A Transformation of the Growth Cycle into the Industrial Cycle in a Four-Dimensional Goodwinian Model with Leontiev Technology

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Abstract

A three-dimensional Goodwinian model L-1, containing the greed feedback loops, reflects destabilizing cooperation and stabilizing competition of investors. The growth rate of output per worker directly depends on growth rate of fixed capital whereas the capital-output ratio is constant. Oscillations imitating growth cycles are endogenous. A crisis is a manifestation of relative and absolute over-accumulation of capital. A knife-edge limit cycle maintains a growth cycle with the Kondratiev duration; a more solid one with a period of about 7.5 years reflects business cycle except reduced net output in a recession. These limit cycles result from the subsequent supercritical Andronov – Hopf bifurcations. The transformation of the growth cycle into industrial cycle gives credit to raising status of capital-output ratio from auxiliary in L-1 to the level status (phase variable) in four-dimensional L-2.

The latter includes new 11 intensive feedback loops. The model implements proportional and derivative control over the capital-output ratio by owners of fixed capital. A pair of supercritical Andronov – Hopf bifurcations cause two limit cycles. The second is a remote analogue for Kuznets cycle with the period of about 18 years; the first upholds the industrial cycle with period of about 7 years and declining output in the outright crisis.

Keywords: capital accumulation, scale effects. relative wage, surplus value, monopoly capital, phases of industrial cycle, limit cycle, supercritical simple Andronov – Hopf bifurcation

JEL Classification Numbers: E11, E27

Introduction

The industrial cycles are middle-term cycles with oscillations of investments into fixed capital (commonly named after Clément Juglar) with typical duration between roughly 5 and 12 years. They are characterised not only by regular fluctuations of positive growth rates of investment, employment, net output but by negative rates of change of these indicators in crises.

A model denoted as Z-1 in Ryzhenkov (2016), containing the greed feedback loops, reflects the destabilizing cooperation and stabilizing competition of investors. In a system of three ODEs, the rate of capital accumulation has become the new phase variable. A targeted long-term increase of the stationary rate of capital accumulation reduces a stationary profit rate together with a raising stationary relative wage. Here and below ODE briefly stands for an ordinary differential equation.

Oscillations imitating industrial cycles are endogenous. A crisis is a manifestation of relative and absolute over-accumulation of capital, as explained in Ryzhenkov (2021). Limit cycle with a period of about 6 years results from supercritical Andronov – Hopf bifurcation (see, particularly, Gandolfo (2010), Fanti and Manfredi (1998)).

A special case of Z-1 with Leontiev technology in Ryzhenkov (2016) denoted here as L-1, free of standard “neoclassical” assumptions, deserves a careful examination. The problem is that unlike the mother model it possesses a constant capital-output ratio – consequently the industrial cycles are not generated any more, and only growth cycles can run their course.

This limitation requires the augmentation of L-1 by an alternative hypothesis on a partial dynamic law governing the capital-output ratio. The transformation of growth cycle into industrial cycle gives credit to raising status of capital-output ratio from auxiliary in L-1 to level in L-2. A family of appropriate feedback loops with pertinent loop gains is to be revealed.

Section 1 is devoted to the tree-dimensional Goodwinian model (with the Leontiev technology) of growth cycles (L-1). Section 2 develops it into a four-dimensional model of industrial cycles (L-2). In these sections, the Marxist theory of extended capitalist reproduction maintained by vanguard system dynamics methodology and mathematical bifurcation theory provides theoretical guidelines for the research. Both L-1 and L-2 are conditionally structurally stable.

Appendixes 1 and 2 explain details of the essential properties of L-1 and L-2. The reader will find Propositions 1–3 for L-1 in Appendix 1; for L-2, subsection 2.2 contains Proposition 5, and Appendix 2 – Propositions 4 and 6.

1. The model of growth cycles L-1

1.1. The extensive form of L-1

At the present level of abstraction, international economic relations and the state economic activity are not explicit. A growth rate of a variable is indicated by a hat directly above it, whereas its time derivative – similarly by a dot. Table 1 lists variables of L-1 and subsequent L-2, considered in the present paper.

Table 1. Main variables in L-1

Variable	Expression
Net product	q
Fixed production assets	k
Capital-output ratio	$s = k/q$
Employment	l
Output per worker	$a = q/l$
Labour force	n = const
Wage	w
Total wage	wl
Relative wage (unit value of labour power)	$u = w/a = wl/q$
Profit	$M = q – wl = (1–u)q$
Profit rate	$R = (1–u)/s$
Capital accumulation rate	z

For a national economy with a generalised Leontief technology (in the meaning that it permits variable input-output coefficients), net output is equivalently determined either as a product of output per worker and employment or as a product of output-capital ratio and fixed capital

$$q = al = (1/s)k. \tag{1}$$

Balance equation (2) shows the end use of net product q, where C is non-productive consumption, $\dot{k}$ is net fixed capital formation defined in the equation (3).

$$q = C + \dot{k} = wl + (1-z)M + \dot{k}\ ; \tag{2}$$

$$\dot{k} = zM = z(1-u)q, \tag{3}$$

$0 < z_{infimum} < z \le 1$; an explicit $z_{infimum}$ will follow.

Investment delays as well as discrepancies between orders and inventories are not taken into explicit account. In result, net fixed capital formation equals net investment. The attribute ‘net’ will be omitted below for brevity.

The price of produced commodity is identically one. Therefore surplus product $(1-u)q$ equals total profit M that can be not only invested, but also be used to cover personal expenses of the bourgeoisie and via implicit taxes for unspoken public consumption. Consequently, rate of accumulation z, is measured as the investment share of surplus product, or as ratio of investment to profit.

A simplified Phillips equation defines the growth rate of wage

$$\hat{w} = f(v), \tag{4}$$

where $f'(v) > 0,$ for $v \to 1 \quad f(v) \to \infty$.

For certainty, a specification satisfying these requirements is applied in the models

$$f(v) = -g + \frac{r}{(1-v)^2}. \tag{5}$$

Achieving a target employment ratio X requires, as a rule, adding a control parameter in (5):

$$f(v) = -g + \frac{r}{(1-v)^2} + \omega. \tag{6}$$

such as

$$\omega = f(X) + g - \frac{r}{(1-X)^2}. \tag{7}$$

Clearly (5) becomes a particular case of (6) for $\omega = 0$.

As in Boggio (2006), the growth rate of output per worker is assumed to be a linear function of a growth rate of fixed capital

$$\hat{a} = \alpha + \gamma\hat{k}, \tag{8}$$

where $0 < \gamma < 1$. Differently from a similar logistic function considered in Lordon (1995) and Ryzhenkov (2021), this simplification serves avoiding the problem of multiple equilibria in the present paper.

An addition of scale effects in the form of (8) in the original 2-dimensional Goodwin model has turned closed orbits into divergent trajectories in phase space in Boggio (2006). These scale effects will be also present in 3-dimensional model that encompass and refine these preceding models in next subsection. The scale effects are thereby “tamed” with a help of an appropriate non-linear ODE for an endogenous rate of capital accumulation.

