System Dynamics Model of Technology and Economic Growth: A Preliminary Study

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Abstract

The models of technological change and economic growth those have been developed so far do not provide satisfying directions for policy purposes. In this study, a simple system dynamics model based on an integration of micro- and macroeconomic theories is constructed to explore the process of technological change affecting the economic growth. It is hoped that by understanding the process, the developing country may have some directions more clearly how to design its technology policies. The capital-labor ratio change is used to represent the technology change and the mathematical equations of the model are derived from the underlying economic concepts. The main point of deriving the equations is that the production function has a capital intensity which is not constant. The study resulted in an important finding that the capital intensity is affected by the average life of capital in a negative direction. The study shows that the increase in capital intensity is an important source of the economic growth. This increase will strengthen the accelerator mechanism of the economy and creates larger multiplier effects. The increase in capital intensity can be obtained through managing innovation processes base on the development of education and the R&D capacity of the nation.

Keywords: Capital-labor ratio, Capital intensity, Innovation, System dynamics

1. Introduction

It is an accepted view that technological progress is an extremely important, perhaps the most important, determinant in the growth in output per man. In the discussions of the role of technological change in the economy, one of some important questions naturally arise is how does technological change affect different factors (capital and labor). Traditionally, some technological changes are thought of as “labor intensive”, and some as “capital intensive”.

As a milestone in the theory of economic growth literatures, Solow (1956) modeled the technological change through simply multiplying the production function by an exogenous increasing scale factor A(t). The term A(t) in the production function represents all the influences that go into determining output besides capital and labor. Changes in A over time represent technical progress. Thereafter, there are some studies those have been trying to replace the term A by some endogenous variables and to specify the exact real world meaning of those variables. Among others are knowledge accumulation (education), R&D (Research and Development), and human capital. The model those have been developed so far do not provide satisfying directions for policy purposes. For a developing country, the most important question is how to design a robust strategy of technological changes those can be expected to improve her national productivity considerably. It is important for policy design that the model has to have an appropriate policy space to explore the entry points for an evolutionary change. As a basis for design, the model structure and the behavior of the model and its empirical relevance has to be fully understood.

A simple system dynamics model based on an integration of micro- and macroeconomic theories is constucted to explore the process of technological change (technology) affecting the economic growth. It is hoped that by understanding the process, the developing country may have some directions more clearly how to design its technology policies. Firstly, a theoretical framework of technology and economic growth is described as a basis to construct the model. Secondly, the important features of the system dynamics methodology is briefly explained. By using the methodology then, the framework is converted into a system dynamics model of technology and economic growth; described in the third part of the paper. Furthermore, some experiments of the sources of economic growth are simulated using the model; and the long-run growth patterns resulted from the experiments are analyzed. An attempt base on the process oriented approach is made to build an understanding of the role of technology in the economy.

The study shows that, for long term strategies, the process of a sustainable increase in capital intensity of the economy is an important source of the economic growth. The process, in which the increase in capital intensity of the economy can be maintained in the long-run, may be the direction of the robust strategies for developing nations to improve their national productivity considerably. The increase in capital intensity of the economy can be obtained through managing innovation processes base on the development of education and the R&D capacity of the nation.

2. Theoretical framework

Hicks and Harrod among others, have proposed the different definitions of a neutral technical change. If the change of relative shares is used as the measure of bias of technological change, then the Hicks definition measures the bias along a constant capital-labor ratio while the Harrod definition measures the bias along a constant capital-output ratio. In the growth literature, Harrod neutrality has played a more central role. It has often been alleged that technological change in fact is Harrod neutral. Stiglitz and Uzawa observed an “almost constant capital-output ratio with an almost constant rate of interest” (Stiglitz and Uzawa, 1969). Bach (1968) has showed the validity of their observations. It is based on the patterns of the economic growth in the US that showed that capital-output ratio and the interest rate were roughly flat in trend between 1900-1965. On the other hand, the capital-labor ratio was steadily increased in trend. The fact of the increase in capital-labor ratio can be also observed in the discussions of the relationship between labor productivity and capital-labor ratio (Sumanth, 1985).

In this study, using the Harrod definition of technological change, the capital-labor ratio is used to represent the technology (technology embeds in capital and labor). Thus, the changes of capital-labor ratio in an economy represent the technological changes in the economy. An increase in the capital-labor ratio of an economy means the increase in technology level of the economy. Therefore, the main focus in constructing the structure (physical and decision making structures) of the model is the mechanism of changes in capital-labor ratio that may affect the economic growth.

For such purpose, the mathematical equation of the model is derived from the underlying economic concepts (see Appendix A). The first main point of deriving the equations is that the production function, for society's output as a whole, has a capital intensity which is not constant. The second is that the study uses the standard neoclassical assumption of profit-maximizing behavior to model the decision making structures of the acquisition of production factors i.e. capital and labor.

The important equations used to develop the model are as follows (taken from Appendix A).

Equation (1) the production function:

$$q = q_0 * \left(\frac{K}{Ko}\right)^{\alpha} * \left(\frac{L}{Lo}\right)^{\beta}$$

q, q_0 = production, initial production [unit/year]
K, K_0 = capital, initial capital [unit]
L, L_0 = labor, initial labor [person]
α = capital intensity [dimensionless], **not constant**
β = labor intensity [dimensionless], **not constant**.

Equation (5) the optimum capital K:

$$K = \frac{\alpha * q}{\left(\frac{1}{alk} + R\right)}$$

alk = average life of capital [year], not constant
R = real interest rate [1/year], constant.

Equation (8) the optimum labor L:

$$L = \frac{\beta * q}{rw}$$

rw = real wage rate [unit/year/person].

Equation (9) the capital intensity α:

$$\alpha = KOR * \left(\frac{1}{alk} + R\right)$$

KOR = capital-output ratio, constant [year].

And Equation (24) the production (economic) growth rate:

$$G_q = \alpha\, G_K + (1 - \alpha)\, G_L + KOR * \frac{rw}{KLR} * ln\left(\frac{KLR}{KLRo}\right) * [G_{KLR} - G_{rw}]\ ,$$

where G_q is the growth rate of production q, G_K is the growth rate of capital K, G_L is the growth rate of labor L, G_{KLR} is the growth rate of capital-labor ratio KLR, and G_{rw} is the growth rate of real wage rw.

3. A brief description of system dynamics methodology

Some references in system dynamics literatures, concerning the structure (physical and decision making structures) of system dynamics model, are considered in constructing the model of this study as follows.

- "System dynamics is a methodology for studying and managing complex feedback systems, such as one finds in business and other social systems." [System Dynamics Home Page.htm]

- Forrester (Forrester, 1990, pp. 4-2 - 4-5):
  "In concept a feedback system is a closed system. Its dynamic behavior arises within its internal structure. Any interaction which is essential to the behavior mode being investigated must be included inside the system boundary. Within the system boundary, the basic building block is the feedback loop. The feedback loop is a path coupling decision, action, level (or condition) of the system, and information, with the path returning to the decision point. Every decision is made within a feedback loop. The decision controls action which alters the system levels which influence the decision. There are two fundamental types of variable elements within each loop--the levels, and the rates. The level variables accumulate the results of action within the system. As flows influencing the levels, the rates are the results of action that cause the level to change."
  [Therefore, in constructing a model for policy analysis using the system dynamics methodology, the model has to reflect the way decision is actually made in the system.]

- Sterman (Sterman, 1981):
  "1. Desired states and actual states must be distinguished." [p. 50]
  "The variables and relationships should have real world meanings; equations should balance dimensionally without the addition of scaling factors or parameters." [p. 52]

- Richardson & Pugh III (Richardson & Pugh III, 1981):
  "The system dynamics approach to complex problems focuses on feedback processes. It takes the philosophical position that feedback structures are responsible for the changes we experience over time. The premise is that ***dynamic behavior is consequence of system structure*** and will become meaningful and powerful. At this point, it may be treated as a postulate, or perhaps as a conjecture yet to be demonstrated." [p. 15]
  [There are two structures: physical and decision-making structure.]

