

Fuzzy Modeling of Linguistic Variables in a System Dynamics Context¹

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Abstract

This paper builds on a previously proposed approach where fuzzy logic is used to incorporate linguistic variables in system dynamics modeling. The motivation for this approach is to include vague yet dynamic variables that are combined in a meaningful way. The essence of our approach requires the definition of membership functions as representations of the degree to which specific variable attributes hold, the application of a max-min direct inference approach as a way to combine two or more fuzzy variables, and the use of a defuzzification method that captures (summarizes) the joint effect of the linguistic variables. The objective of this paper is to study the implications of using two alternative defuzzification methods (largest of maximum and center of area) and to highlight various interpretation and modeling challenges associated with each defuzzification method. For illustrative purposes we use a variant of a sales and service model that is based on the concepts of product diffusion, backlog accumulation and personnel adjustments and their respective existing modeling representations in the literature. As a heuristic solution, we suggest substituting the Max-Min operator and eliminating inconsistencies among the fuzzy rules, so that the defuzzified values behave reasonably for both defuzzification methods.

Key Words and Phrases: Linguistic variables, System Dynamics, Fuzzy Set Theory, Uncertainty, Defuzzification

1.0 Introduction

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This research addresses two specific needs identified in literature, i.e., the need to explicitly model linguistic (qualitative) variables in a dynamic modeling context using alternative formulations and the need to explicitly consider dynamic behavior in conjunction with a fuzzy reasoning framework. When considering linguistic variables, one typically encounters two separate issues. The first deals with the ambiguity that surrounds the linguistic variable in question which is best described by the lack of information about the variable itself. The second issue arises from the way the uncertain variable is described (Kikuchi 2005) that is the focus of this paper.

In order to ensure that relevant variable characteristics along with the way they are perceived (or measured) have been accurately captured, one way is to use fuzzy logic (Kikuchi 2005). Furthermore, measuring how the variables change over time poses an additional challenge since one would wish to have a reasonable representation of the changes of the state of the uncertain variable through time. Consequently, it is important to investigate the feasibility and issues that arise when using a fuzzy logic based representation of linguistic variables in a dynamic modeling context.

This paper builds on the method developed by Liu, Triantis et al. (2010) where a fuzzy logic approach was proposed as a way to incorporate linguistic variables in a sales and service model. In this model, the growth of the market share depends upon the attractiveness of the product among potential customers and a favorable word of mouth that makes more potential customers buy the product. There are several factors that determine product attractiveness including for example ‘delivery timeliness’ and ‘customer service’ available for the product. However, the perceptions with respect to ‘delivery timeliness’ and ‘customer service’ have an inherent vagueness that makes a certain or crisp representation difficult. Altogether, we can describe the perceptions of customers in linguistic terms such as low, medium and high, which all have vague definitions. Furthermore, these perceptions change over time especially since there are forces at work (e.g., the training of the sales force) that can affect these perceptions. Hence, the representation of perceptions such as these over time provides the key motivation for exploring the fuzzy logic concepts in the context of dynamic modeling

This paper was motivated in part by the implementation and interpretation difficulties associated with the use of largest of maximum de-fuzzification method within a fuzzy logic inference approach where the dynamic consequences of congestion pricing were studied (Liu 2007). Consequently, the objective of this paper is to study the implications of using two alternative de-fuzzification methods (Largest of Maximum (LOM) and Center of Area (COA)) within a similar dynamic modeling context and to highlight various interpretation and modeling challenges associated with each de-fuzzification method in a system dynamics modeling context.

The remainder of the paper is organized as follows. Section 2 provides a brief literature review on applying fuzzy logic in the system dynamics modeling paradigm. Then in Section 3, the

model that is used for illustrative purposes is explained in detail. In Section 4, the fuzzification process and the inference method used in the developed model is described. The results from the two defuzzification approaches including the Largest of Maximum (LOM) and Center of Area (COA) approaches are studied thoroughly in Section 5. Some counter-intuitive results are also highlighted in this section. Some suggestions for improving the results of the two defuzzification approaches is provided in Section 6. Finally, in Section 7, we conclude with a summary of the main results along with recommendations for future research.

2.0 Background

There have been several attempts to bridge fuzzy logic with dynamic modeling. Levary (1990) proposes applying the concept of fuzzy sets introduced by Zadeh (1965) in system dynamics methodology to deal with imprecision or vagueness, but does not actually implement the suggestions in an actual system dynamics model. Maeda, Asaoka et al. (1996) propose a reasoning method that incorporates a vague time delay into fuzzy if-then rules. Ortega, Sallum et al. (2000) propose the combination of fuzzy logic and non-linear dynamical systems, in order to treat some of the uncertainties and imprecision present in epidemic problems. Furthermore, Polat and Bozdağ (2002) compare and contrast fuzzy and classical crisp rules by running a system dynamics simulation for a simple heating model that is controlled by a human operator. Similarly, Chang, Pai et al. (2006) present an application of fuzzy arithmetic representations in a system dynamics context. However the combined effect of the fuzzy variables is not studied by any of the authors.

In a paper that is similar to ours in spirit, Kunsch and Springael (2008) provide a system-dynamics model of carbon tax design on the residential sector using fuzzy rules. The authors neither provide complete details about how they implement their approach, nor describe the details of the behavior of fuzzy parameters during the simulation of the model. In a different approach, Campuzano, Mula et al. (2010) have demonstrated that applying the possibility theory and using fuzzy numbers to estimate demand and orders in a supply chain system dynamics model; can be very useful, under conditions of demand uncertainty.

Overall from the literature review in this area, we find that there are a limited number of studies that address the issue of combining two or more fuzzy variables in a System Dynamics modeling framework. Furthermore, these papers do not explicitly provide the implementation details associated with their respective approaches. To the best of our knowledge, there is no study that addresses the implications of considering alternative fuzzification and defuzzification methods.