1.2. The intensive form of L-1

The following soon ODE (11), first, takes into account, in agreement with the views of K. Marx in the first and third volumes of "Capital", that net change of the share of investment in the surplus product has an opposite sign in response to relative wage gains. The negative feedback of the 3rd order containing the rate of accumulation z, employment ratio v, and labour value u, was implicitly expressed by Marx (1863–1883, p. 634).

Net change of the share of investment in surplus product, first, has an opposite sign in response to profitability gains as surmised in (11).This equation, second, reflects capitalists' soft targeting of the rate of capital accumulation at $z_b = z_{goal}$; restrictions $p > 0$ is a prerequisite for proportional control. Third, the product $z(Z - z)$ reflects logistical dependence of $\dot{z}$ on z that bounds trajectories in the phase space while a magnitude of Z codetermines amplitude of fluctuations. It permits accounting for the real long-term tendency of capital accumulation rate z to decline.

The intensive form of L-1 is a system of three ODEs for the relative wage, employment ratio and rate of capital accumulation

$$\dot{u} = \left[f(v) - \alpha - \gamma \frac{z(1-u)}{s} \right] u , \tag{9}$$

$$\dot{v} = \left[(1-\gamma) \frac{z(1-u)}{s} - (\alpha + \beta) \right] v . \tag{10}$$

$$\begin{aligned} \dot{z} &= -b\hat{R}z\left(Z - z\right) + p(z_b - z) = \\ &= -b\frac{\dot{u}}{1-u} z\left(Z - z\right) + p(z_b - z), \end{aligned} \tag{11}$$

where s = const, $b \ge 0$, $p > 0$, $z_{infimum} < z_b < Z \le 1$.

The system (9)–(11) has non-trivial stationary state

$$E_b = (u_b, v_b, z_b) , \tag{12}$$

where $0 < u_b = 1 - \dfrac{ds}{z_b} < 1$, $z_{infimum} = sd < z_b < Z \le 1, v_b = f^{-1}(\hat{a}_b) = X$, where (6) is applied.

Notice that for $sd \ge z_b$ there is a violation of the socio-economic requirement $0 < u_b$; $u_b < 1$ is true only if $d > 0$.

The stationary rate of growth of output per worker, capital intensity and wage is defined as

$$\hat{a}_b = (k \hat{/} l)_b = \hat{w}_b = (\alpha + \gamma\beta) / (1 - \gamma) . \tag{13}$$

The stationary rate of growth of fixed production assets and net product is determined

$$\hat{k}_b = \hat{q}_b = \hat{a}_b + \beta = (\alpha + \beta) / (1 - \gamma) = d. \tag{14}$$

The stationary capital-output ratio and profit rate are specified as

$$s = const , \tag{15}$$

$$R_b = (1 - u_b) / s = d / z_b . \tag{16}$$

There is the stationary employment ratio – stationary relative wage trade-off in L-1: the higher γ, the higher is the first and the lower is the second. For specification (5) of (4), we have $\frac{\partial v_b}{\partial g} > 0$ and $\frac{\partial v_b}{\partial r} < 0$.

An increase in stationary growth rate of net output d, *ceteris paribus*, affects relative wage u_b negatively. The higher is rate of capital accumulation z_b, the higher is stationary relative wage u_b and the lower is stationary profit rate R_b. Similarly, the higher is output-capital ratio $1/s$, the higher is stationary relative wage u_b and the lower is stationary profit rate R_b.

Figure 1 as well as Tables 2 and 3 reflect a condensed stock-and-flow structure of L-1 near unstable E_b (12) undergoing the simple Andronov – Hopf bifurcation (AHB). Otherwise on the place of R2 there would be a negative feedback loop of the same order and length. Initial vector x_0 is not depicted for brevity.

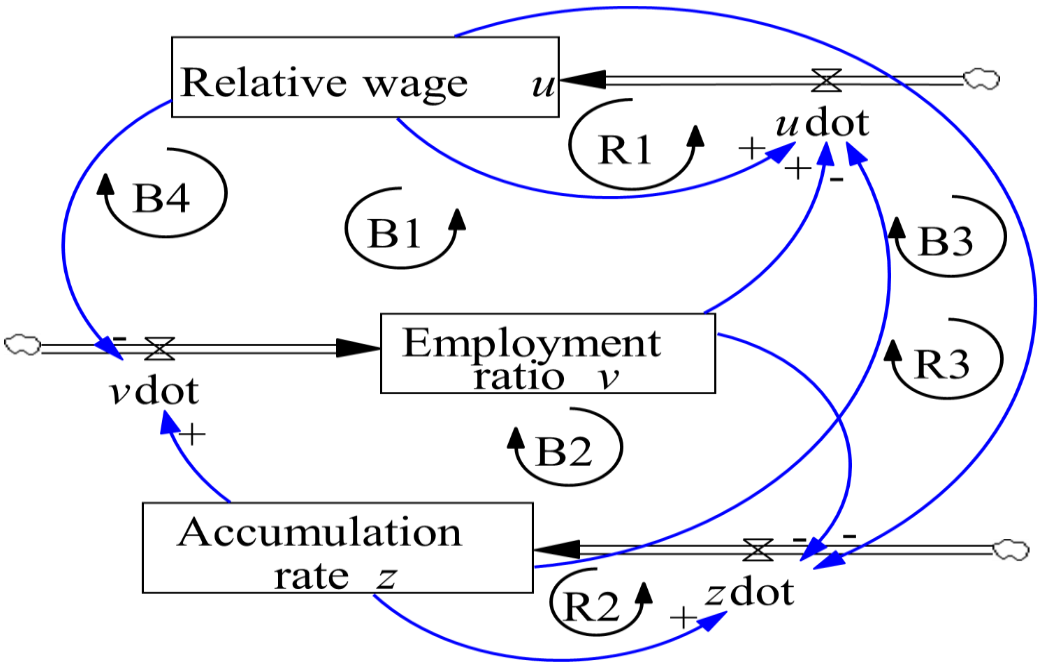

Figure 1 – A condensed stock-and-flow diagram of L-1 at E_b (12);
a total number of feedback loops – 7, among them: 1st order – 2 (*positive*), 2nd order – 3 (2 – negative, 1 – *positive*), 3rd order – 2 (2 – negative)

Table 2. The signs of partial derivatives at unstable E_b (12) undergoing AHB in L-1

Net change (flow variable)	Phase (level) variable		
	u	v	z
$\dot{u}$	1	1	–1
$\dot{v}$	–1	0	1
$\dot{z}$	–1	–1	1 (or –1)☺

☺ 1 is for the 1st AHB with $b_{critical} = 150 > b_0$; –1 is for the 2nd AHB with $b_{critical} = 0.6745$ with weakly sensitive z.

Table 3. The intensive feedback loops in L-1 at unstable stationary state E_b (12) undergoing AHB with $b_{\text{critical}} > b_0$ for the shorter cycle

Quantity	Order	Feedback loops
2	1st	R1 of length 1 $u \to \dot{u}$ R2 of length 1 $z \to \dot{z}$
3	2nd	B1 of length 3 $u \xrightarrow{-} \dot{v} \to v \to \dot{u}$ B2 of length 3 $v \xrightarrow{-} \dot{z} \to z \to \dot{v}$ R3 of length 3 $u \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{u}$
2	3rd	B3 of length 5 $u \xrightarrow{-} \dot{z} \to z \to \dot{v} \to v \to \dot{u}$ B4 of length 5 $v \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{u} \to u \xrightarrow{-} \dot{v}$

Note. R2 and R3 have been named greed feedback loops in L-1 and L-2 as in Z-1 in Ryzhenkov (2016).