- Saeed (Saeed, 1994):
  "Empirical evidence is the driving force both for delineating micro-structure of the model and verifying its behaviour, although the information concerning the behavior may reside in the historical data and that concerning the micro-structure in the experience of the people [Forrester 1979]." [p. 22]
  "The dynamic hypothesis must incorporate causal relations based on

> information about the decision rules used by the actors of the system, and not on correlations between variables observed in the historical data." [p. 22]
> "The model structure must be "robust" to extreme conditions and be "identifiable" in the "real world" for it to have credibility, where real world consists both of theoretical expositions and experiential information." [p. 22]
> "When a close correspondence is simultaneously achieved between the structure of the model and the theoretical and experiential information about the system, and also between the behavior of the model and the empirical evidence about the behavior of the system, the model is accepted as a valid representation of the system. [Bell & Senge 1980, Forrester & Senge 1980, Richardson & Pugh III 1981]." [p. 23]

Based on the mentioned references above, the main (important) features of the system dynamics methodology in constructing the structure of the model are summarized as follows.

(a) Is the model structure consistent with relevant descriptive knowledge of the system?
(b) Does the model conform to basic physical laws?
(c) Do the decision rules capture the behavior of the actors in the system?
(d) Is each equation dimensionally consistent without the use of parameters having no real world meaning?
(e) Do all parameters have real world counterparts?

4. Model description

By using the system dynamics methodology mentioned above, the theoretical framework of the model is converted into a simple system dynamics model of technology and economic growth. The model is formulated from a familiar theoretical model, the multiplier-accelerator model of Samuelson (Samuelson, 1939), with several minor modifications. The model also considers the inventory adjustment model of Metzler (Metzler, 1941). The model consists of a single-sector two-factor production system that incorporates national income accounting at an aggregate level and an important mechanism to determine capital-labor ratio. There are 4 sub-models namely Income Sub-model, Labor & Unemployment Sub-model, Wage Sub-model, and Innovation Sub-model as shown in Figure 1. In this preliminary study, the model uses a constant price measures (real terms) so that the price is not included in the model; therefore there is no need to consider the money balance in the model. It means that there is no influence of money availability to economic decisions. The model does not also consider imports and exports.

The Innovation Sub-model has not yet been developed. In developing a model of technology and economic growth one has to include the innovation activities because innovations (basic innovations) create a new type of human activity as stated by Mensch (Mensch, 1979, p. 47):

> "innovations which produce new markets and industrial branches…or open new realms of activity in the cultural sphere, in public administration, and in social services. Basic innovations create a new type of human activity."

In the real world the above statement can be interpreted that the innovations will produce new products (goods) in turn, in macro (global) term, making the average life of goods (including capital) becoming more shorter (empirical evidences support this interpretation). In the developed model, due to the innovation sub-model has not been yet constructed; the life of capital is treated as an exogenous variable and becoming one of the growth sources. Besides this, there are 2 more exogenous variables i.e. population and government spending fraction as shown in Figure 1.

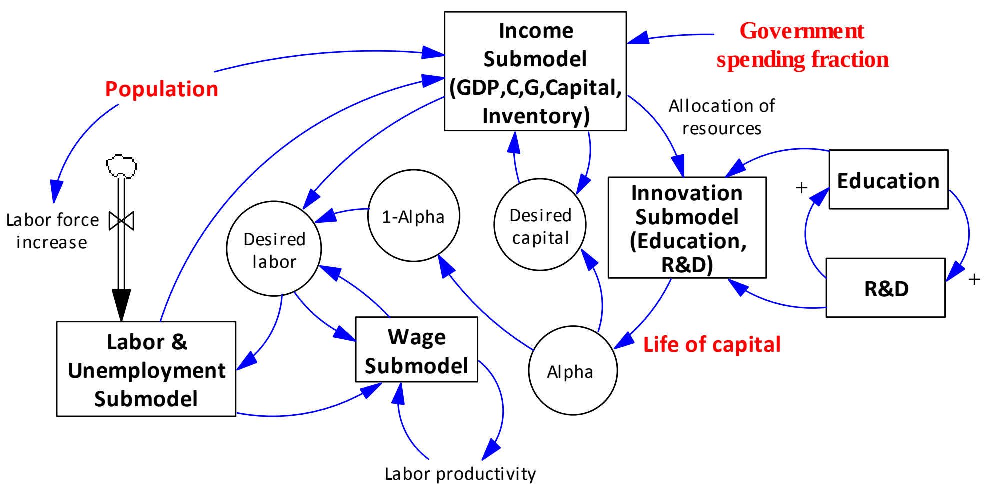

Figure 1: The global structure of the model

In the Innovation Sub-model there are 2 important sectors those have to be included into the technology and economic growth model, i.e. education sector and R&D (Research and Development) sector. The nation education level produced by the education sector represents the “repetition capability” of the nation, meanwhile the effective R&D activities of a nation represents the “generating capability” of the nation. There is a positive feedback relationship between those two sectors. In system dynamics methodology those two sectors create a growth behavior of the system and becoming a powerful structure that has to be considered seriously by a developing country.

Figure 2 is the causal loop diagram of Income Sub-model. As shown in Figure 2, the multiplier-accelerator principle is represented through 2 positive feedback loops namely M/+ loop and A/+ loop respectively. These two positive loops are the growth engine of the economy. When there is an increase in aggregate demand through the increase in investment, or consumption, or government spending; these increases will be multiplied and accelerated by those two loops. Due to the assumption in the model that capital intensity Alpha (α) can vary (not constant) caused by the change in life of capital (as illustrated in Figure 2), an increase in Alpha will augment the positive accelerator loop through the increase in desired capital (illustrated in Figure 2). The change in Alpha is determined by the change in life of capital alk which can be explained through Equation (9)

$$\alpha = KOR * \left(\frac{1}{alk} + R\right).$$

In this equation capital-output ratio KOR and real interest rate R are constant. Based on this equation a decrease in life of capital, caused by the effective innovations, will increase the α. Meanwhile Desired capital is determined through Equation (5)

$$K = \frac{\alpha * q}{\left(\frac{1}{alk} + R\right)}.$$

Production q is replaced by long-run expected demand accommodating the main important features of system dynamics model (described before) in constructing the structure of the system dynamics model. The change of Alpha that is considered in the model of technology and economic growth can be thought as a mechanism how the technology which is produced by the effective innovations affecting the economic growth.

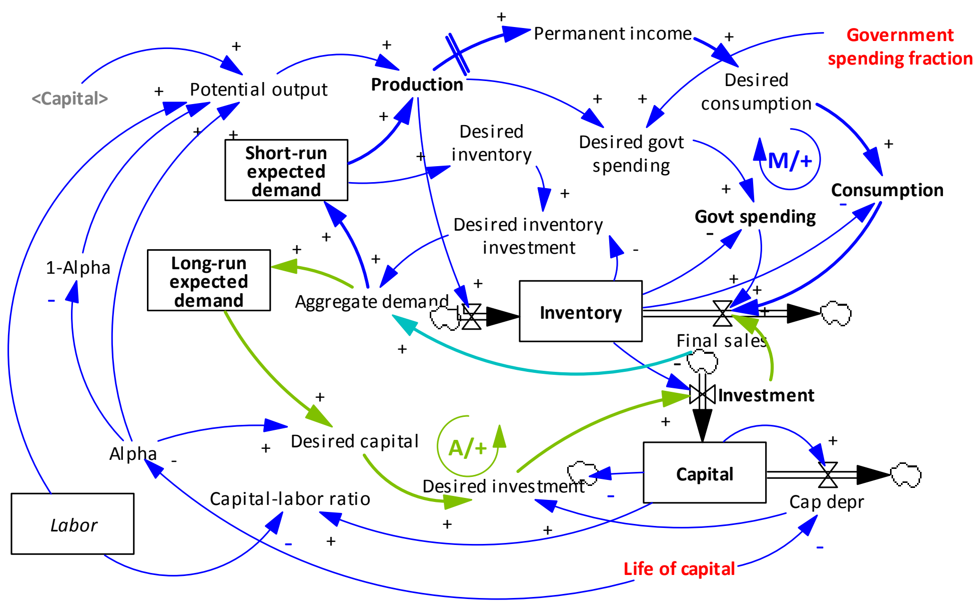

Figure 2: Causal loop diagram of Income Sub-model

The following figure of Figure 3 is the causal loop diagram of Labor & Unemployment Sub-model and Wage Sub-model. As shown in Figure 3, an increase in Alpha may reduce desired labor when at the same time there is also a decrease in short-run expected demand and an increase in wage. In turn, the reduced desired labor will increase the unemployment rate and then will decrease the wage. At the end this will increase the desired labor (balancing behavior of the negative feedback loop between desired labor through the pool of unemployment). This behavior can be explained through Equation (8) which is used to determine the desired labor as follows. Equation (8) is

$$L = \frac{\beta * q}{rw}$$

where beta is 1-Alpha represents the labor intensity of the economy. The output q in Equation (8) in the model is replaced by short-run expected demand. The increased unemployment rate due to the increase in Alpha can be prevented if the augmented economic growth caused by this increased Alpha (technology) is able to maintain the growth of the aggregate demand.

