Self-Organizing Market Structures, System Dynamics, and Urn Theory

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Abstract

This article argues that the tools of system dynamics and urn theory can be used to model self-organizaning markets. A fundamental characteristic of self-organizing markets is that size of firms by rank order follows the Zipf distributions. While complex industrial structures of this kind are hard to describe with conventional theories, system dynamics and urn theory are equipped with adequate tools to deal with this kind of evolutionary phenomena.

1. Introduction

It has been shown that the distribution of sizes of many enormously complex physical, biological, and socioeconomic phenomena can be well described by a very simple *power law*: the number of objects whose size exceeds S is proportional to s^{-a}, where the integer *a* usually is a round number, like 1 or 2. Among the most spectacular examples of a power law is one that involves economics: the size distribution of firms. I this paper, I argue that the tools of system dynamics and urn theory can provide a more complete picture of the economics of the growth of the firm and stronger and more general conclusions about the evolution of self-organizing market structures. This paper has three additional sections. In section two, I discuss the self-organizing nature of firms' size. In section three I analyze how the tools of system dynamics can be used to better understand the sources of increasing returns to the growth of the firm. In the last section, I develop some ideas about the convenience of using urn theory to formalize mathematically self-organizing complex systems.

2. Self-Organization of Industries

Self-organizing systems, systems that start from an almost homogeneous or almost random state, spontaneously form large-scale patterns. Initially, these systems show imperceptible differences, but over time those small differences become magnified through a process of self-reinforcement. One of the most evident attributes of firms is that their size distribution exhibits properties of *self-organizing systems*. This implies that firm sizes in modern industrial countries are highly skew, such that a very small number of large firms coexist with a very large number of smaller firms. The interest in the distribution of company sizes started with Zipf (1949), who established that USA corporation assets approximately follow the law

$$s_r = 1/r \tag{1}$$

where s_r is the size of the company ranked *r* in a list of firms ordered by asset size, beginning with the largest. The same law has been found to describe the distribution of words in a variety of languages, s_r being then the number of occurrences of the *r*th word in a list ordered by number, beginning with the most frequent (Zipf, 1932). Empirical studies have found that Zipf's law describes phenomena in various fields, including cities (Gabaix and Ioannides, 2003), immune system response (Burgos and Moreno-Tovar 1996; Li 2001), and aspects of Internet traffic (Breslau et al. 2000).

To visualize how the distribution of firm sizes follows Zipf's law, we take the firms of a country and order them by size[1]. We then draw a graph; on the y-axis we place the log of the rank, *r*, and on the x-axis the log of the size of the corresponding firms

[1] Size can be measured in number of ways, and these arguments have variously applied to measure of annual sales, current employment, and total assets. Though we might in principle expect systematic differences between the several measures, such differences have not been a focus of interest in the literature. An interesting property of firm size distributions noted in the studies of large firms is that qualitative character of such distributions is independent of how size is defined.

($s_1 \succ s_2 \succ ... s_N$)). When we draw log-rank against log-size, we get a straight line, with a slope that is very close to -1. Furthermore, if we run the regression

$$\ln r = \beta_1 - \beta_2 \ln s_r + \varepsilon_1 \quad (2),$$

an expression of Zipf's law is that the slope of this regression line (β_2) is very close to -1. In terms of the distribution, this means that the probability that the size of a firm is greater than some *S* is proportional to $1/S$:

$$P(Size \succ S) = 1/S^{\beta_2} \quad (3),$$

with $\beta_2 = 1$. However, Mandelbrot (1954, 1954) established that Zipf's law was a special case of a more general relation, the so-called simplified canonical law (scl)[2]:

$$s_r = P(r + \rho)^{-1/\beta_2} \quad (4)$$

when $\rho = 0$ and $\beta_2 = 1$.