Only one feedback loop in L-1 at the unstable stationary state for the 2nd AHB at $b = b_3$ differs from the 1st AHB at $b = b_0$: the negative first-order feedback loop of length 1 $z \xrightarrow{-} \dot{z}$ substitutes R2 $z \to \dot{z}$. Table 4 reports on quantitative differences in the most relevant partial derivatives.

Table 4. Magnitudes of partial derivatives of the accumulation rate for unstable stationary state E_b (12) undergoing 1st AHB for $b_{\text{critical}} = 150$ (1) or 2nd AHB for $b_{\text{critical}} = 0.6745$ (2)

$\frac{\partial \dot{z}}{\partial u}$		$\frac{\partial \dot{z}}{\partial v}$		$\frac{\partial \dot{z}}{\partial z}$	
1	2	1	2	1	2
–0.046	–0.0002	–7.368	–0.033	0.029	–0.199

Relative and absolute over-accumulation of capital

Positive declining profit rate $R = \frac{1-u}{s}$ ($\hat{R} < 0$) is the indicator for a relative excess of capital. The latter can be circular and/or cyclical.

A deeper Marx's analysis in the third volume of "Capital" distinguishes two forms of absolute excess of capital in Marx (1863–1883):

1) of type 1, if the fall in the profit share (unit surplus value) is not compensated through the mass of surplus labour, when the increased capital produced just as much, or even less, surplus value than it did before its increase;

2) of type 2, if the fall in the rate of profit is not compensated through the mass of profit, when the increased capital produced just as much, or even less, profit than it did before its increase.

We will establish that relative and even absolute capital over-accumulation is the necessary yet not sufficient conditions for a slump in a proper crisis of industrial cycle.

1.3. Supercritical Andronov – Hopf bifurcations and self-sustained growth cycles

Parameter b from (11) has been taken as a bifurcation parameter. *Propositions* 1 and 2 have been proved that E_b (12) is locally asymptotically stable for $0 < b_3 < b < b_0$ and that AHB takes place in the system (9)–(11) at $b_{critical} < b_3$ and $b_{critical} > b_0$ (see Appendix 1 based on Ryzhenkov (2016)).

Consider the first AHB.[1] According to simulation runs, a supercritical AHB occurs at $b = b_{critical} > b_0$. The period of oscillations near E_b is about $2\pi / \sqrt{a_1(b_0)}$. Then growth cycles shape the economic dynamics on the transient to a closed orbit as a periodic attractor in the phase space. Different stable limit cycles differ from each other by period and amplitude depending on a particular magnitude of the chosen control parameter.

Notice that the second AHB brings about a remote analogue of the Kondratiev cycle. In this case, the limit cycle is the knife-edge property of dynamics for a particular magnitude of the control parameter very close to b_3 (see Table 10). The shorter limit cycle in L-1 is stronger structurally stable than the longer one.

1.4. The simulated growth cycle

The plausible common parameters' magnitudes have served in simulation runs: $\alpha = 0.00586$, $\beta = 0$, $\gamma = 0.4043$, $p = 0.2$, $g = 0.06828$, $r = 0.0005$, $d = 0.00985$, $u_0 = 0.7804 > u_b = 0.6749$, $v_0 = 0.9127 < v_b = 0.92$, $z_0 = 0.1014 < z_b = 0.06575 < Z = 0.25$, $s = 2.17$.

Consider simulation experiments on middle-term growth cycles brought about in L-1 by the first AHB at $b_{\mathrm{critical}} > b_0$. Remember that under the second AHB related to $0 < b_{\mathrm{critical}} < b_3 = 0.67455$, there is other closed orbit with a substantially longer period in the range of the Kondratiev cycle (see Table 10).

In the absence of exogenous shocks supposed, fixed assets and net output do not absolutely decline in L-1. Phases of the growth cycle will be delineated based on the profit. This aggregate reaches its local maximum on completion of the boom with the onset of the recession. Ending its fall expresses completion of recession, whereas achieving the pre-recession peak completes recovery. Depression is defined as a phase starting at the end of the recession and ending before recovery when unemployment ratio $1 - v$ becomes (locally) maximal.

For chosen $b = b_{critical} = 150 > b_0 = 120.2$, there is a movement along limit cycle from the initial phase vector x_0. The period of oscillations either close to the initial vector or near E_b is about $7.5 < 2\pi / \sqrt{a_1(b_0)} \approx 8.843$ (years).

Net investment is at the peak in 1.75 y. The booms ends with highest profit in 2.5 y., recession continues until 5.5 y., whereas depression, as next phase, continues until the locally minimal employment ratio is reached in 7.5 y., passing of the pre-recession local maximum of profit happens at the very end of recovery in 8.25 y. Boom continues until next maximum of profit in 10 y. The previous locally maximal employment ratio of 3.25 y. is observed during the recession in 10.75 y., this phase lasts until 13 y.

Why investment $z(1–u)P$ decline before profit? It may be a shortcoming of L-1. Still capitalists can reduce investment in their anticipation of a soon onset of the recession. A competing

[1] The literature on applications of AHB in economic modeling is fast growing and cannot be fully reviewed here. Few references must suffice. Brøns and Sturis (1991), Lordon (1995), Fanti and Manfredi (1998), Asada and Yoshida (2003), Ryzhenkov (2013, 2016, 2021) applied Hopf bifurcation in analysis of the economic long waves and other fluctuations in models reduced to two-, three- and four-dimensional systems of non-linear ODEs. The additional contributions deserve examination beyond the limited scope of the present paper.

view, not fully excluding the previous one, understands such a reduction as the manifestation of a hidden over-production and over-accumulation.

The bottoming of investment opens the way to increases in profitability, surplus value, profit and employment. Figure 2 and Table 5 reflect these processes.

Investment behaviour of capitalists looks like anticipatory – peak of investment in 1.75 y. precedes onset of relative over-accumulation in 2 y., accompanied by absolute over-accumulation judged by surplus value in 2.25 y. and absolute over-accumulation judged by profit in 2.5 y. The employment ratio lags behind these four indicators (investment, profit rate, surplus value, and profit).

The upward arc of the profit cycle comprises 60 per cent of the cycle length (20.5–25 y.), the downward arc (17.5–20.5 y.) – the remaining 40 per cent. Such asymmetry is a common property of realistic business cycles models emphasised by Blatt (1983).

According to Table 5, the completed cycle stretches from the 1 quarter of 10 y. through 2nd quarter of 17.5 y. for about 7.5 years.

Table 5. Duration of phases of the two adjacent growth cycles (years) in L-1

Boom	Recession	Depression	Recovery	Boom	New cycle
8–10	10–13	13–15	15–15.75	15.75–17.5	10–17.5
2	3	2	0.75	1.75	7.5

Relative capital over-accumulation encompasses 17.25–20.5 y., absolute capital over-accumulation of type 1 presides over 17.25–20.75 y., absolute capital over-accumulation of type 2 continues during 17.5–20.5 y. A succession of local extrema of indicators' growth rates over 2018.25–27.5 y. is presented in Figure 2.