3.0 The Model

The model is based on Liu, Triantis et al. (2010) where an alternative version of the model was used to introduce the idea of how to incorporate multiple linguistic variables in system dynamics

modeling. The model focuses on the introduction of a new product into the market where potential customers become aware of the new product and then, based on word-of-mouth information and their perception of the product’s attractiveness are converted to customers. Since the model is too complex to be described in full, each component will be described separately and the interested reader can study the full model provided in Appendix 1.³

The molecule ‘Product Diffusion’ developed in VENSIM (Hines 2004) is included to represent the dynamics associated with word-of-mouth when generating new customers and the growth of the market share (Figure 1). Based on the concentration of the potential customers in the whole population, a fraction of these contacts are with potential customers. Then based on the probability that a contact will generate a new customer, these contacts will convert non-customers to customers, which is defined as the ‘Word-of-Mouth Conversions’.

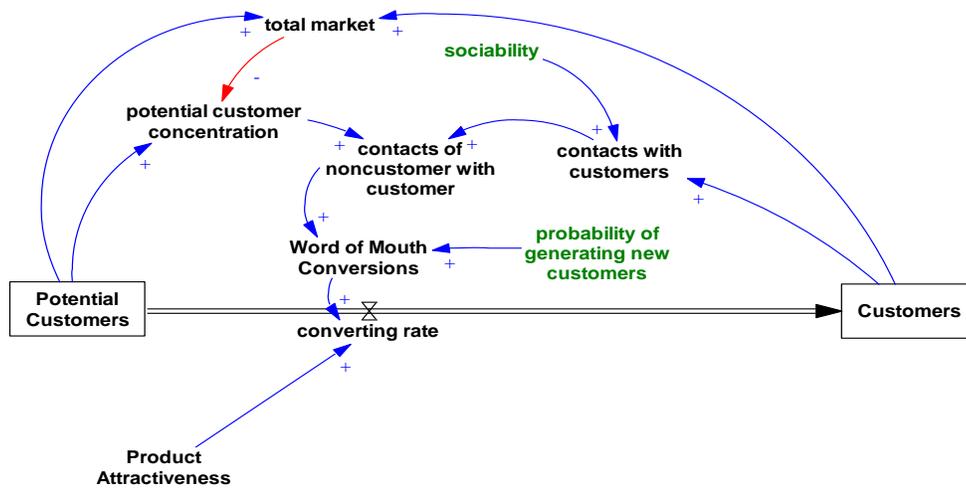


Figure 1- Product Diffusion based on Word of Mouth and Product Attractiveness

Additionally, the conversion rate from ‘Potential Customers’ to ‘Customers’ is also affected by ‘Product Attractiveness’ that is typically affected by several factors including customer service and delivery timeliness. Delivery timeliness is dependent upon the inventory and backlog level of the firm. This section is modeled by using the ‘Backlog Shipping Protected by Flow’ molecule developed in VENSIM (Hines 2004) (Figure 2).

Delivery Timeliness which is determined by the ‘Normalized Delivery Timeliness’ variable is calculated as the ratio of ‘Fulfilling Orders’ over ‘Indicated Shipping from Backlog’. The former is the actual delivery rate of orders and the latter is the desired delivery rate based on the time delay in processing and shipping orders. The ‘Normalized Delivery Timeliness’ is the basis for estimating ‘Perceived Delivery Timeliness’ that in turn affects the attractiveness of the product.

³ The code is available from the authors on request.

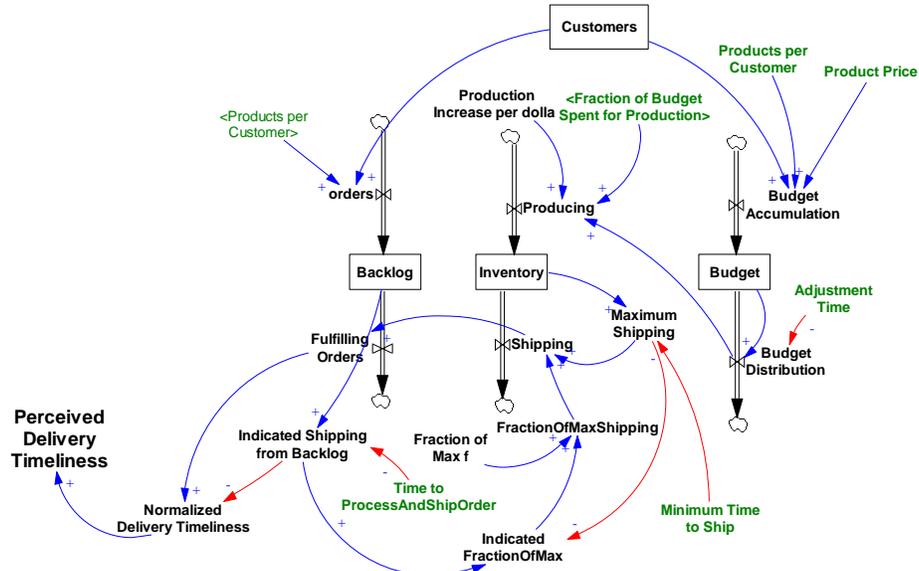


Figure 2-Backlog Shipping Protected by Flow

The number of workers is changed by hiring extra sales personnel, so that on average each salesperson is serving the ‘Desired Sales Personnel per Customer’ (chosen arbitrarily without qualitatively affecting the results). The ratio of the number of salespeople hired (i.e. ‘Workforce’), over the number of the desired workforce (i.e. ‘Desired Total Salespeople’) will define ‘Normalized Customer Service’ (a value between zero and one). Consequently, the customers’ perception about customer service (i.e. ‘Perceived Customer Service’) can be determined by the normalized customer service value. In conclusion, ‘Perceived Customer Service’ and ‘Perceived Delivery Timeliness’, together determine the variable ‘Product Attractiveness’ (Figure 3).