Recently, Ramsden and Kiss-Haypál (2000) found that the equation (4) fits the data for the different countries they studied. Specifically, their analysis of the data on the largest 500 U.S. firms gives a β_2 close to 1.25. For other countries, β_2 ranges from 0.44 for South Africa and 0.65 for Netherlands to 1.4 for Hungary and 1.2 for China. In contrast with what Ramsden and Kiss-Haypál (2000) found, Axtell (2001), using data on the entire population of tax-paying firms in the United States, shows that the Zipf distribution characterized firm size: the probability a firm is larger than size *s* is inversely proportional to *s*. These results hold for data from multiple years and for various definitions of firm size. Specifically, Axtell (2001) proves that data from USA Census including firms with 1 employee are approximately Zipf-distributed ($\beta_2 = 1.059$), as determined by ordinary least squares (OLS) regression in log-log coordinates. But firms having a single employee are not the smallest economic entity in the United States economy. Although there were approximately 5.5 million firms that had at least one employee during 1977, there were another 15.4 million entities in that year with no employees. These are predominately individuals and partnerships, and are called "nonemployees" firms by USA Census. These firms account for nearly $600 billion in receipts in 1977. If these firms are included in the overall firm size distribution, the Zipf distribution still fits the data well. Here, OLS yields an estimate of $\beta_2 = 1.098$ (SE = 0.064) and the adjusted R^2 = 0.977. Furthermore, Riemer et al (2002), using data from 70 markets, found that the *market shares* by rank order follow the Zipf distribution.

These empirical studies, therefore, have shown that there is no reason to expect the size distribution of firms to take any particular form for the general run of *countries*. Empirical investigations from the 1960s onward have also thrown doubt on whether any single form of size distribution can be regarded as "usual" or "typical" for the general

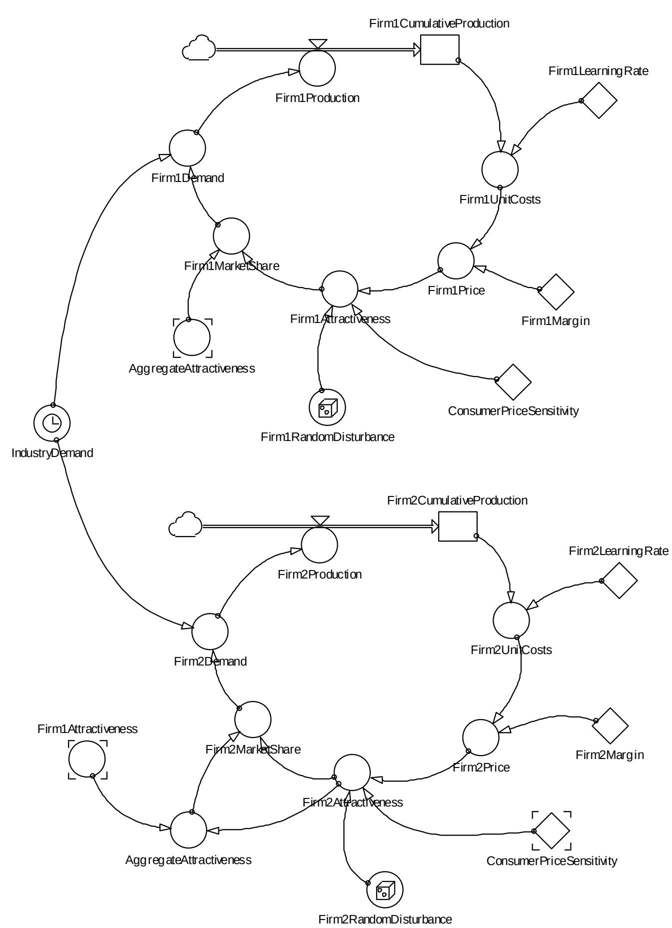
[2] *P*, ρ and β_2 are parameters of the distribution.

spectrum of *industries*[3]. Independently of the fact that market structure varied in systematic way from one country to another and from one industry to another, an interesting property of the distribution of firm sizes is its high level of skewness, which constitutes a clear-cut target that any accurate theory of the firm must try to hit. The following section addresses

3. System Dynamics, Increasing Returns, and the Growth of the Firm

System dynamics was originally developed to help corporate managers better understand and control industrial systems (cf. Sterman, 2000). Later, system dynamics is used to address problems related to systems that change over time, be they physical, biological, or socioeconomic systems. More recently, Radzicki (2003) argue that system dynamics computer simulation modeling can be useful to describe evolutionary economic processes. An example of a system dynamics model that exhibits evolutionary behavior is the competition for market share between firms which are subject to a significant learning curve. Figure 1 shows the system dynamics stock-flow diagram for the learning curve model.

Figura 1. Learning Curve Model



[3] See Schmalensee (1989).