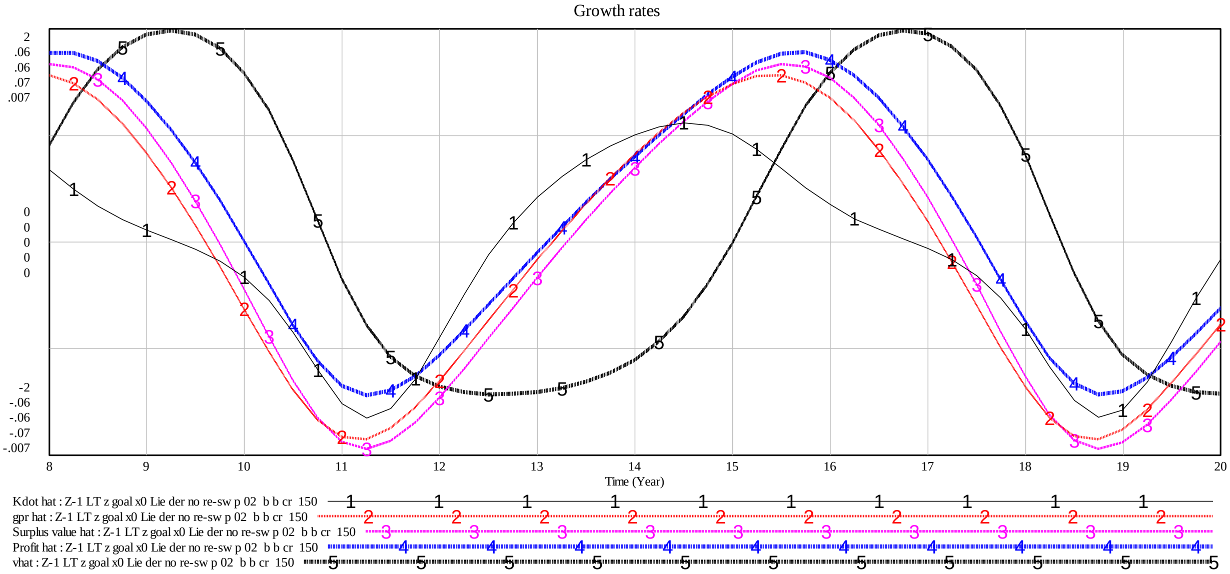

Figure 2 – Growth rates of investment (1), profit rate (2), surplus value (3), profit (4), and employment ratio (5) over the growth cycles along the limit cycle (related to the 1st AHB) in L-1, years 8–20

2. The industrial cycle in L-2

2.1. The L-2 intensive form and properties of its stationary state

The well-known fact of macroeconomics is close negative relation between growth rates of the employment ratio and the capital-output ratio: slack in employment is accompanied by low rate of capacity utilization (reflected by output-capital ratio); tight labour market and high capacity utilization also complement each other. Besides that, similar to the target rate of capital accumulation, a target output-capital ratio suggests itself. Here the proportional control is likely weaker. These working hypotheses determine a new equation for the growth rate of the capital-output ratio (17) and correspondent differential equation (18)

$$\hat{s} = j_1\hat{v} + j_2(s_b - s), \tag{17}$$

$$\dot{s} = [j_1\hat{v} + j_2(s_b - s)]s, \tag{18}$$

where $-1 < j_1 < 0,\ 1 > j_2 > 0.$

The ODE for the relative wage u is not affected by this extension. The latter transforms two ODEs for v and z:

$$\dot{v} = \left[\frac{1-\gamma}{1+j_1}\frac{z(1-u)}{s} - \frac{j_2}{1+j_1}(s_b - s) - \frac{\alpha+\beta}{1+j_1}\right]v, \tag{19}$$

$$\begin{aligned}\dot{z} &= b\left(-\frac{\dot{u}}{1-u} - \hat{s}\right)z\left(Z - z\right) + p(z_b - z) = \\ &= b\hat{R}z\left(Z - z\right) + p(z_b - z),\end{aligned} \tag{20}$$

where the growth rate of profitability is

$$\hat{R} = -\frac{\dot{u}}{1-u} - \hat{s}. \tag{21}$$

An intensive deterministic form of L-2 uses one equation of intensive form of L-1 for relative wage u (9), it replaces equations (10) with equation (19) for employment ratio v, the equation (20) substitutes (11), and finally, L-2 becomes four-dimensional after gaining the new level variable s that represents the capital-output ratio in (18).

A positive non-trivial stationary state is defined in L-2 as

$$X_b = (u_b, v_b, s_b, z_b), \tag{22}$$

It has equivalent counterparts (12) in L-1. It is assumed for an illustrative purpose that s_b in (22) and s = const in L-1 are equal to each other. The qualitative characteristics of (12) and those of (22) are mostly the same. Particularly, the deep rooted interest of monopoly capital in lowering targeted z_b and increasing targeted s_b can be easily made bare again.

The control (bifurcation) parameter b in L-2 plays the similar role as b in L-1. The stationary growth rates of labour force, employment, output per worker, capital intensity, net output, fixed capital, wage, profit and surplus value are the same as in L-1 for $s = s_b$. Tables 6 and 7 inform the reader about the new feedback loops for level s in L-2 in relation to L-1.

Table 6. Signs of partial derivatives at X_b in L-2 (applied in Table 7)

Net change (flow variable)	Phase (level) variable			
	u	v	s	z
$\dot{u}$	1	1	1	–1
$\dot{v}$	–1	0	–1	1
$\dot{s}$	1	0	1	–1
$\dot{z}$	–1	–1	–1	1 (or –1)☺

☺ The 1st AHB with $b_{critical}$ = 40 implies 1, –1 is for the 2nd AHB with $b_{critical}$ = 1.39.

Table 7. The new feedback loops involving level *s* in L-2

Loop		
1st order	2nd order	3rd order
Number 1 of length 1 – positive $s \longrightarrow \dot{s}$	Number 2 of length 3 – positive $s \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{s}$	Number 4 of length 5 – positive $s \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{u} \to u \to \dot{s}$
	Number 3 of length 3 – positive $s \to \dot{u} \to u \to \dot{s}$	Number 5 of length 5 – positive $s \to \dot{u} \to u \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{s}$
		Number 6 of length 5 – negative $s \xrightarrow{-} \dot{v} \to v \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{s}$
		Number 7 of length 5 – negative $s \xrightarrow{-} \dot{v} \to v \to \dot{u} \to u \to \dot{s}$

Table 7 (continued).

4th order
Number 8 of length 7 – negative $s \xrightarrow{-} \dot{v} \to v \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{u} \to u \to \dot{s}$
Number 9 of length 7 – negative $s \to \dot{u} \to u \xrightarrow{-} \dot{v} \to v \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{s}$
Number 10 of length 7 – negative $s \xrightarrow{-} \dot{z} \to z \to \dot{v} \to v \to \dot{u} \to u \to \dot{s}$
Number 11 of length 7 – negative $s \xrightarrow{-} \dot{v} \to v \to \dot{u} \to u \xrightarrow{-} \dot{z} \to z \xrightarrow{-} \dot{s}$

Note. Strongest greed feedback loop R2 in L-1 is present in L-2 too.

