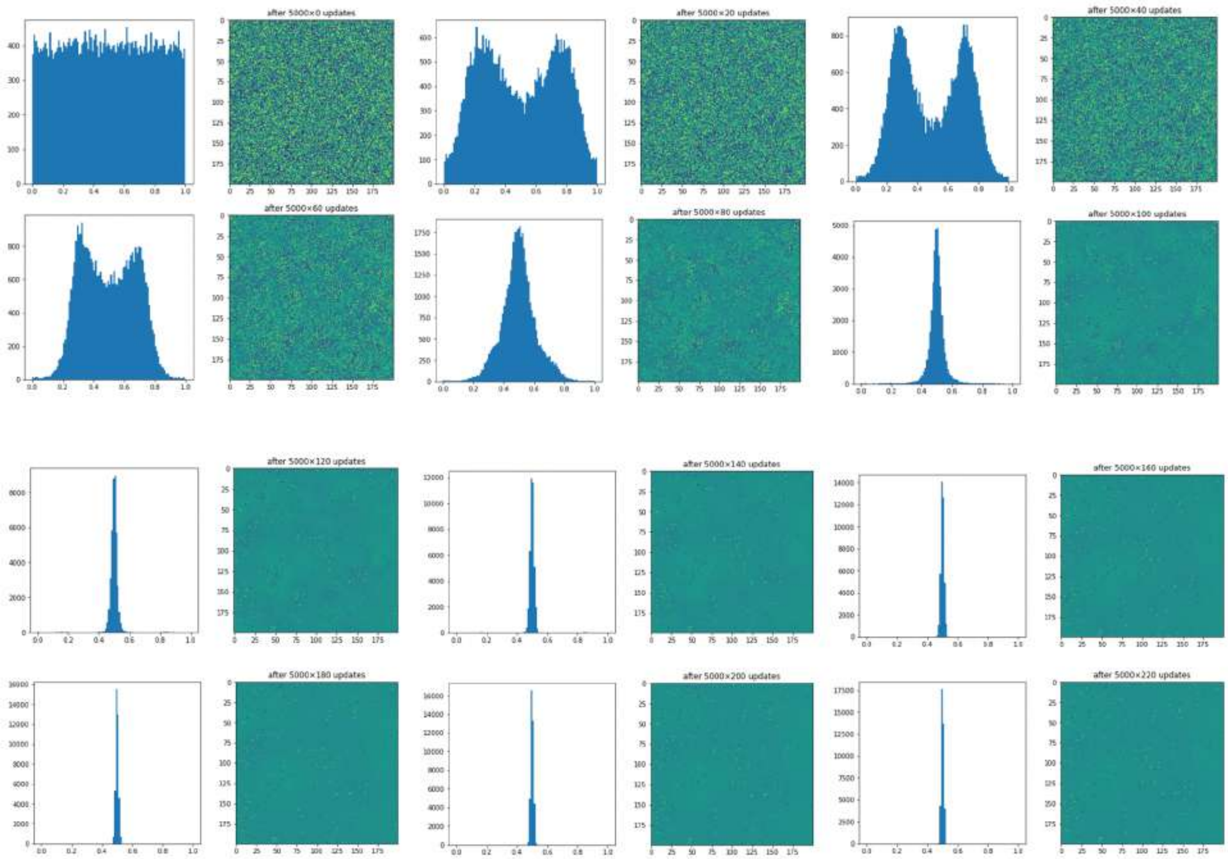


Figure 35: The histogram and spatial distribution after 220 iterations of 5000 updates at ε=0.3, R=5 - the “nuclei” of agglomerations appear to be evenly scattered across the map, and such nuclei are not as distinct as in Figure 33 and Figure 34.

It is hypothesized that nucleation of centrists in the spatial distribution is the cause of the transition from an initial twin cluster to an “almost consensus” state with a single centrist cluster. To further investigate this hypothesis, the value of ε was changed so that the histogram exhibits shallower trench between the initial twin cluster and the population having more nuclei of centrists.

The dynamics of a population with ε=0.24, R=5 (Figure 34) is compared with that with ε=0.30, R=5 (Figure 35). The nucleation events in Figure 35 occur in numerous locations across the spatial distribution. These agglomerations merge during the early phase of growth. While in Figure 34, the nucleation events occur in fewer locations and agglomerations merge during their later phase of growth.