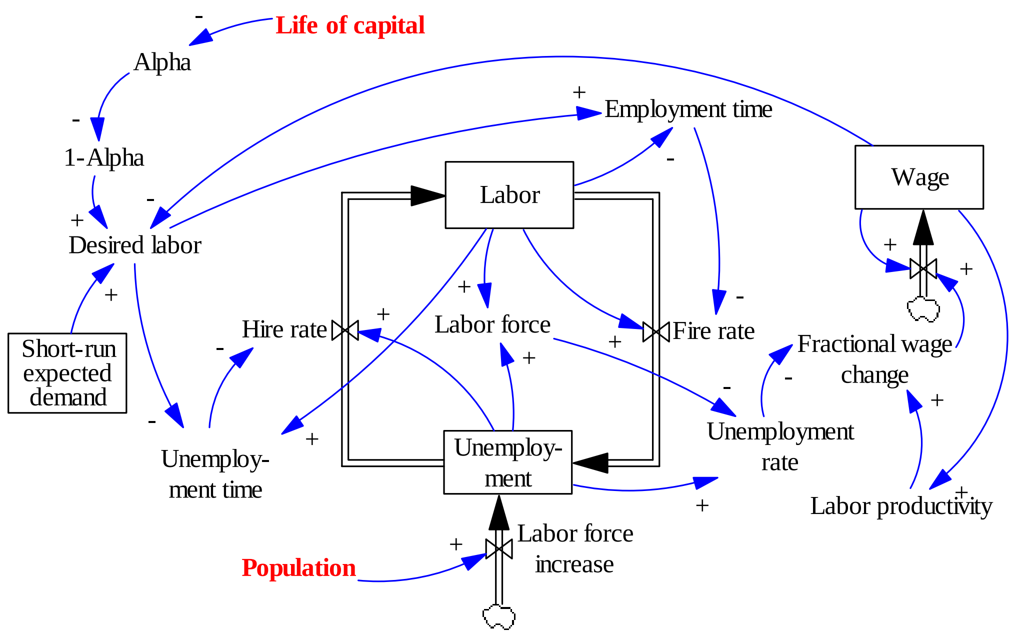

Figure 3: The causal loop diagram of Labor & Unemployment Sub-model and Wage Sub-model

5. Simulation results and analysis

As an attempt to explore the process of technological change affecting the economic growth that the developing country may have some directions more clearly how to design its technology policies; there are three sources of growth (growth scenarios) are considered in this study i.e.: population, government spending, and innovation (through life of capital). The model is initialized in the full equilibrium. In this equilibrium the population growth rate is equal to zero, the government spending fraction is 0.15, and life of capital is 14 years providing the capital intensity Alpha is 0.25. The model is simulated for 350 years, and the changes of the source of growths are introduced into the model in year 50.

In trying to understand the paths of the economic growth due to technology development in the economy, the first experiment of 3 scenarios are simulated using the model to show whether those three sources of the growth considering in this study can produce an increase in production. First, in the absence of changes in life of capital and government spending fraction, the model is simulated with introducing 1% per year growth in the population namely “Population”. Second, only the change in government spending fraction is introduced into the model (becoming 20% from 15%) namely “Government”. And the third, only the gradually change in life of capital is introduced into the model from 14 years to 10 years in the period of 100 years, namely “Technology”. The behavior comparisons of those 3 scenarios are shown in Figure 4, Figure 5, Figure 6, Figure 7, Figure 8, Figure 9, and Figure 10.