The following peculiarities attract attention: there are opposite signs in partial derivatives of $\dot{u}$ and $\dot{z}$, $\dot{v}$ and $\dot{s}$; the columns for *s* and *u* have the same signs; opposite signs in the columns for same *s* and *u* vs the column for *z* at unstable equilibrium undergoing AHB. Besides, there is a single difference in signs of $\frac{\partial \dot{z}}{\partial z}$ for the two considered AHBs – look at Table 11 and Figure 6 in Appendix 2.

The positive partial derivative $\frac{\partial \dot{z}}{\partial z}$ is an indicator of investors' destabilizing co-operation; the negative one expresses their stabilizing competition. Whereas stabilizing competition is a characteristic of the second limit cycle, the first limit cycle involves transition of competition to its opposite (co-operation) and back.

The same more or less plausible magnitudes for in simulation runs with the following modifications: r = 0.0004, g = 0.05266, and extensions: j_1 = –0.82, j_2 = 0.001. The magnitudes of control parameter *b* are posted in Table 12.

2.2. Two typical Andronov – Hopf bifurcations in L-2

The parameter b is chosen as the control parameter again. Using the Liénard – Chipart criterion, the conditions of asymptotic local stability of X_b are determined after routine calculations. Using the Liénard – Chipart criterion as in Liu (1994), the author has established analytically for $0 < s_b d < z_b < 1$ the following mathematical statement. Recall that the 'simple' Hopf bifurcation means that all the characteristic roots except a pair of purely imaginary ones have negative real parts.

Proposition 5. When a magnitude of the control parameter b becomes critical (twice), inequality (23) turns into equity, formerly steady state X_b loses stability and a closed orbit is born as a result of a simple Andronov – Hopf bifurcation (twice). A mathematical proof of this Proposition applies the results from Liu (1994).

$$\Delta_3 = a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0, \tag{23}$$

where a_1, a_2, a_3 and a_4 are parameters of the given polynomial (see Appendix 2).

Table 12 informs about the roots of the characteristic equation related to unstable stationary state X_b in L-2.

The trajectories, in result of the 1st simple AHB at X_b, approach a stable limit cycle with a period of a middle-term cycle for a very wide set of initial values on multiple simulation experiments maintained by *Vensim*.

2.3 The industrial cycle in a selected simulation experiment

Investment lead boom in crisis, yet profit rate, surplus value, profit slightly leads investment out of crisis – this lead and lag are within 1 quarter. Also within 1 quarter, employment ratio v leads net output q from boom into crisis; q slightly leads v out of crisis.

Depression is defined now as a cycle's phase starting at the end of the crisis and ending before recovery when capital-output ratio s is (locally) maximal (Figure 3).

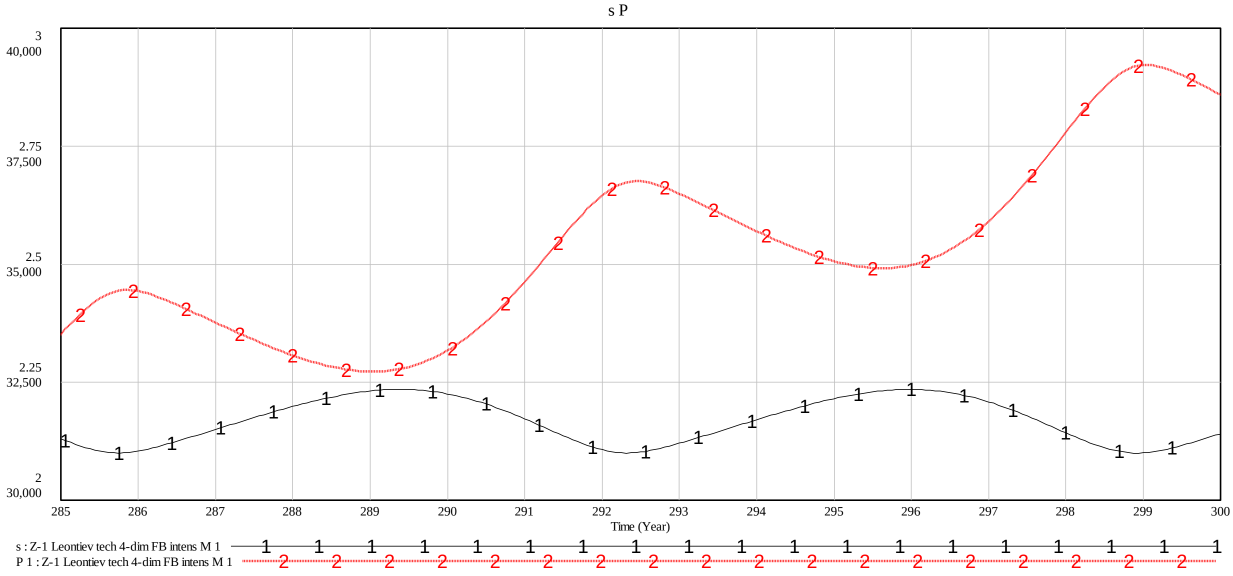

Figure 3 – Counter-phases of bet output q (2) and capital-output ratio s (1) in industrial cycles resulting from the 1st AHB in L-2, years 285–300

The leads and lags of the indicators in L-2 are in good agreement with the scientifically held view (Figures 4 and 5). The duration of a particular cycle and its phases are in the required bounds (Table 8).

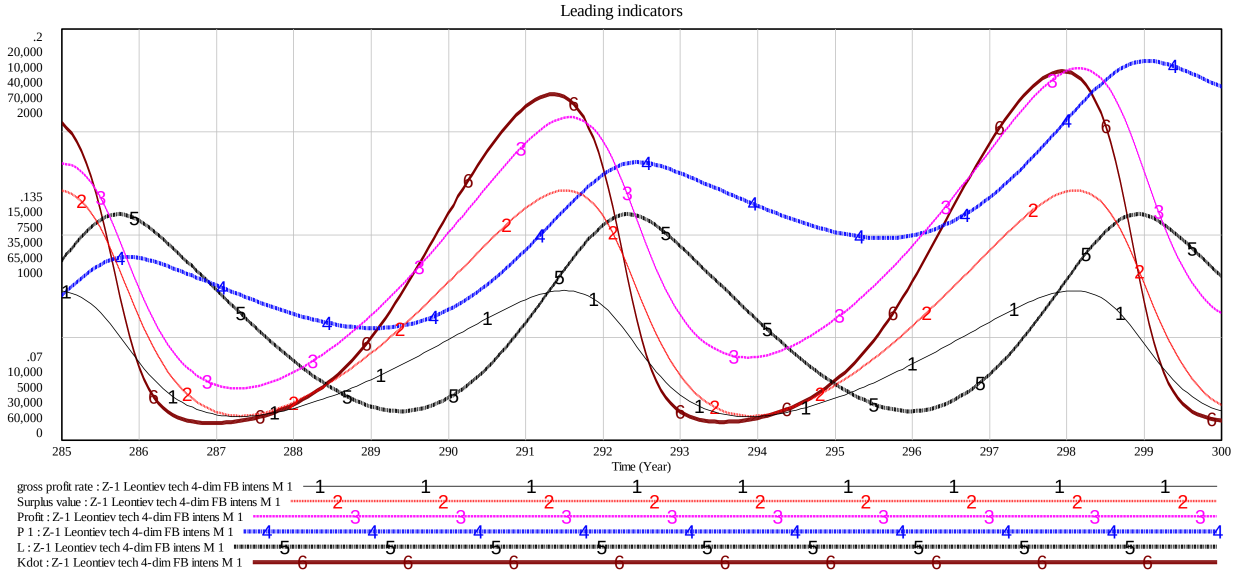

Figure 4 – Leading, coinciding and lagging indicators of industrial cycles resulting from the 1st AHB in L-2: profit rate (1), investment (6), surplus value (2), profit (3), net output (4), and employment (5), years 285–230