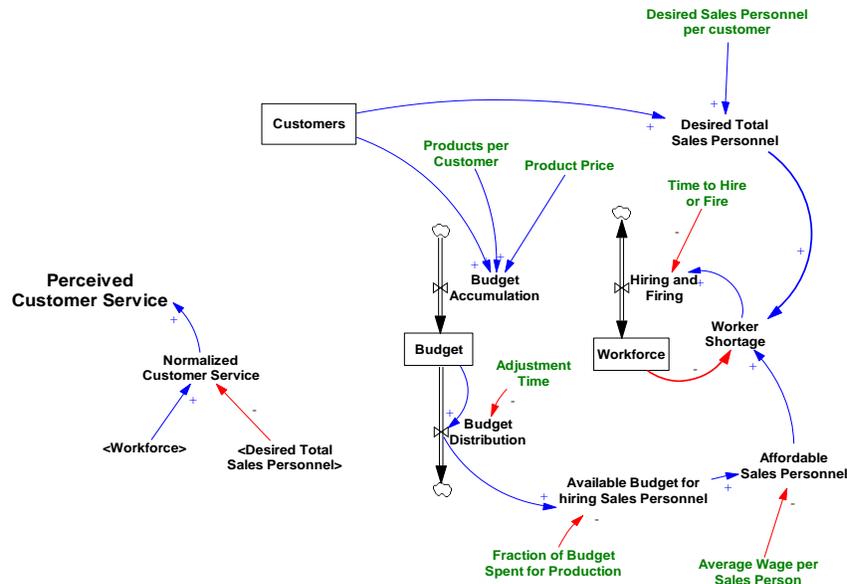


Figure 3-Workforce Change

4.0 The Fuzzification Process

As described in the introduction section, the perception variables are evaluated based on fuzzy logic and for this purpose, fuzzy membership functions have been incorporated to capture the vagueness inherent in the linguistic variables ‘Perceived Delivery Timeliness’ and ‘Perceived Customer Service’. The Fuzzy Rule-Based Inference System using Mamdani’s method (Mamdani 1977) is applied to infer conclusions.

For the sake of illustration we have assumed that both of the linguistic variables, ‘Perceived Delivery Timeliness’ and ‘Perceived Customer Service’ are represented by three membership function that represent how much delivery timeliness and customer service is perceived as high, medium and low. The membership functions associated with the high, medium, and low characteristics are defined as linear triangular functions as depicted in Figure 4. Depending on the need it is possible to develop a model to allow for more or fewer membership characteristics without any loss of generality.

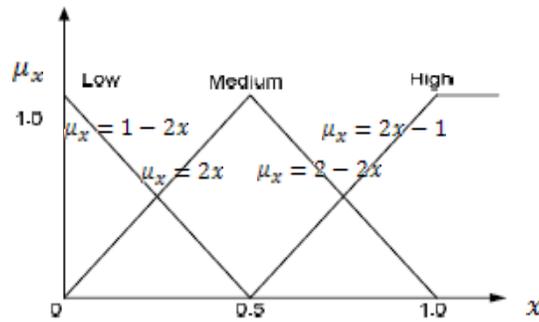


Figure 4-The Membership Functions for the Three Characteristics Associated with Perception Variables

In our example, if the actual shipping rate gets close enough to the desired shipping rate, then the normalized delivery timeliness (i.e. x) in Figure 4 would be near 1. If for instance x is equal to 0.9, then the high membership function of ‘Perceived Delivery Timeliness’ will have the highest value (degree of truth) among all membership functions, which is 0.8, while the medium membership function would have a value of 0.2 and the low membership function would have a value equal to zero.

In order to find the combined effect of the two fuzzy perception variables on ‘Product Attractiveness’, a fuzzy inference method is applied. A fuzzy inference method consists of all the steps required to map some input to a crisp output by using fuzzy logic. One of the most common inference methods used is the Mamadani’s method because of its simple structure of min-max operations which is also applied for our application.

In the first step, a certain number of rules are chosen that represent different combinations of the two linguistic variables. For the sake of illustration, this set of rules are defined, however, others

can also be used. Since each fuzzified variable has the same number of characteristics (Low, Medium and High), nine rules need to be evaluated in order to find the “Product Attractiveness” (See Table 1).

Table 1-The 9 Rules

	Perceived Customer Service	Perceived Delivery Timeliness	Product Attractiveness
1	Low	Low	Low
2	Low	Medium	Low
3	Low	High	Medium
4	Medium	Low	Medium
5	Medium	Medium	Medium
6	Medium	High	High
7	High	Low	Medium
8	High	Medium	High
9	High	High	High

There is not a unique way to define the fuzzy rules and could be based on expert advice or available data. In order to illustrate the affects of different types of rules, we have chosen a combination of an optimistic and pessimistic decision maker to define the rules. However, some rules depicted in Table 1, lack consistency.

In order to find the value of each rule, based on Mamdani’s Max-Min inference method, the minimum of the two perception values for that membership range (i.e. Low, Medium or High), is calculated. Then in order to find the union of the rules, the maximum value of all rules that result into a low, medium and high membership domain is found separately. As a result, in this approach, in order to evaluate the union, the maximum between rules 1 and 2 is found which is the result for representing the low membership function, the maximum for rules 3, 4, 5 and 7 is the result for representing the medium membership function and for the high membership function the maximum of rules 6, 8 and 9 is found. For example, the union of rules 2, 5 and 6 which have low, medium and high membership representations respectively is shown in Figure 5 as the shaded area of the graph.