However, the growth of the firm and the emergence and evolution of industrial dominance is not the result of a simple cause-effect relationship between two or three variables, no matter how important these variables may be. There is abundant evidence suggesting that the firm's growth and industrial concentration take place from the complex network of relations that result from the mutual causality between numerous variables. As with other dynamic systems, both the growth of the firm and the evolution of its industrial structure are subject to both negative and positive feedbacks. Negative feedbacks stem from *decreasing returns to the growth of the firm*. These diseconomies —reductions of benefits due to scale diseconomies— may occur because the firm becomes "bureaucratically" congested or administratively limited. Decreasing returns to the growth of the firm are stabilization forces that hinder the growth of the firm and prevent the eventual emergence of an infinite-size firm. The growth of the firm and the concentration of the industry where it competes depend to a great extent on positive feedbacks; that is, from *increasing returns to the growth of the firm*. Concerning firm size and market structure, many interactions between variables may cause this form of increasing returns, but from consistent findings in the literature, we can identify the following as the most important:

Scale economies. With his seminal article *The Economies of Scale*, Stigler (1958) laid the foundations of an increasing-returns based theory of how the firm grows. His argument is that the more rapid the rate to which a firm loses its share of the industry's output (or capacity) the higher is its private cost of production relative to the cost of production of firms of the most efficient size.

Scope and integration economies. Chandler's (1966, 1977, and 1990) historical approach. Chandler's main intellectual contribution was to recognize that, in order to achieve the lower unit costs, firms had to do a lot more than simply build large plants. They had to be able to maintain a high rate of throughput through their factories —that is, to keep their plants operating consistently at high levels of capacity utilization. In order to maintain a high rate of throughput, firms had to insure that shortfalls in supply did not disrupt their production processes and that output did not pile up in their warehouses unsold. The solution, as Chandler saw it, was for firms to bring these activities under their direct control by integrating backward into raw-material production and forward into distribution, and by building a managerial hierarchy capable of coordinating smoothly the flow of inputs and outputs from raw material to final sale. Therefore, through his historical theory of large business, Chandler has provided empirical evidence of the existence of what theoretically can be called *economies of integration*. Nevertheless, large firms could exploit not only economies of scale and economies of integration, but also economies of scope. According to Chandler, large firms can reap economies of scope by investing large quantities of financial resources in research and development, which allows them diversify their operations into other industries. Chandler claimed that firms that reaped scale economies, integration economies and scope economies improved upon the workings of the market, captured the resulting efficiency gains, obtained enormous competitive advantages, and over time brought under their managerial authority larger and larger portions of the economy. The only firms that could compete with them head to head, he argued, were those that completely duplicated their vertically integrated structures and managerial hierarchies. Because relatively few firms could raise the enormous amounts of capital required, these kinds of industries quickly took on oligopolistic structures.

Expansion economies. Buendía (2006) develops a model that shows how and under what circumstances firms can realize expansion economies. In order to exploit economies of expansion while augmenting benefits, manufacturing firms tend to expand their economic activities to different locations, regions and countries. The level to which firms spread out in a region depends on the local demand. When such local demand is completely satisfied by the firm, it may decide to star business activities in other region. Firms' expansion process is limited by the size of the market and the world demand. The growth of the firm through expansion economies depends on the scale of the different elements that form the basic business unit. Expansion economies are a special case of increasing returns that cause dominant firm highly concentrated industrial structures.

Schumpeterian learning. The most widely accepted theory of technological change among neoclassical economists is Schumpeter's (1949). In a Schumpeterian world, scale economies are present as well, but technology is not a constant. Here the creative role of the entrepreneurs allows for the introduction of new technologies capable to displacing the establish ones. Most of Schumpeter's discussion stresses the advantages of concentrated market structures involving large firms with considerable market share. According to this economist, it is more probable that the necessary scale economies in R&D to develop new technologies be achieved by a monopolist or by the few large firms of a concentrated industry. Large size firms, besides, may increase their rate of innovation by reducing the speed at which their transient rents and entrepreneurial advantage are eroded away by imitators. In the absence of patent protection large firms may exploit their innovations on a large scale over relatively short periods of time —and in this way avoid rapid imitation by competitors— by deploying their productive, marketing and financial capabilities. Large firms may also expand their rate of innovation by imitating and commercializing other firms' technologies.