It is observed that a deeper trench between the initial twin clusters results in nucleation of centrists occurring from fewer locations. This may be a characteristic phenomenon of populations with values of ε and R close to the threshold shown on Figure 26.

When ε is small but R is large, the population becomes polarized on the histogram. (Figure 26) Even though the histogram shows a distinctly polarized pattern, no distinct spatial patterns are observed in the spatial opinion distribution.(Figure 36) In addition, it is already shown that a population in the initial transient twin clusters do not show spatial polarization either.(Figure 33, Figure 34 and Figure 35)

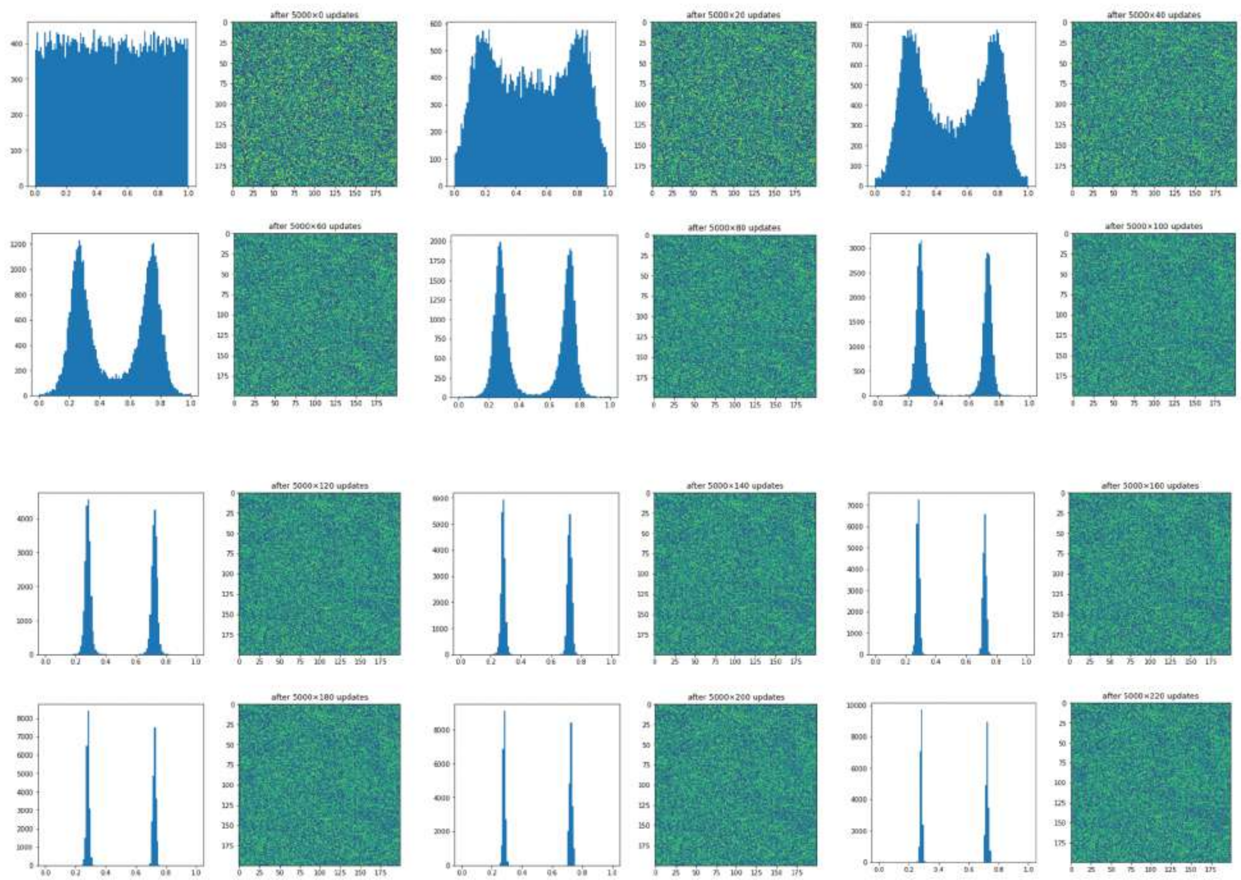


Figure 36: Evolution of clusters and spatial distribution at ε=0.24, R=10 over 220 iterations of 5000 updates

To summarize, polarization of opinions on the histogram does not induce spatial polarization of opinions.