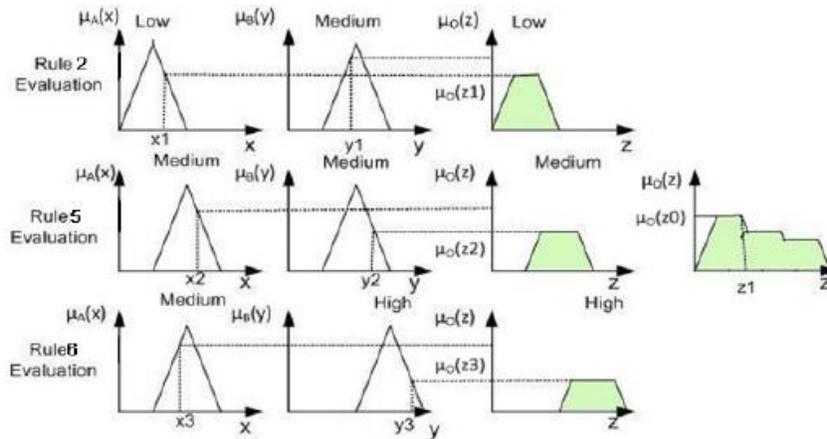


Figure 5-Largest of Maximum Defuzzification of the fuzzy set

5.0 Defuzzification Process

In order to find a crisp value for the combined effect of the two perception variables that constitute ‘Product Attractiveness’, defuzzification algorithms need to be applied. In our approach, two different defuzzification methods are applied in the model, i.e., the ‘Largest of the Maximum (LOM)’ and the ‘Center of Area (COA)⁴’. The process used to calculate the final value and the results obtained are described extensively in the following sections.

5.1 Largest Of Maximum Defuzzification (LOM) Method

In this method, in order to find the domain with the maximum truth, the maximum value among the nine rules values and the corresponding domain, is found. For example, the maximum truth in Figure 5 belongs to the membership function representing the low characteristic. Since this point is ambiguous (i.e., it lies along a plateau), the LOM method selects the farthest edge of the plateau and drops a plumb line to resolve the conflict. As an example, in Figure 5, in order to defuzzify the maximum value μ_{z0} which belongs to the membership function representing the low characteristic, the defuzzified value z_1 is selected which is at the farthest edge and has the maximum value among all the z values belonging to this plateau value.

The fuzzification and LOM-defuzzification section of the model is shown in Figure 6. At each time step of the simulation, the maximum membership value for the rules 1 and 2 representing the *low* characteristic the maximum membership value for rules 3, 4, 5, and 7 representing the *medium* characteristic, and the maximum membership value for rules 6, 8, and 9 representing the *high* characteristic is modeled by the three variables ‘max valueL’, ‘max valueM’, and ‘max valueH’, respectively.

⁴ In the literature sometimes the term ‘Center Of Gravity (COG)’ is used, even though it is computed only for a two-dimensional area.

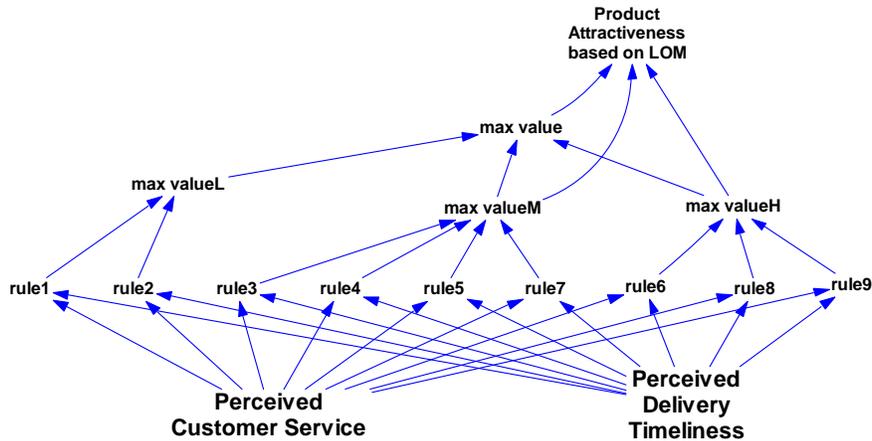


Figure 6-Fuzzification and Largest of Maximum Defuzzification

After running the simulation for the LOM defuzzification method, the ‘Product Attractiveness’ behavior over a time horizon of 300 weeks is shown in Figure 7.

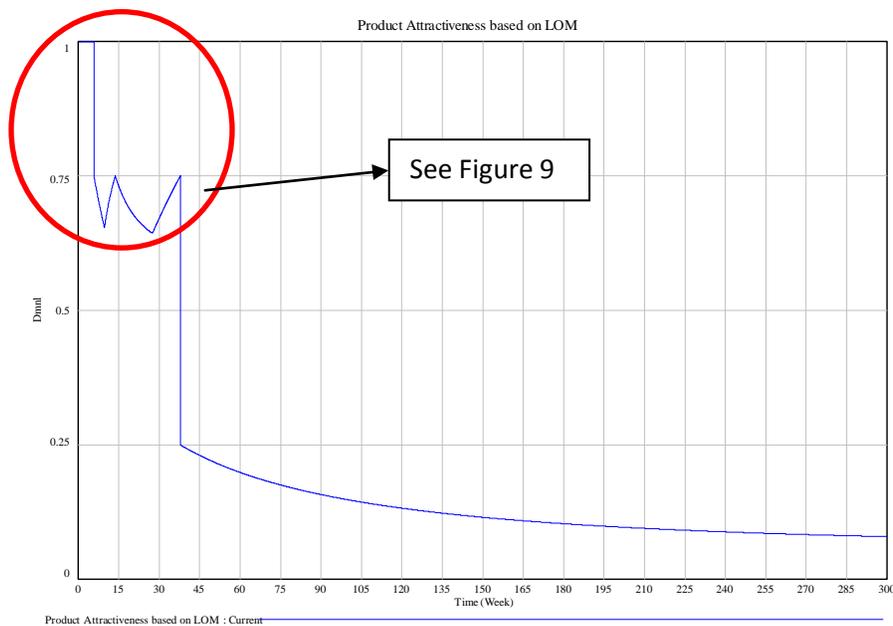


Figure 7-Defuzzified Effect of the Perceived Delivery Timeliness and Perceived Customer Service on the Product Attractiveness based on the LOM Defuzzification Method

Even though at first glance, the behavior of the defuzzified value (i.e. Product Attractiveness) may seem reasonable, there are counter-intuitive results at certain time intervals (the time interval shown in the red circle in Figure 7). In order to better understand why this behavior is unreasonable, we observe that ‘Normalized Delivery Timeliness’ and ‘Normalized Customer Service’ are decreasing continuously during the simulation (See **Error! Reference source not**

found.) This suggests that the perceived product attractiveness should be also continuously decreasing.