Costs Reducing Learning. An important aspect of technological change is costs reducing in nature. Henderson (1975), in the strategic field, pioneered the notion of experience curve as a source of costs reductions. In economics, Hirsch (1956) has underlined the importance of repetitive manufacturing operations as a way of reducing direct labor requirements, while Arrow (1962) has explored the consequences of learning-by-doing (measured by the cumulative gross investment, which produces a steady rate of growth in productivity) on profits, investment, and economic growth. However, the historical study on the pattern of growth and competitiveness of large corporations of Alfred D. Chandler (1990) is a major and detailed contribution to our understanding of the way firm grow by diminishing costs.

4. System Dynamics, the Growth of the Firm, and Urn Theory[4]

Urn theory or Polya processes (Arthur 1994, Arthur et al. 1987, Dosi and Kaniovski 1993) have been considered an important analytical tool to model dynamic economic systems. To understand the relevance of this analytical tool, we can start with the simple model, where a new technology is adopted in each period of time and randomly chosen from two different formats. This technological adoption process can take two different paths depending on whether there are increasing returns or not. If *decreasing returns* are present then the

[4] This section draws heavily from Buendia and Eccius (2010)

process of adopting a technology depends on the probability *i*. We can start by assuming that both technologies have a probability of 0.5 to be adopted per period of time. The long-term behavior of a model with decreasing returns is clear: As random adoptions at each period are independent from one another, the law of large numbers applies, so each technology's share of adoptions has to converge toward a constant assignment probability for this technology; that is to say, $I = 0.5$. If the random process is repeated an infinite number of times, the process will fluctuate in its early phase, but it will always converge toward the long-term share of $I = 0.5$. The fluctuations in the early part of the process result from the fact that the addition of one adoption has a larger impact on the share with a small total number of adoptions than in a technology with a greater number of adoptions. However these fluctuations disappear over time. This corresponds to the typical growth process of the traditional neoclassical growth theory.

When there are *increasing returns*, random assignments at each period are dependent both on one another and on the accumulated numbers of adoptions, so that technology's share converges, toward a different value in the long-term. From a mathematical point of view this process can be described by assuming that the assignment probability at a certain point in time is equal to the shares at that time, which is known as a Polya process. Under this condition the process of adopting a new technology converges to a stable set of proportions in the long run. But although this proportion settles down and eventually becomes constant, it does so to a constant vector that is selected randomly from a uniform distribution over all possible shares that sum to 1.0. As the process will settle down to a certain distribution and then remain constant over time, each possible outcome is equally likely. In other words, we know that this process will produce a stable spatial structure but we do not know a priori what this structure will be. As in the model without increasing returns, there are strong fluctuations early on. A technology that is more adopted early on in the process because of luck will end up with a higher market share in the long run, while the technology that is less adopted on will end with a lower market share.

In this paper we analyze Arthur (1994), Arthur et al. (1987) and we show that the condition that they establish to obtain a monopoly outcome is artificial and ad hoc. Parting from this fact we determine the conditions that are necessary to introduce into the conventional Pólya scheme, to produce tipping results. Specifically we establish the condition of strong network externalities that have to be fulfilled to obtain strict monopoly.

Many social and economic phenomena have the fundamental feature of self-organizing systems. This kind of systems has positive and negative feedbacks and their outcomes follow a skewed distribution. The specific conditions and results of dynamic systems stem from positive feedback, which generates multiple equilibria and the equilibrium when is reached depends on initial conditions as well as transitory incidents and small historical happenings. In the economy and the society there are many kinds of positive feedbacks that have different kinds of consequences. Network externalities are a well known reason for positive feedback, which may make the inferior standard emerge as the dominant technology. One of the most often cited account of tipping and lock-in in the inefficient technology dominating the market is the battle between QWERTY and DVORAK keyboard format. Another classical example is the rivalry between VHS and Betamax among video-recorder formats.

Arthur (1994), Arthur et al. (1987) developed a model of technology adoption where consumers choose among many incompatible technologies. A consumer picks up a technology according to the benefits they obtain from the technology that have been chosen by previous adopters. Arthur's model is based on a Polya-process and explains lock-in and tipping in terms of non-linear feedback driving technology adoption. Under these conditions, some threshold market share and future adoption rate due to interdependencies between consumer preferences, the system will be stable only with monopoly; that is to say, one of the shares will converge to 100% market share and the other to 0%.