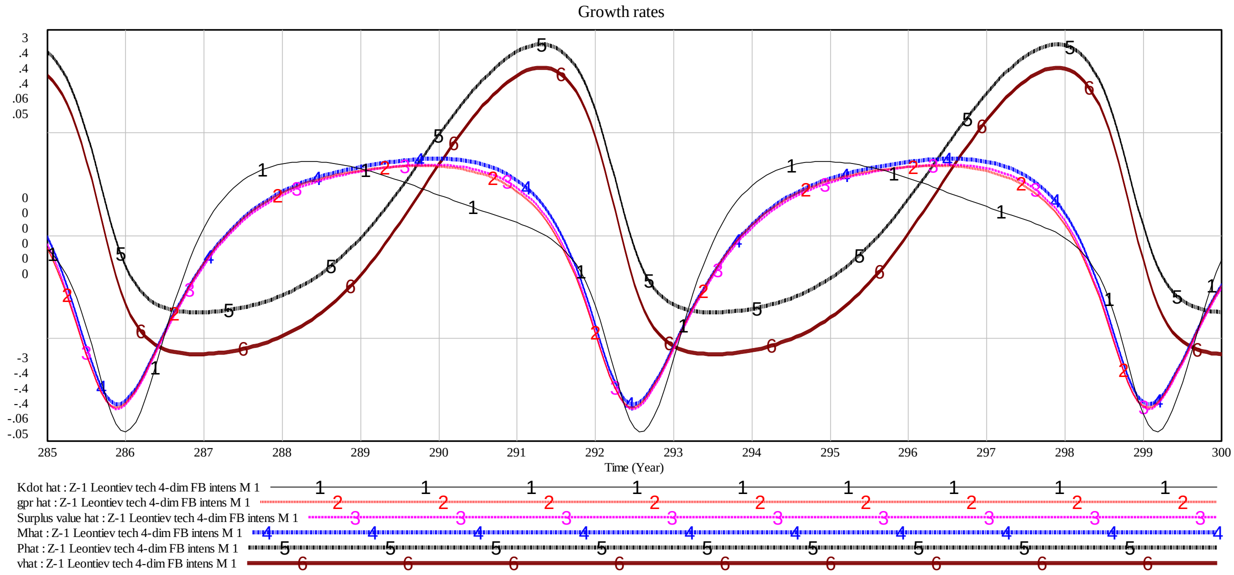

Figure 5 – The growth rates of economic indicators in industrial cycles resulting from the 1st AHB in L-2: investment (1), profit rate (2), surplus value (3), profit (4), net output (5), employment ratio (6), years 285–230

Table 8. Duration of phases of the two adjacent industrial cycles (quarters/ years) in L-2

Boom	Crisis	Depression	Recovery	Boom	New cycle
291–292.5	292.5–295.5	295.75	295.75–297.5	297.5–299	292.5–299
6/1.5	12/3	1/0.25	7/1.75	6/1.5	26/6.5

The drop of employment ratio *v* heralds the onset of the crisis (within 1 quarter) with a decline in net output *q*; on the other hand, the bottoming of net output opens the way to increases in employment ratio (within 2 quarters). The time measures in L-2 are independent of those in L-1.

Relative capital over-accumulation encompasses 291.5–294 y.; absolute capital over-accumulation of type 1 presides over the same period absolute capital over-accumulation of type 2 continues during 291.5–293.75 y. (Table 8).

A succession of local extrema of indicators' growth rates over 2018.25–27.5 y. is presented in Table 9. The phases of cycles are presented fragmentally for condensing the essentials.

Table 9. Extremes of indicators' growth rates for the abbreviated phases of subsequent industrial cycles in L-2

	Boom started in 291					Crisis 292.5–295.5		Depression 295.75	Recovery 295.75–297.5			Boom 297.5–299	
	289	291.25	291.5	292.25	292.5	293.5	294	295.75	23.25	296.5	23.75	298	298.25
q	0	max	↓	↓	0	min	↑	0	↑	↑	↑	max	0
$(1–u)/s$	↑	max	0	↓	min	↑	0	↑	↑	max	↓	↓	0
$(1–u)l$	↑	max	0	↓	min	min	0	↑	↑	max	↓	↓	0
$(1–u)q$	↑	max	0	↓	min	min	0	↑	↑	max	↓	↓	0
v	↑	max	↓	0	↓	min	↑	↑	↑	0	↑	max	↓

Conclusion

The Marxist theory of capital accumulation has to be maintained by vanguard system dynamics methodology and by mathematical bifurcation theory (see Analytical Methods (2015)). The research carried out on these foundations ascends from growth cycle to more relevant industrial cycle. The proposed models explain where their substantial differences come from by demonstrating specific stock-and-flow structures, revealing particular feedback loops and by going through the elaborated simulation experiments for the theoretical models.

All these outcomes have prepared new, more empirically oriented, strides forward; particularly, the outlined crude reality checks of the practical relevance of L-1 and L-2 are to be developed into more elaborated statistical tests in enduring research.

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Appendix 1

The standard characteristic equation of the third order related to Jacobi matrix $J(E_b)$ is written as

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where the parameters are calculated based on the corresponding values of some Jacobi matrix J_X

$$a_0 = -|J_X| = -(J_{11} J_{22} J_{33} + J_{12} J_{23} J_{31} + J_{21} J_{32} J_{13} - J_{13} J_{22} J_{31} - J_{23} J_{32} J_{11} - J_{12} J_{21} J_{33}),$$

$$a_1 = -[J_{23} J_{32} + J_{12} J_{21} + J_{13}J_{31} - J_{11} (J_{22} + J_{33}) - J_{22} J_{33}],$$

$$a_2 = -Trace(J_X) = -(J_{11} + J_{22} + J_{33}).$$

Lemma 1. The quadratic equation based on the above characteristic polynomial corresponding to E_b (12) in L-1 is

$$a(b) = a_1(b)a_2(b) - a_0 = 0, \tag{24}$$

where

$$a_1(b) = e + ob, \tag{25}$$

$$a_2(b) = c - hb, \tag{26}$$

$$b_1 = -\frac{e}{o} < 0, \tag{27}$$

$$b_2 = \frac{c}{h} > 0. \tag{28}$$

This quadratic equation has *typically*, i.e., for a rather wide area of plausible parameters' magnitudes, two real roots:

$$b_{0,3} = \frac{oc - eh \pm \sqrt{(oc-eh)^2 - 4oh(a_0 - ec)}}{2oh}. \tag{29}$$

Assume that these two roots are real indeed. Then significant statements follow.