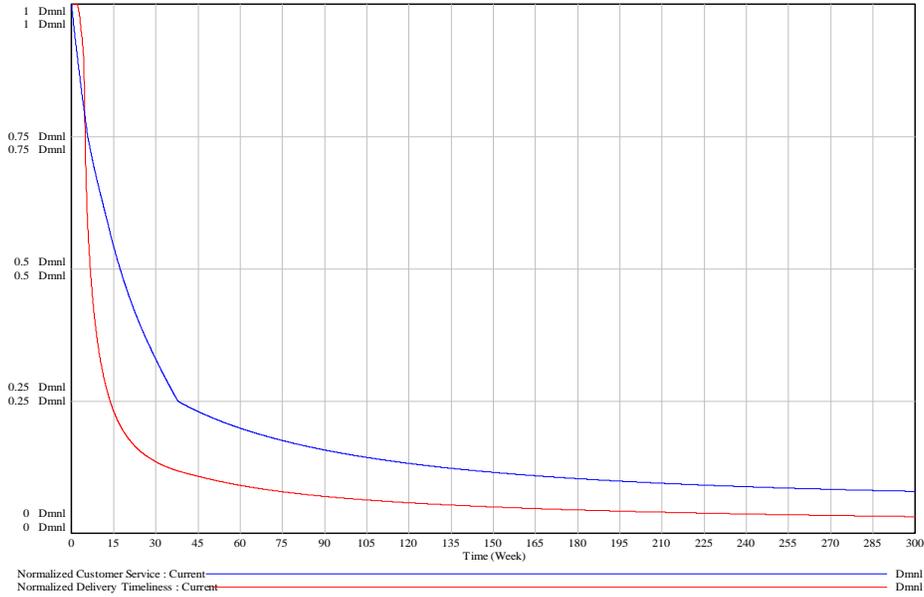


Figure 8-Normalized Delivery Timeliness & Normalized Customer Service

However, initially, when the simulation starts, until time (measured in weeks) 5.75, the ‘Product Attractiveness’ remains constant at 1. Furthermore, for some time steps of the simulation, we observe increasing defuzzified values of ‘Product Attractiveness’ (See Figure 9).

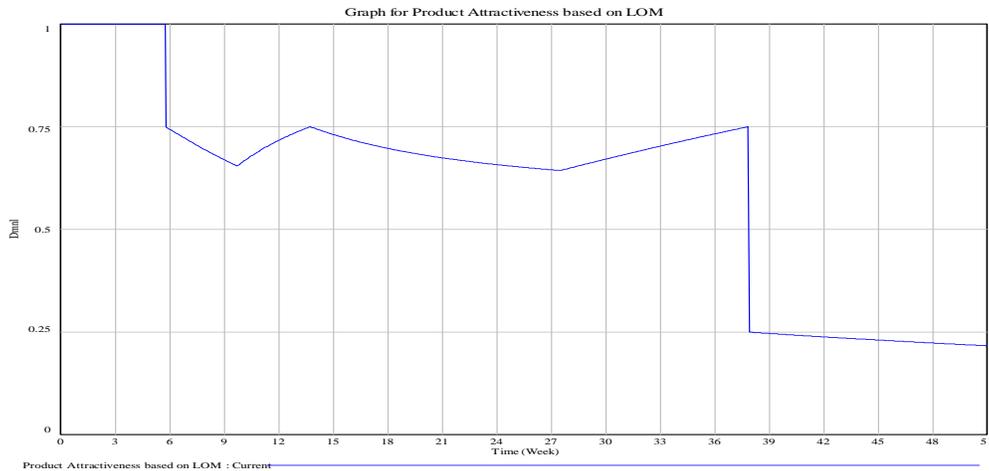


Figure 9-Product Attractiveness based on the LOM - Increasing Behavior at Some Time Intervals

Most of this unreasonable behavior is due to the underlying LOM calculations, For example, when the simulation begins, specifically until time 4.8, rule number 9 (the outcome of which is a

perceived High product attractiveness) that influences the calculation of the value of the defuzzified variable, has calculated decreasing values over time, due to decreases in the membership values associated with the ‘Perceived Customer Service-high membership function’ over time. This means that the calculated defuzzified values should also be decreasing from 1 to 0.79 over time or graphically should be moving in the direction of the red arrow (see Figure 10). However, the defuzzified values remain constant at 1 according to the LOM calculations, which assigns the largest value to the maximum membership value found. This result is not reasonable for the observed conditions described above.