5. Discussion

5.1. The mechanism of reaching a consensus

The results from simulations of populations with different confidence bound ε and globality R indicate that achieving "almost consensus" is actually dependent on the likelihood of nucleation of spatial agglomerations of centrists. Also, the likelihood of nucleation is determined by the depth of trench between the initial transient twin cluster on the histogram. When the depth reaches a level that spatial nucleations of centrists cease to occur, the population ends in polarization. This polarization occurs without spatial polarization.

It is observed that the evolution of a population from the initial transient twin clusters to the growth of a new centrist cluster, is accompanied by the growth of centrist agglomerations on the spatial distribution. It may be hypothesized that agents unidirectionally updating their opinions to the centrist direction occurs primarily at the active interfaces of growing centrist agglomerations. This also explains why initial twin clusters do not move toward each other to merge at the center, but instead form a new centrist cluster that "sucks" agents from the twin peaks.(Figure 25) The small number of agents found between the new centrist cluster and the initial twin cluster act as a conduit for agents to become centrists.

The opinion updates may be occurring at different locations where there is no agglomeration, but pulled by the polarized opinions of the initial twin clusters, updates occur bidirectionally. This puts the agents in an equilibrium and therefore do not show up on the spatial distribution.

On the contrary, the population reaching a stable configuration with a polarized opinion is actually a case where the sequence of events started by nucleation of centrist agglomeration failed to happen and the initial twin cluster is preserved.

In addition, it is generally observed that time taken to reach a stable configuration is independent of whether the population reaches consensus. Meanwhile, higher values of the confidence bound ε or globality R appear to result in quicker attainment of stable configuration. Further confirmation of the relationship between the time it takes to reach a stable configuration and the values of ε and R needs to be investigated further.

5.2. Agents outside clusters

Some agents remain outside of the major clusters or in between the clusters even after a stable configuration is reached. Their opinions are at least ε away from the major clusters. We shall call these agents not belonging to any major clusters as "remainders."

The number of the remainders is found to be affected by the confidence bound and the globality. As the confidence bound increases, the number of the remainders also increases until a certain threshold is reached. However, after crossing that threshold, the number of the outlying agents starts to decrease.(Figure 37)