Lemma 2. It is true that the relations $-\infty < b_1 < \min(b_3, b_0) \le \max(b_3, b_0) < b_2 < \infty$ are true.

Proposition 1. The dynamics of the system (9)–(11) linearized in the neighbourhood of its hyperbolic stationary state E_b (12) are locally asymptotically stable (LAS) provided that $0 \le b < b_0 < b_2 < \infty$ if $b_3 < 0$. Then stationary state E_b is also LAS in the non-linear system (9)–(11). Stationary state E_b is *not* stable for $b \ge b_0$ in the linearized system (9)–(11). Besides that, if $0 < b_3$, E_b is stable for $0 < b_3 < b < b_0$ in the linearized system (9)–(11).

As E_b is hyperbolic and LAS, it is LAS also in the non-linear system.

Proposition 2. (a) For b_3 and b_0 defined by (29), the AHB does take place in the system (9)–(11) in a local vicinity of E_b (12) only at $b_{critical} > b_0 > 0$ if $b_3 < 0$. Then, according to the Hopf theorem, there exists some periodic solution bifurcating from E_b and the period of fluctuations is about $2\pi/\beta_0$ ($\beta_0 = \lambda_2(b_0)/i$).

(b) If additionally $b_3 > 0$, the AHB does take place in the system (9)–(11) in a local vicinity of E_b (12) at b_3 as well. Then, according to the Hopf theorem, there exists some periodic solution bifurcating from E_b at $0 < b_{critical} < b_3$ and the period of fluctuations is about $2\pi/\beta_0$ ($\beta_0 = \lambda_2(b_3)/i$).

If a closed orbit is an attractor, it is called a *limit cycle*. The Hopf theorem establishes only the existence of closed orbits in a neighbourhood of E_b at b_0 or also at positive b_3, still it does not clarify the stability of orbits. This stability is revealed by simulation experiments.

Table 10. Roots of the characteristic equation related to unstable stationary state E_b (12) in L-1

Bifurcation	b_0 and b_3	λ_1	Re(λ_2, λ_3)	Im(λ_2, λ_3)	$b_{critical}$	Period of limit cycle at $b_{critical}$
1st AHB	120.16959	–0.0070	0.02189	±0.79058	150	7.9465
2nd AHB	0.67455	–0.1907	0.0000128758	±0.151542	0.67450	41.4662

The proof of these Propositions in Ryzhenkov (2016) for $1 \le Z$, $b_3 < 0$ and $b_0 > 0$ remains valid for $Z \le 1$, $b_0 > 0$ and also for $b_3 > 0$ in this paper. New informational gains comprise the following Proposition 3 with two parts.

Proposition 3. (a) The duration of the second limit cycle related to $b_3 > 0$ in L-1 drops with increases in target rate of capital accumulation z_b, whereas the duration of the first limit cycle related to $b_0 > 0$ increases. Theoretically for sufficiently high $z_b < Z$ a limit cycle of the second kind disappears and only a limit cycle of the first kind remains – this happens at the border of real solutions of equation (29) when positive b_3 and b_0 become equal to each other

$$b_3 = b_0 = \frac{oc - eh}{2oh} \tag{30}$$

for $1 \ge Z \ge z_b > sd$ such that $(oc - eh)^2 - 4oh(a_0 - ec) = 0$. This border magnitude of z_b is the only relevant solution of the past quadratic equation for z_b considered as its variable.

(b) A limit cycle of the first kind exists for appropriate $z_b < Z$ and $b_{critical} > \dfrac{oc - eh}{2oh}$ even if the roots $b_{0,3}$ are complex-conjugate for $(oc - eh)^2 - 4oh(a_0 - ec) < 0.$

Appendix 2

The characteristic equation related to Jacobi matrix $J(X_b)$ is written as

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{31}$$

where for realistic parameters' magnitudes

$$a_1 = a_{11} + a_{12}b,$$
$$a_2 = a_{21} + a_{22}b,$$
$$a_3 = a_{31} + a_{32}b,$$
$$a_4 = \text{const} > 0,$$

and $a_{11} > 0,\ a_{12} < 0, a_{21} > 0,\ a_{22} > 0, a_{31} > 0,\ a_{32} > 0.$

Based on Asada and Yoshida (2003), it is clear that the polynomial equation (31) has a pair of pure imaginary roots and two roots with negative real parts if and only if the following set of conditions is satisfied:

$$a_1(b) > 0, a_3(b) > 0, a_4 > 0,$$

$$\Delta_3(b) = a_1(b)a_2(b)a_3(b) - a_1(b)^2 a_4 - a_3(b)^2 = 0.$$

Proposition 4. If theses set of conditions is satisfied, stationary state X_b (22) is LAS.

For the given polynomial (23) the cubic equation $\Delta_3(b) = 0$ may have:

1) three different real roots, 2) three real roots of which two are the same, 3) the same three real roots, 4) one real and two complex conjugate roots.

For the plausible ranges of the parameters' magnitudes only the third case is irrelevant, whereas the first plays the main role in the present paper. Hereby two positive roots b_3 and b_0 are accompanied by $b_{negative}$ that is not economically relevant. The second and fourth cases are possible for sufficiently high Z and z_b necessarily satisfying $1 \geq Z \geq z_b > s_b d$.

Let two limit cycles exist. The following statements follow for plausible ranges of the parameters' magnitudes.

Proposition 6. (a) The duration of the first limit cycle at unstable X_b (for $b_{critical} > b_0$) increases with increases in target rate of capital accumulation z_b, whereas the duration of the second limit cycle drops at unstable X_b (for $0 < b_{critical} < b_3$). Theoretically for sufficiently high Z and z_b a limit cycle of the second kind disappears and only a limit cycle of the first kind remains – this happens when two positive b_3 and b_0 become equal to each other, while a strongly negative solution $b_{negative}$ continues to exist. A good (qualitatively and quantitatively) approximation takes place:

$b_{negative} \approx -a_{31}/a_{32} << 0.$

(b) A limit cycle of the first kind at unstable X_b may exist for sufficiently high Z and z_b necessarily satisfying $1 \geq Z \geq z_b > s_b d$ even if there is a single real solution $b_{negative} \approx -a_{31}/a_{32} <<$ 0 for $\Delta_3(b) = 0$ accompanied by two complex-conjugate roots, given that $\Delta_3(b_{critical}) < 0$, where $b_{critical} > b_3 = b_0$. The latter pair is from the previous part of this Proposition. A complete mathematical proof goes beyond the scope of this paper.