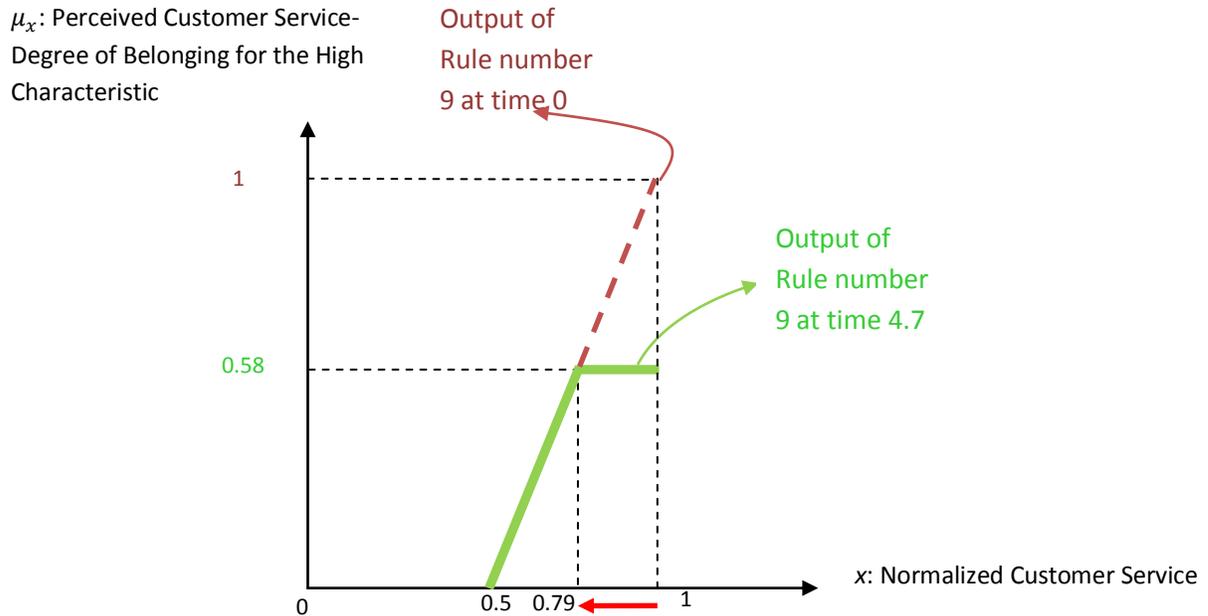


Figure 10-Impact of Rule 9 on the Defuzzified Value at the Beginning of the Simulation

5.2 Center Of Area Defuzzification (COA) Method

The Center of Area (COA) defuzzification method evaluates the final crisp value based on the following formula:

$$z_0 = \frac{\int \mu_x x dx}{\int \mu_x dx}$$

Based on this equation, we need to calculate the weighted mean of the fuzzy area displayed in the shaded green area in Figure 5 which is defined by the union of the maximum membership values for each domain value that is associated with the interface of the nine rules. For this purpose, and in order to incorporate the COA defuzzification method in the VENSIM model, we need to devise some approach to find the area under the boundary that is shown with the red color curve in Figure 11.

In order to get the red boundary of Figure 11, at each time step of the simulation, the maximum membership value associated with the rules 1 and 2 representing the *low* characteristic, the maximum membership value associated with rules 3, 4, 5, and 7 representing the *medium* characteristic, and the maximum membership value associated with the rules 6, 8, and 9 representing the *high* characteristic is modeled by the three variables ‘max valueL’, ‘max valueM’, and ‘max valueH’, respectively (see Figure 6). These variables represent the membership values for each domain value shown by the two-headed arrows in Figure 11. Then the minimum between the values max valueL, max valueM, and max valueH and the corresponding membership values for the functions is found (See Figure 11). Finally the value of Center of Area is calculated by using the approximation of $z_0 = \frac{\sum \mu_x x}{\sum \mu_x}$ for the integral $z_0 = \frac{\int \mu_x x dx}{\int \mu_x dx}$ at each time step.

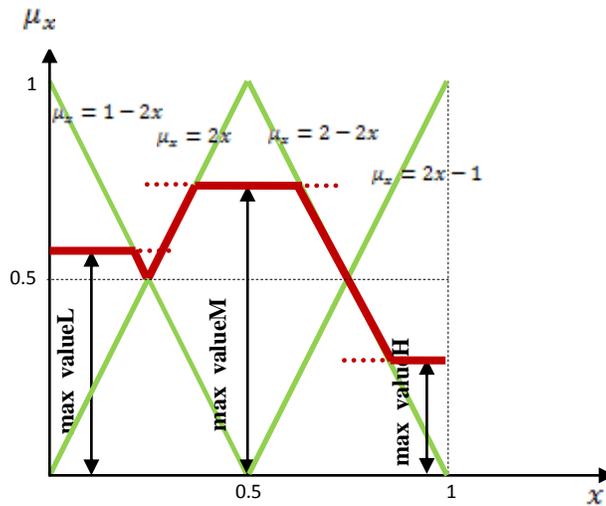


Figure 11- COA Calculation

One would expect that the overall behavior of the defuzzified value represented by ‘Product Attractiveness’ is consistent with the behavior of the two linguistic variables in the sense that when the delivery timeliness decreases and the customer service deteriorates, the COA value should also generally decrease (see Figure 12).