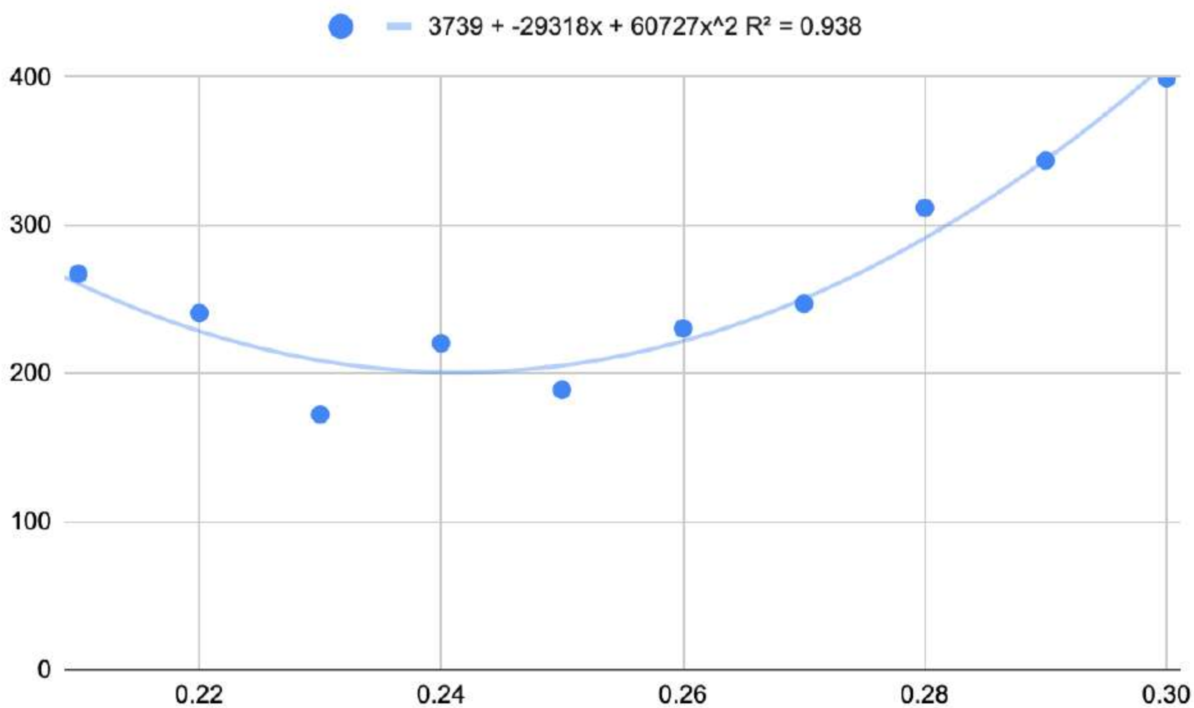


Figure 37: The relation between the value of ε and the number of agents in non-centrist satellite clusters at “almost consensus” states, showing a certain minimum - a population of 200*200 agents and R=3 was assumed.

5.3. Potential real world application

To apply the above studies to the real world, we have to examine the real world meaning of the confidence bounds and the globality.

Like how the majority rule model successfully predicted the result of several political events, the Deffuant model could also be applied to simulate the opinions dynamics in real-world scenarios. For example, the political parties in the United States and Japan could be used to illustrate the concepts of the confidence bound and the globality by the application of this study.

In the United States, the political model is bipartisan between the Republican Party and the Democratic Party. It may be hypothesized that the political opinion distribution and landscape is actively shaped by influential vocal citizens, and as they often have very established political positions and opinions that they equate to agents with small confidence bound. From our study, we find that the small value of the confidence bound leads to polarization of opinions.(Figure 32)

Conversely, Japan's post-WW2 political landscape has been characterized by the dominance of the Liberal Democratic Party winning most of the elections. This could also be hypothesized by people’s tendency to follow the surrounding opinions and their general indifference toward politics, resulting in large confidence bound. A population with a large confidence bound is more likely to converge into a consensus than ending in polarization.(Figure 29)

Another possible example is the impact of social networks on opinion dynamics. Polarization of opinions in modern nations, possibly caused by the use of social networks, is actively discussed.(Casal Bértoa and Rama 2021) Before the age of social networks, interactions between agents were more local and limited to a smaller group of individuals. This may be represented as a small value of globality, but the recent introduction of social network services may have greatly increased the value of globality by establishing lines of influence between large numbers of individuals with a wider range of opinions. This increased globality may be leading to the polarization of opinions.

6. Conclusion

In this work, the two-dimensional Deffuant model was modified to include the local connectivity of agents. The dynamics were visualized by the opinion distribution histogram and the map of spatial opinion distribution. Such visualizations allowed for a clearer understanding of the spatial effect on opinion dynamics. This was achieved by introducing a new parameter of “globality”, the distance within which an agent can interact with neighbors. The existing parameters included the confidence bound and the convergence parameter. In addition, the state of “almost consensus” was newly defined to save computational resources and better represent a real society, in which 100% exact consensus is not required.

It was found that the degree of globality and confidence bound have an impact on the thresholds for reaching “almost consensus”. In addition to the previous findings that larger confidence bound increases the probability of the population attaining a state of “almost consensus”, it was found that smaller number of neighbors, or “low globality” in other words, increased the likelihood of attaining “almost consensus”, regardless of the value of the confidence bound.

As a population capable of attaining “almost consensus” evolves toward this state, agglomerations of centrists were found to appear and grow in size until the whole population is covered. There may be minority agents with outlying opinions scattered across the two-dimensional space, but the overall trend is not affected.

It was hypothesized that conversion of agents to centrist agents occur at the interface of growing agglomeration of centrists. It was also hypothesized that small confidence bound and large globality prevents nucleation of centrist agglomerations, resulting in polarization of the population. It must be noted that polarization of opinion occurs without spatial polarization.

As the next step, the spatially defined phenomenon such as the growth of centrist agglomerations can be quantified and investigated in more detail. This can be achieved by employing image processing methods such as clustering algorithms and dimensionality reduction. These techniques may allow us to identify properties of nucleation phenomena, give us insights into what local agent clusters might qualify as a nucleus to agglomeration, and to help us gain a deeper understanding of the underlying dynamics of opinion formation.

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