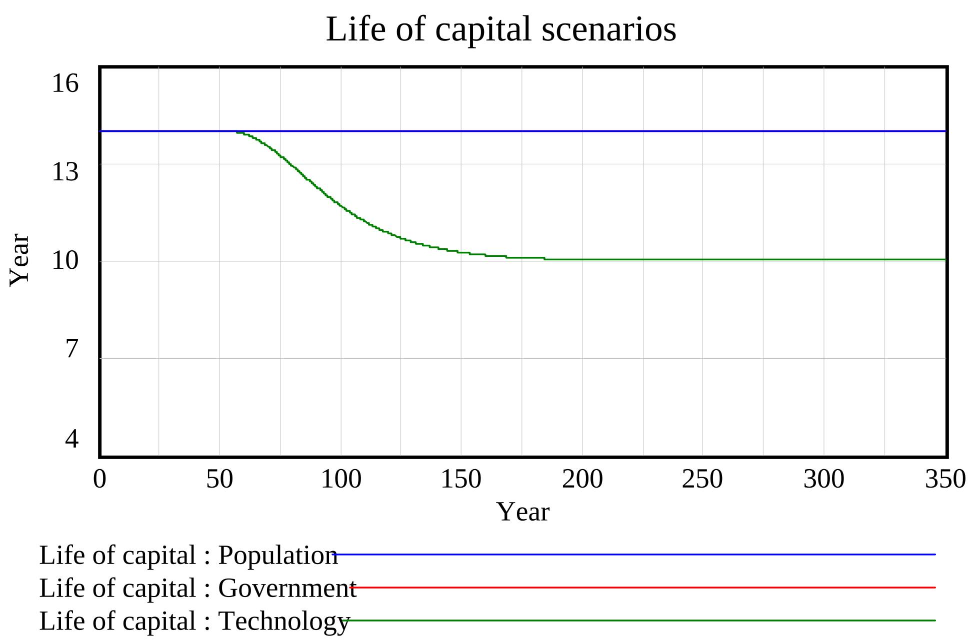

Figure 4: Life of capital scenarios

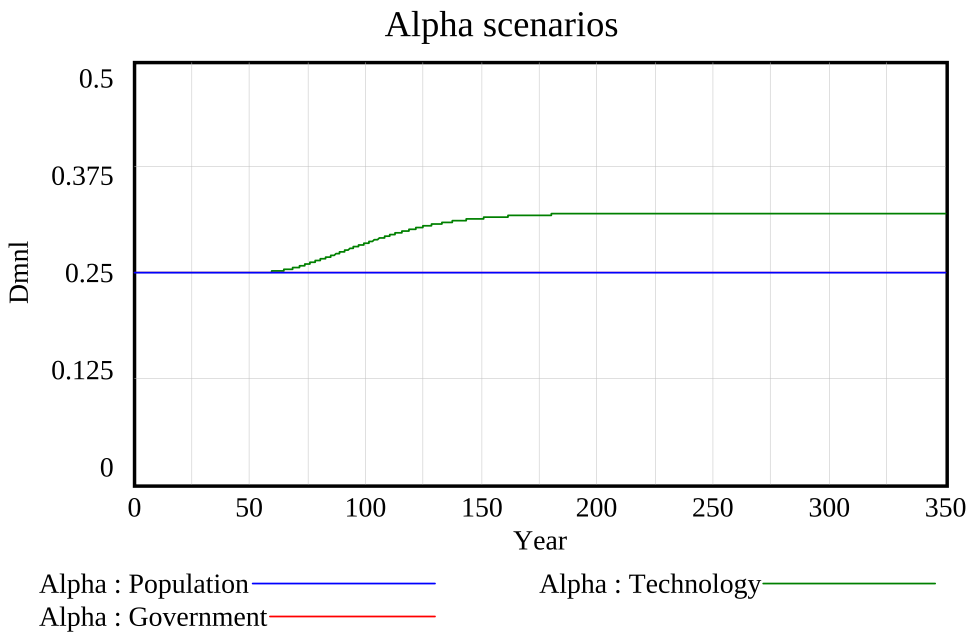

Figure 5: Alpha scenarios

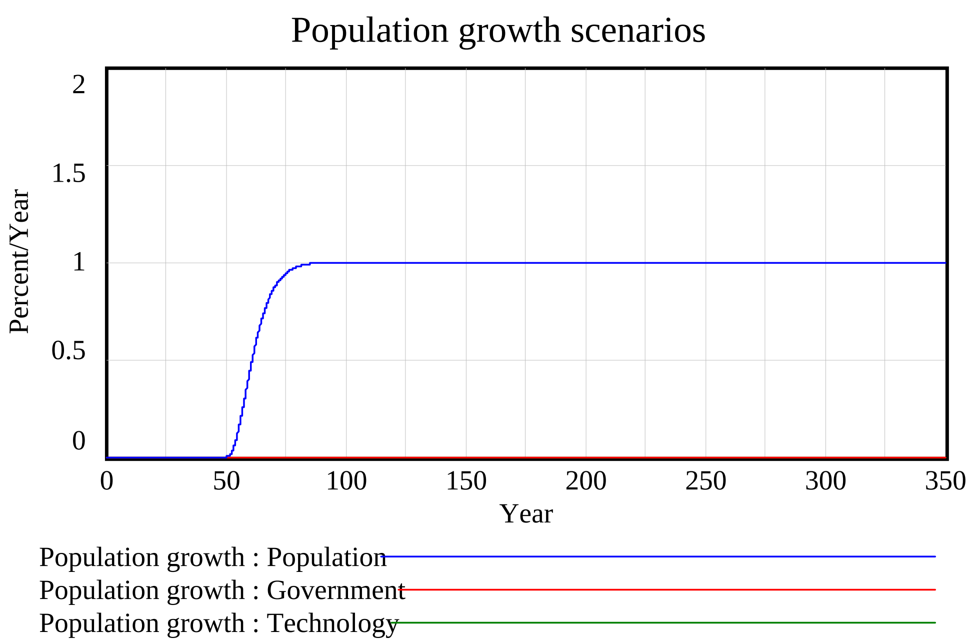

Figure 6: Population growth scenarios

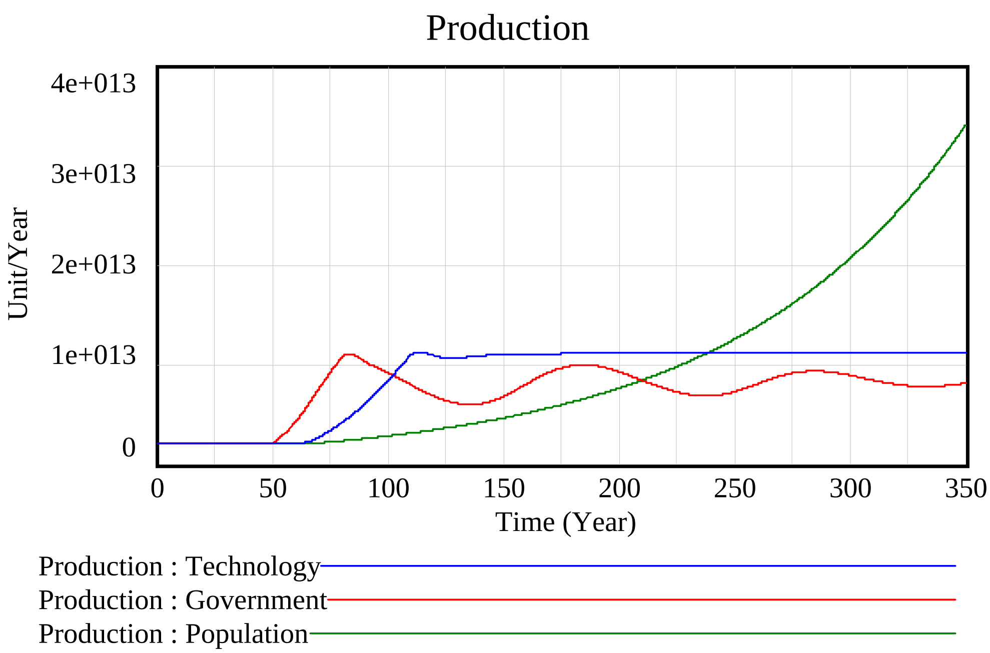

Figure 7: Production

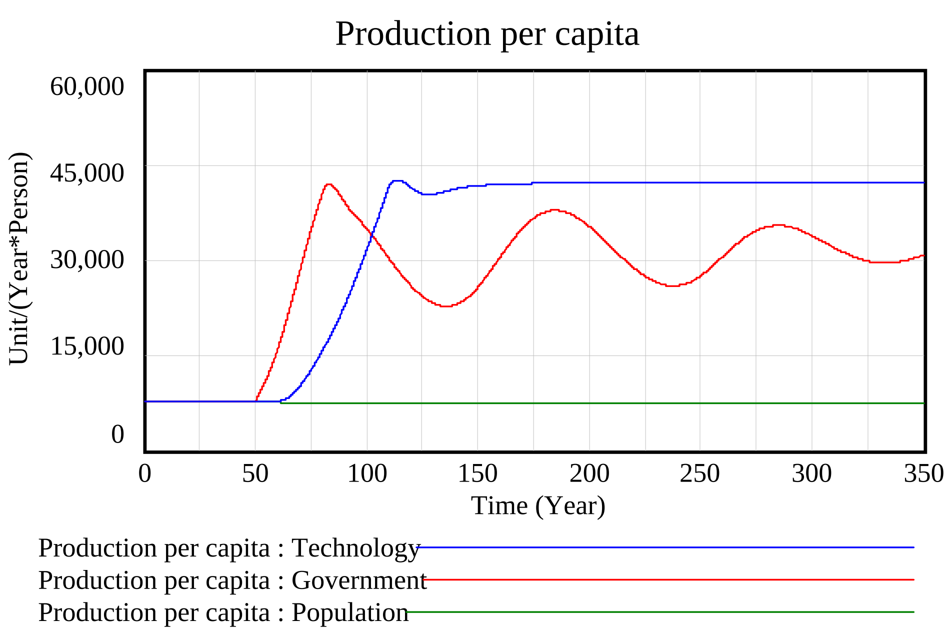

Figure 8: Production per capita

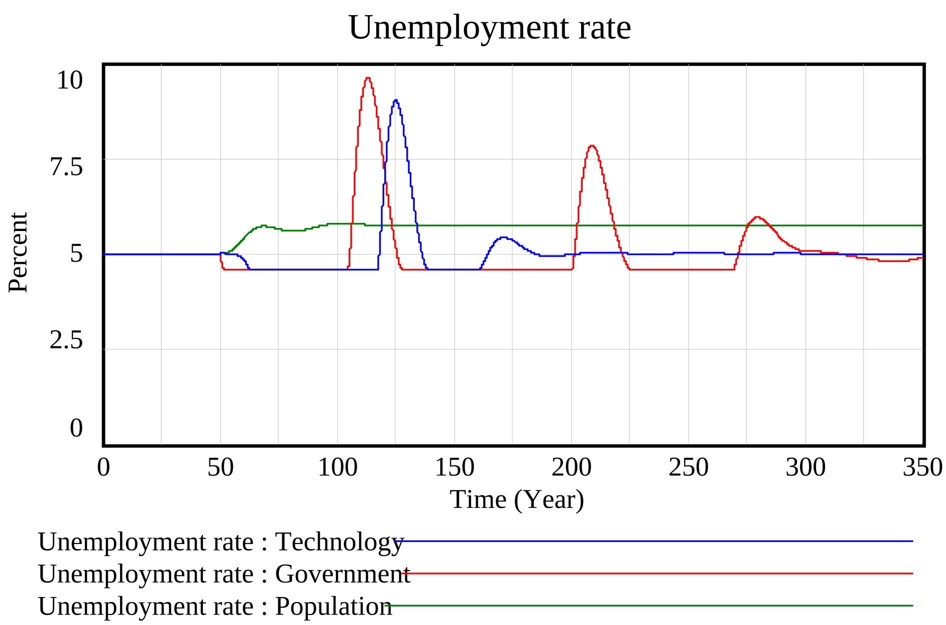

Figure 9: Unemployment rate

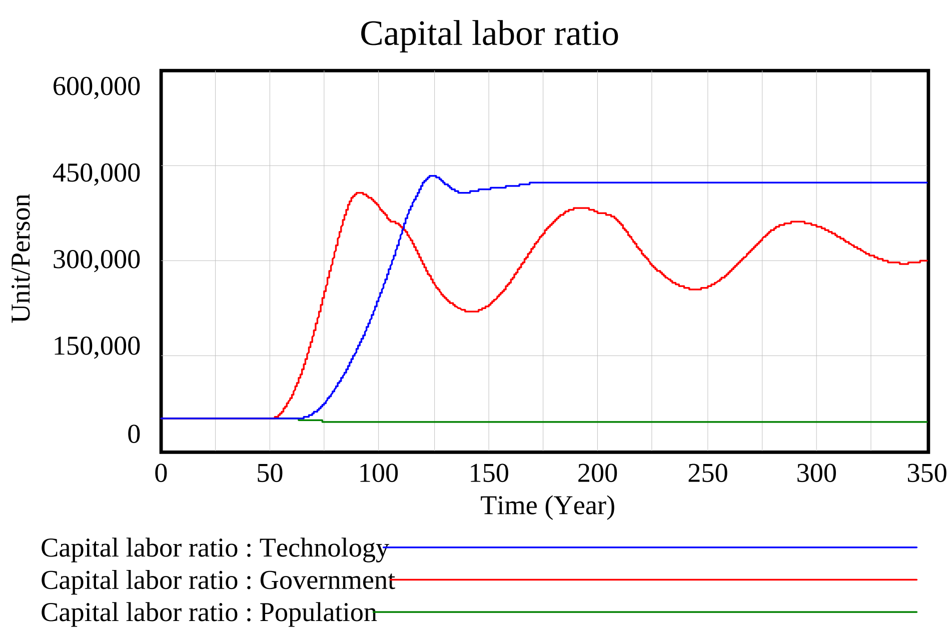

Figure 10: Capital labor ratio

Figure 4, Figure 5, and Figure 6 illustrate the changes of input scenarios of those three scenarios: the life of capital followed by its effect on the Alpha (the capital intensity), and the population growth respectively. As sources of economic growth, those three scenarios show that they can produce an increase in production with different growth paths (Figure 7). However, the increase in production per capita (income per capita) can be only obtained by "Government" scenario and "Technology" scenario (Figure 8). These mean that in "Population" scenario the growth rate of production is equal to the population growth rate. Besides this constant production per capita, the long-term unemployment rate of this scenario is higher compared with its initial and the other two scenarios (Figure 9). The level of technology (the capital-labor ratio) is also constant in the "Population" scenario, meanwhile in the "Government" and the "Technology" scenarios the technology levels are increasing (Figure 10).

The second experiment of 5 scenarios is done to understand the power of technology (innovation) in producing a higher sustainability of economic growth. In this experiment, for those 5 scenarios, the population growth rate is set at 1% per year. Those 5 scenarios are namely: (1) "Population" (population growth rate of 1% per year only), (2) "Population+Government" ("Population" scenario plus an increase in government spending fraction from 15% to 20%), (3) "Population+Technology 10" ("Population" scenario plus a gradually change in life of capital from 14 years to 10 years), (4) "Population+Technology 10+Government", and (5) "Population+Technology 7" ("Population" scenario plus a gradually change in life of capital from 14 years to 7 years). The simulation results are shown from Figure 11 until Figure 17.