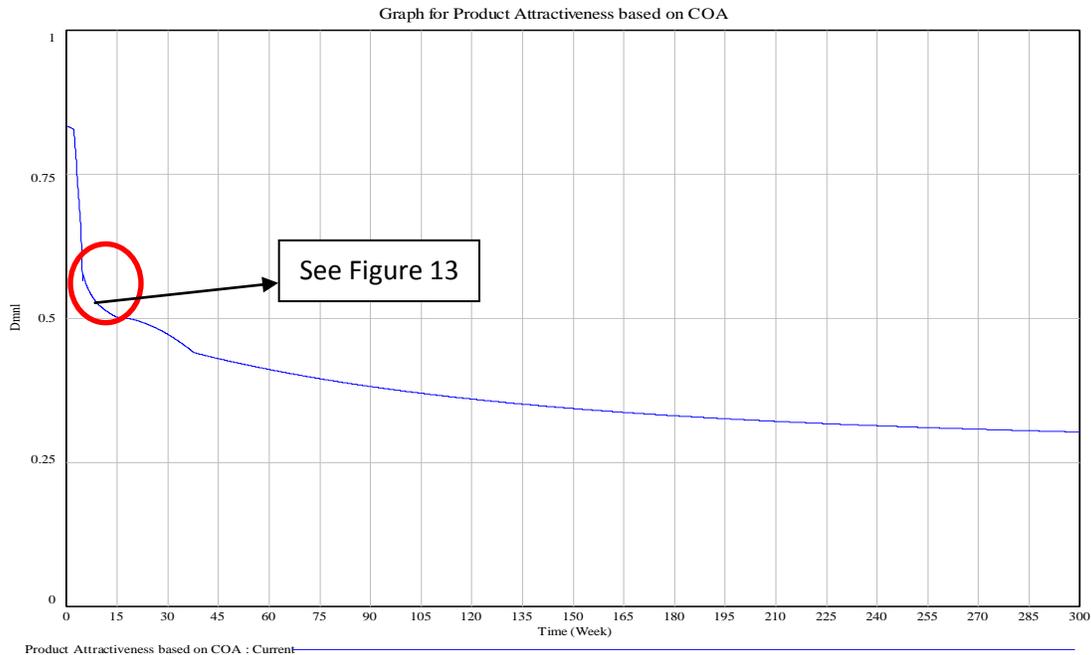


Figure 12- The Defuzzified effect of Product Attractiveness based on the COA Defuzzification Method

However, counterintuitive results are observed for the interval starting at time step 4.85, for which the corresponding value of the COA is 0.5666 which then increases to 0.5755 at time step 5 (See Figure 13).

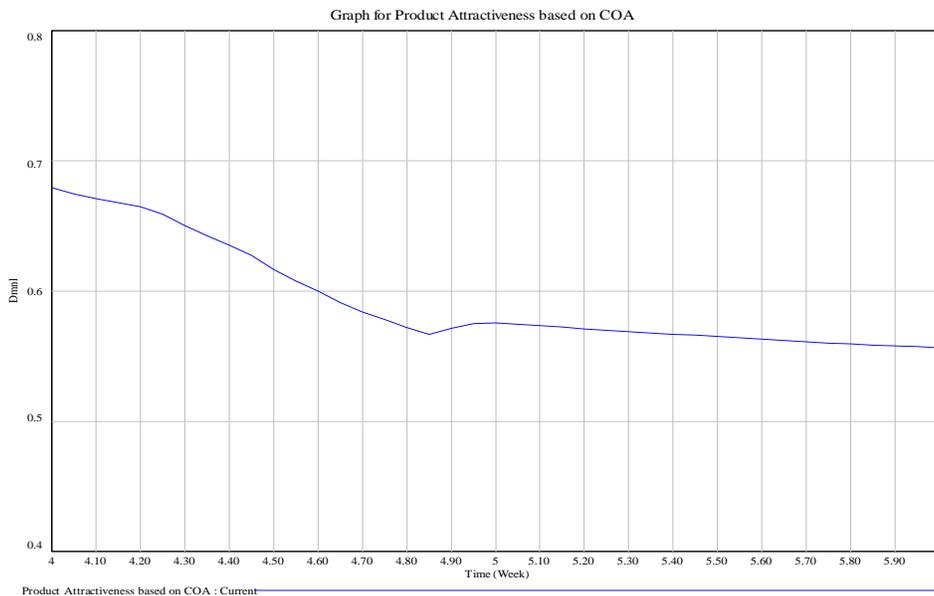


Figure 13- Counterintuitive Results based on the COA Defuzzification Method-

In order to understand the underlying mechanism for this behavior, we need to study the values of max valueL, max valuM and max valuH during the time interval 4.85 to 5. The results show that value of ‘max valueL’ remains zero which corresponds to the maximum value of the Rules 1

and 2 during this period. At the same time, the ‘max valueM’ which is the maximum value of the Rules 3, 4, 5 and 7 increases in this time interval from the value of 0.4286 at time 4.85, to the value of 0.4403 at time 5 (see Figure 14).

However, the ‘max valueH’ which is the maximum value of the Rules 6, 8 and 9 decreases until time 4.85, and then starts to increase from the value of 0.5013 at time 4.85 to the value of 0.5596 at time 5. Also the values of ‘max valueH’ are higher than ‘max valueM’ indicating that we will use the max valueH values in the de-fuzzification process for this time period (See Figure 14).