Table 11. Partial derivatives at X_b (22) for z under the 1st AHB for $b_{critical} = 40$ with strongly sensitive z and under the 2nd AHB for $b_{critical} = 1.39$ with weakly sensitive z in L-2

$\frac{\partial \dot{z}}{\partial u}$		$\frac{\partial \dot{z}}{\partial v}$		$\frac{\partial \dot{z}}{\partial s}$		$\frac{\partial \dot{z}}{\partial z}$	
1	2	1	2	1	2	1	2
–0.0018	–0.0522	–1.572	–0.055	–0.0051	–0.0002	0.058	–0.191

Table 12. Roots of the characteristic equation related to unstable stationary state X_b (22) in L-2

Bifurcation	b_0 and b_3	λ_1	λ_2	Re(λ_3, λ_4)	Im(λ_3, λ_4)	b_{critical}	Period of limit cycle at $b_{critical}$
1st AHB	20.58	–0.02396	–0.00217	0.05338	±0.89958	40	6.984
2nd AHB	1.3936	–0.16581	–0.00217	–0.000044	±0.34259	1.39	18.357

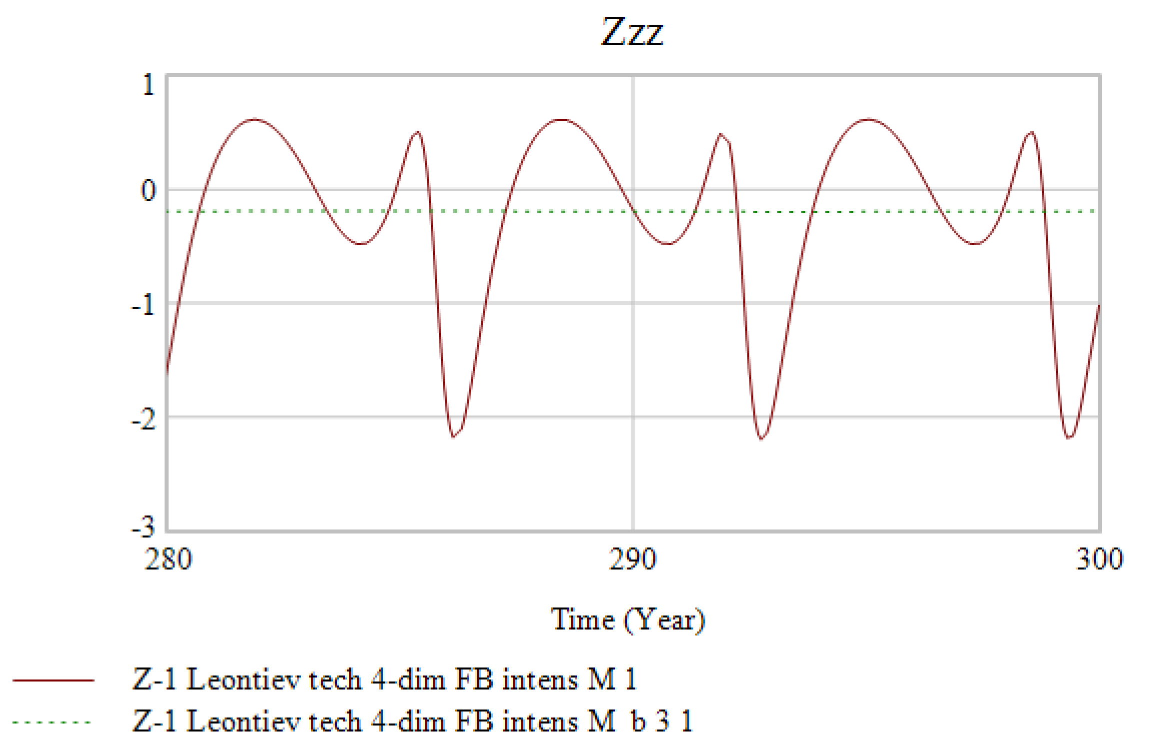

Figure 6 – Signs and amplitude of $\dfrac{\partial \dot{z}}{\partial z}$ for two limit cycles (solid curve for the 1st, dotted curve – for the 2nd) involving the same unstable stationary state X_b in L-2

References

Analytical Methods for Dynamic Modelers. 2015. Ed. by Hazhir Rahmandad, Rogelio Oliva, Nathaniel D. Osgood. Cambridge (Mass.): The MIT Press. 416 p.

Asada T., Yoshida H., 2003. Coefficient criterion for four-dimensional Hopf bifurcations: A complete mathematical characterization and applications to economic dynamics. *Chaos, Solitons and Fractals* 18: 525–536.

Blatt J. 1983. Dynamic Economic Systems. Brighton: Wheatsheaf Books.

Boggio L. 2006. Growth, unemployment and wages: disequilibrium models with increasing returns // Classical, Neoclassical and Keynesian Views on Growth and Distribution / Salvadori N., Panico C. (eds). Elgar. Cheltenham. UK. Northampton, MA, USA. 2006. P. 227–242.

Brøns M., Sturis J. 1991. Local and global bifurcations in a model of the economic long wave. *System Dynamics Review* 7 (1): 41–60.

Fanti L., Manfredi P. 1998. A Goodwin-type growth cycle model with profit-sharing. *Economic Notes* 27: 371–402.

Gandolfo G. 2010. Economic Dynamics. Fourth edition. Heidelberg: Springer. 830 p.

Liu W. M. 1994. Criterion of Hopf bifurcation without using eigenvalues. *Journal Math. Analysis and Applications* 182: 250–256

Lordon F. 1995. Cycles et chaos dans un modèle hétérodoxe de croissance endogène. *Revue économique* 46(6): 1405–1432.

Marx K. Capital. A Critique of Political Economy. Volume I. Book One: The Process of Production of Capital. First published: in German in 1867.

https://www.marxists.org/archive/marx/works/download/pdf/Capital-Volume-I.pdf

Marx K. (1863–1883) 1959. *Capital.* Vol. 3. Ed. by F. Engels. Moscow: Institute of Marxism-Leninism. Chapters 6, 27, 29, 30.

http:www.marxists.orgarchivemarxworks1894-c3index.htm

Mosekilde E., Aracil J., Allen P. M. 1988. Instabilities and chaos in nonlinear dynamic systems. *System Dynamics Review* 4 (1–2): 14–55.

Mosekilde E., Laugesen J. L. 2007. Nonlinear dynamic phenomena in the beer model. *System Dynamics Review* 23 (2–3): 229–252.

Ryzhenkov A. 2013. Employment-centred stabilisation policy propelling the economy to “escape velocity”. In: *The 31st International Conference of the System Dynamics Society*. [online] Cambridge, Massachusetts, USA.

http://www.systemdynamics.org/conferences/2013/proceed/papers/P1170.pdf.
http://www.systemdynamics.org/conferences/2013/proceed/supp/S1170.pdf.

Ryzhenkov A. 2016. Endogenous industrial cycles in a reshaped “neoclassical” model// The 34th International Conference of the System Dynamics Society. Delft, Netherlands. July 17 – July 21, 2016 [Electronic resource]. – Delft, 2016. https://www.systemdynamics.org/assets/conferences/2016/proceed/papers/P1318.pdf

https://www.systemdynamics.org/assets/conferences/2016/proceed/supp/S1318.pdf

Ryzhenkov A. Insights on Profit Sharing and Job Creation in Bi-Dimensional Goodwinian Models. – DOI: 10.1007/978-3-030-67190-7_16 // *Feedback Economics*. Economic Modeling with System Dynamics / eds.: R.Y. Cavana, B.C. Dangerfield, O.V. Pavlov, M.J. Radzicki, I.D. Wheat. – Cham : Springer, 2021. – Online ISBN 978-3-030-67190-7. – Ch. 16. – P. 427-463. – (Contemporary Systems Thinking). (First Online: 01 July 2021). https://link.springer.com/chapter/10.1007%2F978-3-030-67190-7_16