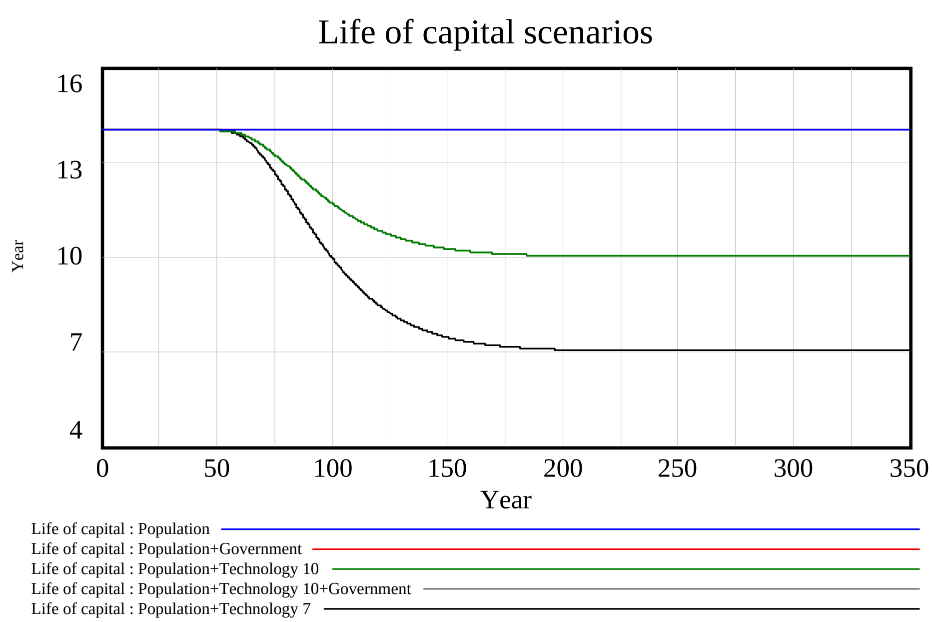

Figure 11: Life of capital scenarios

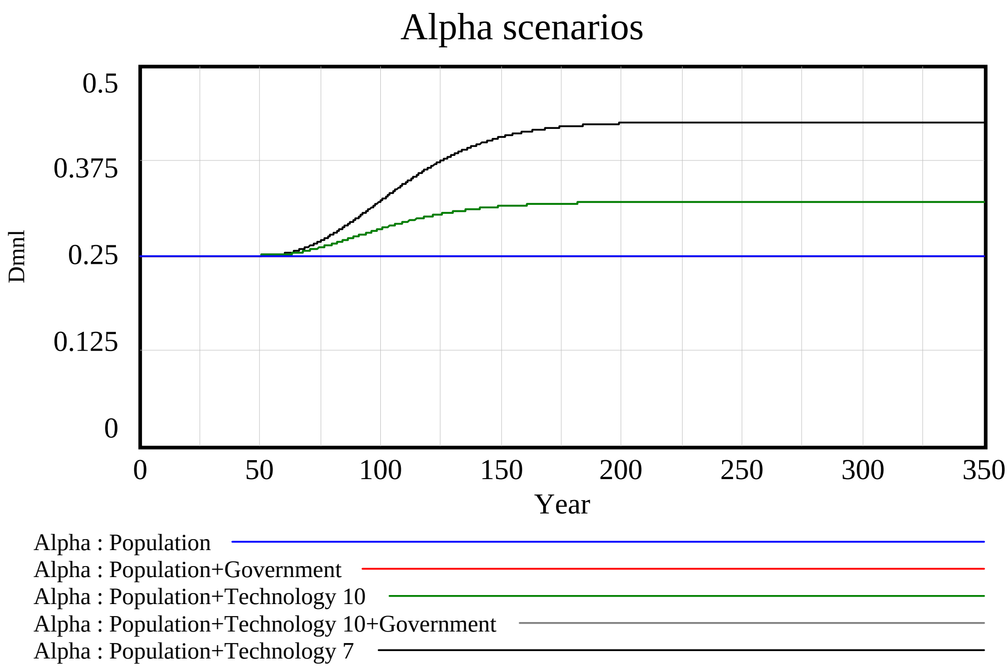

Figure 12: Alpha scenarios

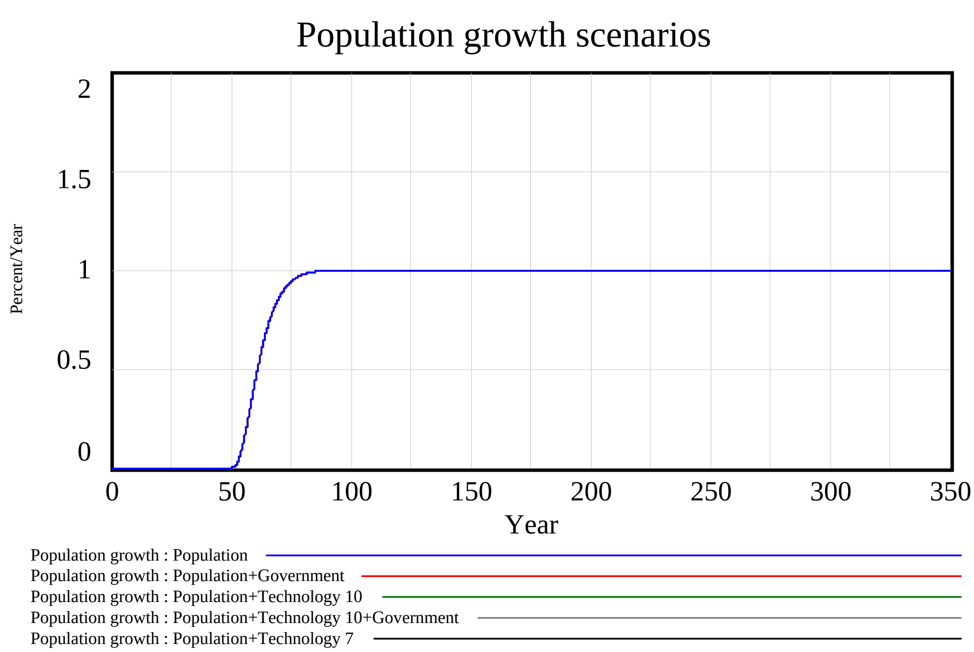

Figure 13: Population growth scenarios

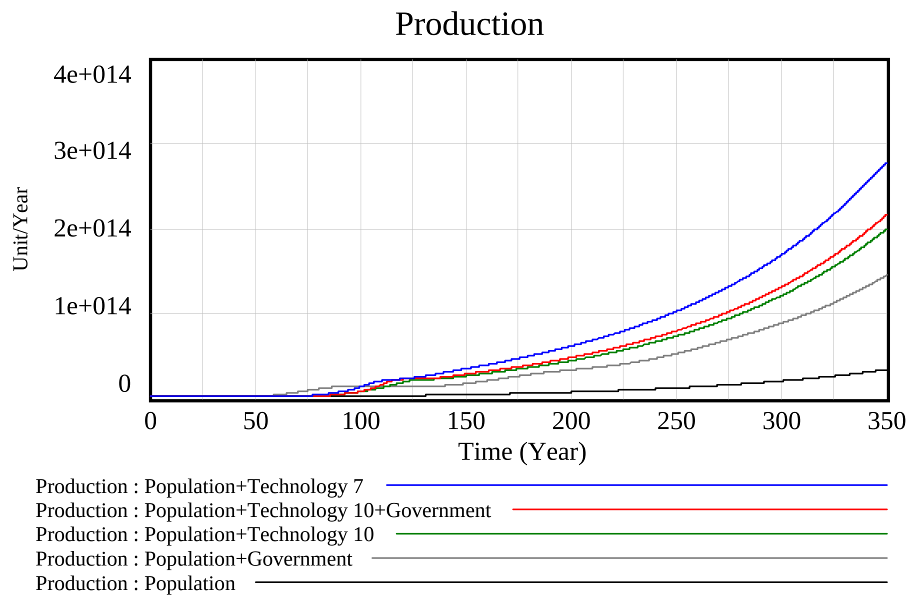

Figure 14: Production

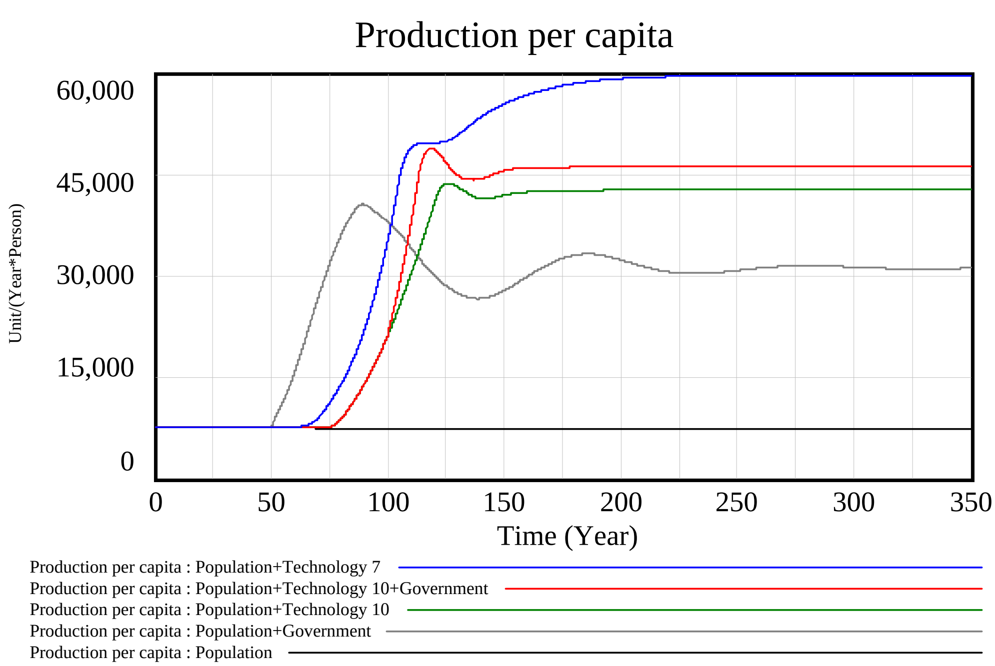

Figure 15: Production per capita

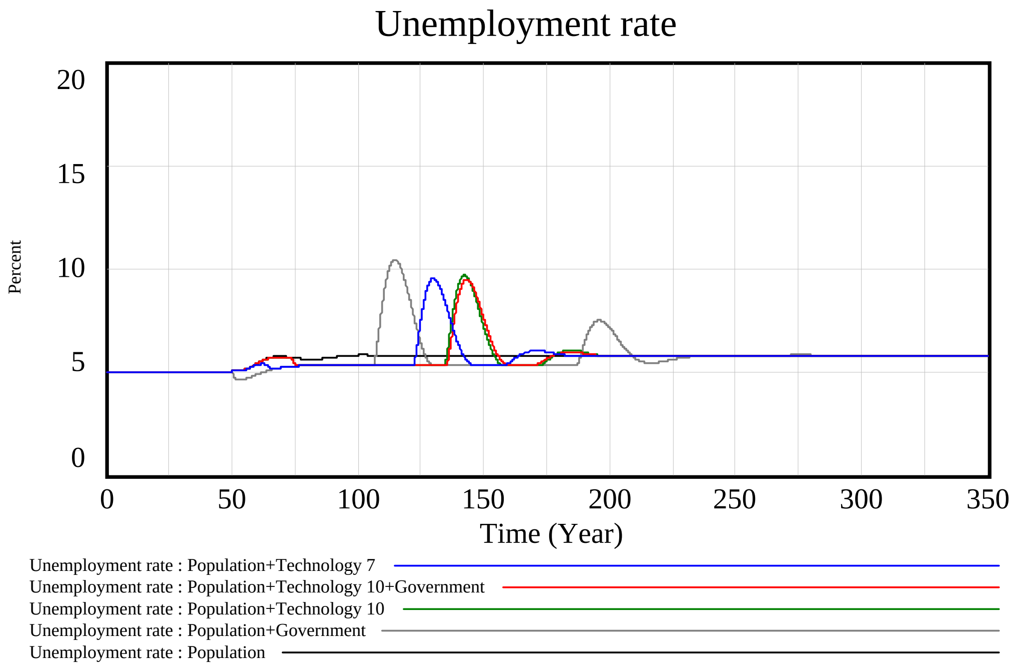

Figure 16: Unemployment rate

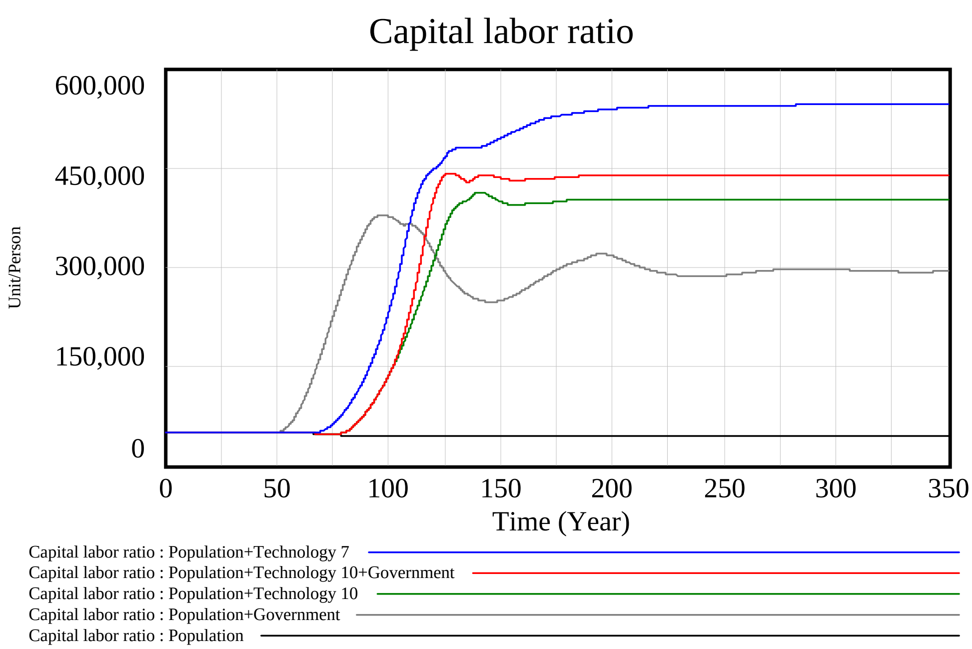

Figure 17: Capital labor ratio

Figure 11, Figure 12, and Figure 13 illustrate the changes of input scenarios of those five scenarios: the life of capital followed by its effect on the Alpha (the capital intensity), and the population growth respectively. The results show differences in production patterns (Figure 14), production per capita (Figure 15), and capital-labor ratio representing the technology level (Figure 17). The differences in unemployment rate patterns (Figure 16) emerge in the transition periods from its initial value to the slightly higher of new equilibrium value. An analysis of the five scenarios reveals that the economic growth (production per capita) resulted by the model in the scenarios those augmented with technology (innovation) surpass those in the other scenarios without technology augmentation. The innovation in the long term will decrease the average life of capital of the economy; in turn this will increase the capital intensity of the economy. The increase in capital intensity will strengthen the accelerator mechanism of the economy and creates larger multiplier effects. Apparently, in this scenario the behavior of the unemployment rate fluctuates in the transition periods from its initial value to the slightly higher of new equilibrium value. This indicates that developing countries have to develop policies for establishing more efficient labor market. The simulation results show the power of the technology (innovation), represented by an increase in the capital-labor ratio of the economy (Figure 17), in producing the higher sustainability growth of the economy.

6. Concluding remarks

This study is an attempt to investigate and to understand the dynamic effects of technological changes (technology development through innovation activities) on the growth of the economy. Using the changes in capital-labor ratio as a representation of technological changes, three sources of economic growth (and its combination) are simulated, i.e.: population growth, increasing in government spending, and innovation. A theoretical framework of the model is derived using the standard assumption of profit-maximizing behavior to model the decision making structures of the acquisition of production factors (capital and labor). The framework is then converted to a system dynamics model which is used to explore the power of the technology in maintaining the higher sustainability growth of the economy.

The study shows that the economy, in which a sustainable increase in capital intensity of the economy can be maintained through innovation activities in the long run, can be expected to improve the economic growth considerably and continuously. Therefore, developing countries have to manage the innovation processes based on the development of education and the R&D capacity of the nation. Besides, to reduce the increase in unemployment rate in the transition periods of the improved of economic growth due to the increase in technology level; developing countries have to develop policies for establishing more efficient labor market.

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APPENDIX A
The Mathematics of Technology-Economic Growth Model

Given a production function for society's ouput as a whole

$$q = q_0 * (\frac{K}{Ko})^{\alpha} * (\frac{L}{Lo})^{\beta} \qquad (1)$$

q, q_0 = production, initial production [unit/year]
K, K_0 = capital, initial capital [unit]
L, L_0 = labor, initial labor [person]
α = capital intensity [dimensionless], **not constant**
β = labor intensity [dimensionless], **not constant**

And profit in the model is total revenues (output times price of output) less the holding cost of capital (capital times the price of capital times the depreciation rate plus the interest rate) less the cost of labor (labor [employment] times the wage rate), as follows.