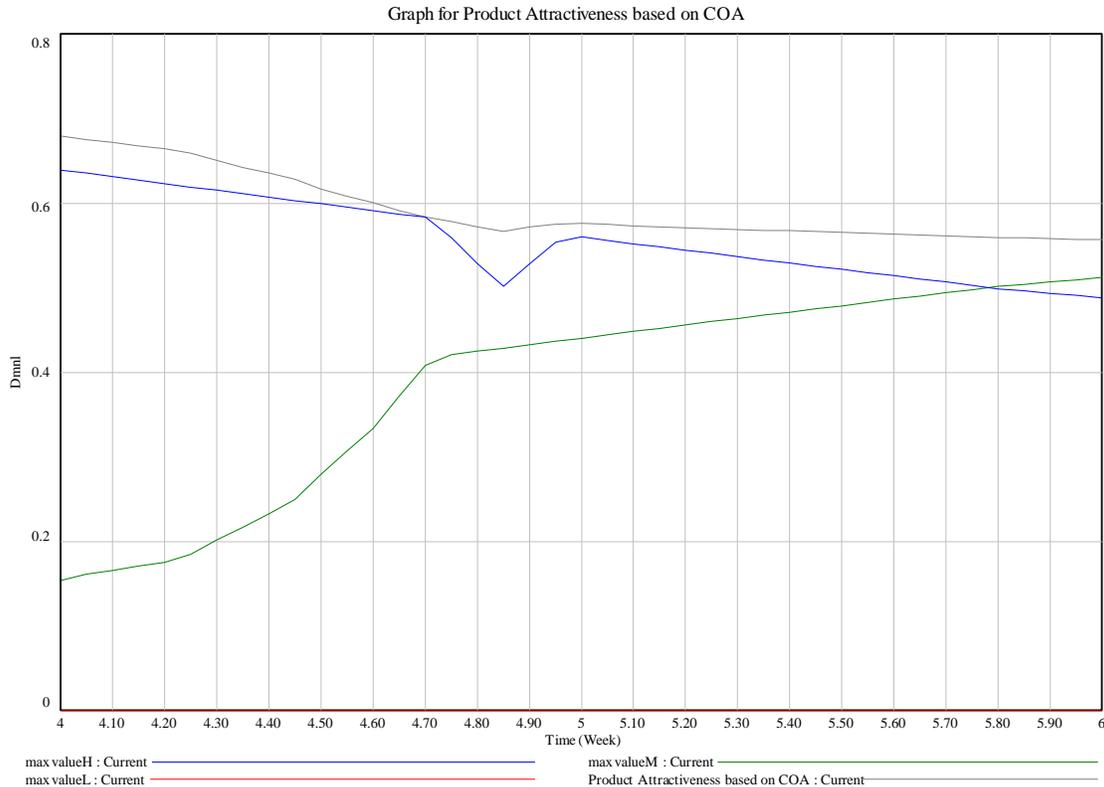


Figure 14- Counterintuitive Results based on the COA Defuzzification Method

Since the increase in ‘max valueH’ is higher than the increase of ‘max valueM’, the Center of Area formulation finds a higher value for the defuzzified effect, because as the high membership value increases more than the medium membership value, a higher weight is placed on the area to the right of 0.5666 (i.e. $x > 0.5666$). This analysis illustrates the challenge of combining two linguistic variables. Although the linguistic variables exhibit a declining behavior (Figure 8), the resulting defuzzified value is increasing (Figure 13) which is counter intuitive. If during the time interval 4.85 to 5, the ‘max valueM’ had increased equivalently to ‘max valueH’ or ‘max valueL’ had not remained constant at zero, then the overall impact would have resulted in lower values of defuzzified values based on the COA calculation which represents product attractiveness.

This observation indicates that the counter intuitive results can be due to the inconsistent definition of the 9 rules and not the shortcoming of the defuzzification method per se. In other words the increase in the resulting value of rules which have high membership values, should be accompanied with a proportional decrease in the resulting values of the rules that have low or medium membership value and vice versa.

6.0 Modification of Fuzzy operators

As we observed in section 5, the LOM method has counterintuitive increases in some regions which are significantly higher compared to the COA method. In the COA method, the defuzzified value decreases more smoothly but still shows unreasonable behavior in some regions. One of the reasons for the discontinuity and unreasonable behavior observed in the plots is due to the Mamdani Max-Min method depicted in Figure 5. Every time the minimum of the two perception variables is found, a discontinuity will result. Also every time that the maximum value between all rules is calculated, the final defuzzified value will be affected by another discontinuity.

In order to remove these discontinuities, we suggest the modification of the Max-Min method. In the first stage, instead of using the minimum among the membership values associated with the perceptions, the result of each rule is found by averaging the membership values associated with the two perception variables. Then at the second stage, instead of finding the maximum among rules for each domain value, the average membership value of the rules associated with each domain (i.e. low, medium and high) is calculated. Thus both modifications serve to smooth out the observed discontinuities. We then execute both de-fuzzification methods after each stage. It is apparent that by applying the average-average operations instead of the initial Max-Min operations, the final defuzzified values for both defuzzification methods have fewer discontinuities.

Furthermore, as was explained in Section 5, another reason for the unreasonable behavior observed in the defuzzified values is related to inconsistencies found among the rules. Inconsistency of fuzzy rules is defined as when some rules are conflicting. Meesad and Yen (2003) describe that “fuzzy rules are conflicting if they have similar antecedents but rather different consequents” (p. 450). Likewise, in order to examine whether the unreasonable behavior of the model is affected by rule inconsistency or not, we propose to change Rule 4, to have a ‘Low’ result instead of ‘Medium’ result, to make it consistent with Rule 2, which converts a Low and Medium premise into a ‘Low’ membership value result (See Table 2).

Table 3-Modified Consistent Fuzzy Rules

	<u>Perceived Customer Service</u>	<u>Perceived Delivery Timeliness</u>	<u>Product Attractiveness</u>
1	Low	Low	Low

2	Low	Medium	Low
3	Low	High	Medium
4	Medium	Low	Low
5	Medium	Medium	Medium
6	Medium	High	High
7	High	Low	Medium
8	High	Medium	High
9	High	High	High

7.0 Conclusion and Future Research

In this paper, fuzzy logic is applied in the System Dynamics framework to enhance the ability of System Dynamic modelers to incorporate the vagueness associated with linguistic variables. Two different defuzzification methods, i.e., the ‘Largest of Maximum (LOM)’ and ‘Center of Area (COA)’ were applied. The LOM defuzzification method is easier to compute and incorporate in a VENSIM model. On the other hand, based on the required precision for the final value, the COA defuzzification method requires the calculation of a vector consisting of hundreds up to thousands of elements, for every time step of the simulation. Therefore, the run time is significantly increased.

As discussed in the paper, the utilization of defuzzification methods in finding crisp values for the fuzzy set in a dynamic framework leads to some counterintuitive results. However, based on our study, we can conclude that compared to LOM, the COA defuzzification method is more reasonable in representing the actual real world conditions, due to the fact that it averages the membership values of the entire domain range. However, as a consequence of the defuzzification process which leads to the reduction of the representational dimensionality of the fuzzy region, a higher loss of information is concluded by the COA method. For instance even in the most favorable situation which is the point (Normalized Total Service Hours, Normalized Timeliness) = (1, 1), the value of COA defuzzified is 0.611; while the LOM defuzzified value is 1

The approach chosen to define the fuzzy rules for the inference method does not affect the outcome of LOM or COA defuzzification methods, unless the rules are inconsistently defined which leads to unreasonable results. We suggest modifying the inference method from using Max-Min operations to Average-Average and also making the defined fuzzy rules set consistent, so that the defuzzified values behave reasonably for both defuzzification methods.

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