$$\text{Profit} = q * P_q - K * P_k * [(\frac{1}{alk}) + R] - L * (rw * P_q) \qquad (2)$$

Profit = [$/year]
P_q = *price of output* [$/unit]
P_k = *price of capital* [$/unit]
alk = *average life of capital* [year]
R = *real interest rate* [/year]
rw = *real wage* [unit/year/person]

The standard neoclassical assumption of profit-maximizing behavior in the acquisition of production factors are obtained by setting the partial derivative of profit with respect to capital equal to zero and setting the partial derivative of profit with respect to labor (employment) also equal to zero, and solving for capital and labor:

$$\frac{\partial Profit}{\partial K} = 0 \quad \text{and} \quad \frac{\partial Profit}{\partial L} = 0.$$

The condition of $\frac{\partial Profit}{\partial K} = 0$ gives

$$\frac{\partial q}{\partial K} * P_q - P_k * \left(\frac{1}{alk} + R\right) = 0 \ . \qquad (3)$$

Based on Equation (1), the partial derivative of output q with respect to capital K is

$$\frac{\partial q}{\partial K} = \alpha\, q_0 \left(\frac{\left(\frac{K}{Ko}\right)^{\alpha}}{\left(\frac{K}{Ko}\right)}\right) * \left(\frac{1}{Ko}\right) * \left(\frac{L}{Lo}\right)^{\beta}$$

where $q_0 * (\frac{K}{Ko})^{\alpha} * (\frac{L}{Lo})^{\beta} = q$ [Equation (1)], hence the derivative can be simplified as

$$\frac{\partial q}{\partial K} = \frac{\alpha * q}{K} \ . \qquad (4)$$

Putting Equation (4) into Equation (3) [(4) ➔ (3)], the Equation (3) can be written as

$$\frac{\alpha * q}{K} P_q - P_K * \left(\frac{1}{alk} + R\right) = 0 \ ,$$

and assuming that $P_q = P_k$ (one goods), hence the equation for capital K is obtained as

$$K = \frac{\alpha * q}{(\frac{1}{alk} + R)} \ . \qquad (5)$$

The condition of $\frac{\partial Profit}{\partial L} = 0$ gives

$$\frac{\partial q}{\partial L} * P_q - (rw * P_q) = 0 \quad . \qquad (6)$$

Based on Equation (1), the partial derivative of output q with respect to labor L is

$$\frac{\partial q}{\partial L} = \beta * q_0 \left(\frac{K}{Ko}\right)^{\alpha} * \left(\frac{\left(\frac{L}{Lo}\right)^{\beta}}{\left(\frac{L}{Lo}\right)}\right) * \left(\frac{1}{Lo}\right)$$

where $q_0 * (\frac{K}{Ko})^{\alpha} * (\frac{L}{Lo})^{\beta} = \mathbf{q}$ [Equation (1)], hence the derivative can be simplified as

$$\frac{\partial q}{\partial L} = \frac{\beta * q}{L} . \qquad (7)$$

Putting Equation (7) into Equation (6) [(7) ➔ (6)], the Equation (6) can be written as

$\frac{\beta * q}{L} * P_q - (rw * P_q) = 0$; hence the equation for labor L is obtained as

$$L = \frac{\beta * q}{rw} . \qquad (8)$$

Assuming that capital-output ratio (**KOR**) [KOR = K/q] is constant, hence Equation (5) $K = \frac{\alpha * q}{(\frac{1}{alk} + R)}$ can be written as

$\frac{K}{q} = \frac{\alpha}{\left(\frac{1}{alk} + R\right)}$ (where K/q = KOR), and then gives

$$\alpha = KOR * \left(\frac{1}{alk} + R\right) \quad . \qquad (9)$$

Dividing capital (K) by labor (L) [Equation (5)/Equation (8)], mentioned as capital-labor ratio, KLR; gives

$$KLR = \frac{\alpha * rw}{\left(\frac{1}{alk} + R\right) * \beta} . \qquad (10)$$

Putting Equation (9) into Equation (10) [(9)➔ (10)], the Equation (10) can be written as

$KLR = \frac{KOR * \left(\frac{1}{alk} + R\right) * rw}{\left(\frac{1}{alk} + R\right) * \beta}$; and then gives

$$\beta = \frac{KOR * rw}{KLR} \qquad (11)$$

From the Equation (11) $\beta = \frac{KOR * rw}{KLR}$ the change rate of β (labor intencity) dβ/dt, can be derived as follows.

$$\frac{d\beta}{dt} = KOR * \left[\frac{\partial\left(\frac{rw}{KRL}\right)}{\partial(rw)} \frac{d(rw)}{dt} + \frac{\partial\left(\frac{rw}{KRL}\right)}{\partial(KLR)} \frac{d(KLR)}{dt}\right]$$

$$= KOR * \left[\frac{1}{KLR} \frac{d(rw)}{dt} - \frac{rw}{(KLR)^2} \frac{d(KLR)}{dt}\right]$$

$$= KOR * \left[\frac{rw}{KLR} \frac{\frac{d(rw)}{dt}}{rw} - \frac{rw}{KLR} \frac{\frac{d(KLR)}{dt}}{KLR}\right]$$

gives
$$\frac{d\beta}{dt} = KOR * \frac{rw}{KLR} * [G_{rw} - G_{KLR}] . \qquad (12)$$

where $G_{rw} = \frac{\frac{d(rw)}{dt}}{rw}$, growth rate of real wage [/year] and

$G_{KLR} = \frac{\frac{d(KLR)}{dt}}{KLR}$, growth rate of capital-labor ratio [/year].

For a Cobb-Douglas production fuction (i.e. α + β = 1 or α = 1 – β), the Equation (12) gives
the change rate of α (capital intensity) dα/dt, as

$$\frac{d\alpha}{dt} = KOR * \frac{rw}{KLR} * [G_{KLR} - G_{rw}]. \tag{13}$$

The growth rate of the output (production) or the economic growth rate (i.e. $\frac{dq/dt}{q}$) can be derived as follows.

Given a Cobb-Douglas production fuction of the economy as

$$q(t) = q_0 * (\frac{K(t)}{Ko})^{\alpha(t)} * (\frac{L(t)}{Lo})^{\beta(t)}. \tag{14}$$

and assuming that f(K,α) = $(\frac{K}{Ko})^{\alpha}$ and g(L,β) = $(\frac{L}{Lo})^{\beta}$, hence the Equation (14) can be written as

$$q = q_0 f g \tag{15}$$

and then gives

$$q_0 = \frac{q}{fg}. \tag{16}$$

[Note: q_0 (initial production) is a constant]

Differentiating Equation (15) with respect to time gives

$$\frac{dq}{dt} = q_0 [\frac{df}{dt} g + f \frac{dg}{dt}] \quad \text{where}$$

$$\frac{df(K,\alpha)}{dt} = \frac{\partial f}{\partial K}\frac{dK}{dt} + \frac{\partial f}{\partial \alpha}\frac{d\alpha}{dt} \quad \text{and}$$

$$\frac{dg(L,\beta)}{dt} = \frac{\partial g}{\partial L}\frac{dL}{dt} + \frac{\partial g}{\partial \beta}\frac{d\beta}{dt} \quad \text{; hence}$$

$$\frac{dq}{dt} = \frac{q}{fg}[\{\frac{\partial f}{\partial K}\frac{dK}{dt} + \frac{\partial f}{\partial \alpha}\frac{d\alpha}{dt}\} g + \{\frac{\partial g}{\partial L}\frac{dL}{dt} + \frac{\partial g}{\partial \beta}\frac{d\beta}{dt}\} f]. \tag{17}$$

Dividing by q gives

$$\frac{dq/dt}{q} = \frac{\partial f/\partial K}{f}\frac{dK}{dt} + \frac{\partial f/\partial \alpha}{f}\frac{d\alpha}{dt} + \frac{\partial g/\partial L}{g}\frac{dL}{dt} + \frac{\partial g/\partial \beta}{g}\frac{d\beta}{dt}$$

$$= \left\{\frac{\partial f/\partial K}{f} K\right\}\frac{dK/dt}{K} + \left\{\frac{\partial f/\partial \alpha}{f}\right\}\frac{d\alpha}{dt} + \left\{\frac{\partial g/\partial L}{g} L\right\}\frac{dL/dt}{L} + \left\{\frac{\partial g/\partial \beta}{g}\right\}\frac{d\beta}{dt}. \tag{18}$$

but

$$\frac{\partial f}{\partial K} = \alpha \left(\frac{K}{Ko}\right)^{(\alpha-1)} \frac{1}{Ko}$$

$$= \alpha \frac{(K/Ko)^{\alpha}}{(K/Ko)Ko} \quad \text{then}$$

$$\frac{\partial f/\partial K}{f} K = \alpha \frac{(K/Ko)^{\alpha}}{(K/Ko)Ko} * \frac{1}{(K/Ko)^{\alpha}} * K \quad \text{or}$$

$$\frac{\partial f/\partial K}{f} K = \alpha; \tag{19}$$

and the same procedure gives

$$\frac{\partial g/\partial L}{g} L = \beta \tag{20}$$

In addition, some terms of Equation (18) are derived as follows:

$$\frac{\partial f}{\partial \alpha} = \left(\frac{K}{Ko}\right)^{\alpha} ln\left(\frac{K}{Ko}\right)$$

where $f = \left(\frac{K}{Ko}\right)^{\alpha}$, and then $ln\, f = \alpha\, ln\left(\frac{K}{Ko}\right)$

$$\frac{\partial\, lnf}{\partial \alpha} = ln\left(\frac{K}{Ko}\right)$$

$$\frac{1}{f}\frac{\partial f}{\partial \alpha} = ln\left(\frac{K}{Ko}\right) \rightarrow \frac{\partial f}{\partial \alpha} = f\, ln\left(\frac{K}{Ko}\right)$$

$$\frac{\partial f}{\partial \alpha} = \left(\frac{K}{Ko}\right)^{\alpha} ln\left(\frac{K}{Ko}\right);$$

hence

$$\frac{\partial f/\partial \alpha}{f} = \frac{(K/Ko)^{\alpha}\, ln\,(K/Ko)}{(K/Ko)^{\alpha}}$$

or $\frac{\partial f/\partial \alpha}{f} = \boldsymbol{ln}\left(\frac{K}{Ko}\right)$; (21)

and the same procedure gives

$$\frac{\partial g/\partial \beta}{g} = \boldsymbol{ln}\left(\frac{L}{Lo}\right). \qquad (22)$$

Putting Equation (19), Equation (20), Equation (21), and Equation (22) into Equation (18); the Equation (18) can be written as

$$\frac{dq/dt}{q} = \alpha \frac{dK/dt}{K} + ln\left(\frac{K}{Ko}\right)\frac{d\alpha}{dt} + \beta \frac{dL/dt}{L} + ln\left(\frac{L}{Lo}\right)\frac{d\beta}{dt} \ ,$$

and using β = 1 – α (Cobb- Douglas production function), hence

$$\frac{dq/dt}{q} = \alpha \frac{dK/dt}{K} + ln\left(\frac{K}{Ko}\right)\frac{d\alpha}{dt} + (1-\alpha)\frac{dL/dt}{L} - ln\left(\frac{L}{Lo}\right)\frac{d\alpha}{dt}. \qquad (23)$$

Some terms of the above equation, Equation (23), are simplified as follows:

$$ln\left(\frac{K}{Ko}\right)\frac{d\alpha}{dt} - ln\left(\frac{L}{Lo}\right)\frac{d\alpha}{dt} = \frac{d\alpha}{dt}\left[ln\left(\frac{K}{Ko}\right) - ln\left(\frac{L}{Lo}\right)\right] = \frac{d\alpha}{dt}\left[ln\left(\frac{\frac{K}{Ko}}{\frac{L}{Lo}}\right)\right] = \frac{d\alpha}{dt}\left[ln\left(\frac{\frac{K}{L}}{\frac{Ko}{Lo}}\right)\right] ,$$

and replacing dα/dt in the above equation by Equation (13) $\frac{d\alpha}{dt} = KOR * \frac{rw}{KLR} * [\boldsymbol{G_{KLR}} - \boldsymbol{G_{rw}}]$, hence the Equation (23) can be written as

$$\frac{dq/dt}{q} = \alpha \frac{dK/dt}{K} + (1-\alpha)\frac{dL/dt}{L} + KOR * \frac{rw}{KLR} * ln\left(\frac{KLR}{KLRo}\right) * [\boldsymbol{G_{KLR}} - \boldsymbol{G_{rw}}] \text{ or}$$

$$\boldsymbol{G_q = \alpha\, G_K} + (1-\boldsymbol{\alpha})\,\boldsymbol{G_L} + KOR * \frac{\boldsymbol{rw}}{\boldsymbol{KLR}} * ln\left(\frac{\boldsymbol{KLR}}{\boldsymbol{KLRo}}\right) * [\boldsymbol{G_{KLR}} - \boldsymbol{G_{rw}}] \ . \qquad (24)$$

where $\boldsymbol{G_K}$ is the growth rate of capital K, $\boldsymbol{G_L}$ is the growth rate of labor L, $\boldsymbol{G_{KLR}}$ is the growth rate of capital-labor ratio KLR, and $\boldsymbol{G_{rw}}$ is the growth rate of real wage.