Polya's original urn process provides a simple explanation of reinforced random processes, which has the sufficient structure to support outcomes other than monopoly. Stable patterns of market sharing with one dominant firm with a large market share and a number of small firms with small market share are most common outcome than strict monopoly even among markets characterized by network externalities. In fact the original Polya scheme explains a range of equilibrium outcomes other than monopoly. Consequently, this original scheme can describe a variety of increasing returns situations, where feedbacks can be positive but the different levels of strength. While very strong positive feedback leads to market dominance by one of the competing technologies, in other cases, competitors may share the market. There is thus a trade-off between market share and feedback strength. Evidently, the conventional Polya scheme supports this kind of results. This process is based on an urn with balls of two colors (white and black), with a sampling and replacement policy which obeys the following rule: draw a ball from the urn, observe its color, return it to the urn (sampling with replacement) along with $S > 0$ balls of the same color. History dependence of the process is an inherent characteristic of the evolution of the distribution of the proportion of balls of different colors. For instance, consider an urn containing balls of two colors, say white and black, which represent, for example, two firms competing for market share. The initial number of balls of different color represents the initial sizes of market share of each firm. The sampling and replacement process may be as follows: from an urn containing n_1 white balls and n_2 black balls, a ball is drawn, and its color noted and the ball is returned to the urn along with additional ball(s) depending on the label of the color of the ball drawn. If a ball labeled i (i = 1,2) is drawn, a_{ij} balls labeled j (j= 1,2) are added. We can generalize an addition of balls with a matrix of integers $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where the rows are indexed by the color selected and its columns are indexed by the balls added. In its simplest form this matrix of integers may take the following values: $a_{11} = a_{22} = 1;$ that is $S = 1$; $a_{21} = a_{12} = 0$ and one white and one black ball in the urn: $n_1 = n_2 = 1$. Under these conditions the random variable x_t converge almost surely to a limit X. When $n_1 = n_2 = S = 1$, the limit variable X is uniform on the interval [0,1].

Arthur (1994) models the patterns of evolution for the adoption of two competing technologies in a market[5], using a basic *urn scheme* with white and black balls, with each color corresponding to a competing product. At its initial state, the urn contains n_w white balls and n_b black balls, and a ball is added at subsequent time instances t = 1, 2, 3, …. The *probability* of this ball being white is given by $f_t(x_t)$, and the probability for a black ball is

[5] Obviously, the model can be applied to the assignment of firms to two different regions, the allocation of innovation to two firms, and other processes.

$1 - f_t(x_t)$, with the *random variable* x_t standing for the proportion of white balls in the urn at time *t*. The dynamics followed by the number of white balls w_t depends on a random binary variable $\xi_t(X_t)$, which is independent of time and takes on values from the sub-set of integer numbers: {1 with *probability* $f_t(x_t)$, 0 with probability 1 -$f_t(x_t)$}. This dynamics is modeled by

$$w_{t+1} = w_t + \xi_t(X_t) \tag{5}$$

where it is established that the number of white balls at each state remains the same (with probability 1-$f_t(x_t)$) or it is incremented by one (with probability $f_t(x_t)$): $w_{t+1} = w_t$ or $w_{t+1} = w_t + 1$. The dynamics that rules the total number of balls γ_t in the urn at time *t* is given by

$$\gamma_{t+1} = n_w + n_b + t\,, \tag{6}$$

and it is incremented by one at each time.

The proportion of white balls X_{t+1} in the urn at time *t+1* is obtained by dividing the number of white balls w_{t+1} by the total number of balls γ_{t+1},

$$X_{t+1} = \frac{w_t + \xi_t(X_t)}{n_w + n_b + t} = \frac{(w_t + \xi_t(X_t))(n_w + n_b + t - 1)}{(n_w + n_b + t)(n_w + n_b + t - 1)} = \frac{(n_w + n_b + t)\, w_t - w_t + (n_w + n_b + t - 1)\, \xi_t(X_t)}{(n_w + n_b + t)(n_w + n_b + t - 1)}\,. \tag{7}$$

In order to have the current value of X_{t+1} expressed in terms of its previous value X_t plus an increment ΔX_t, some algebraic manipulations are performed,

$$X_{t+1} = \frac{(n_w + n_b + t)\, w_t}{(n_w + n_b + t)(n_w + n_b + t - 1)} + \frac{(n_w + n_b + t - 1)\, \xi_t(X_t) - w_t}{(n_w + n_b + t)(n_w + n_b + t - 1)} \quad (8),$$

$$X_{t+1} = \frac{w_t}{n_w + n_b + t - 1} + \frac{\dfrac{(n_w + n_b + t - 1)\, \xi_t(X_t)}{n_w + n_b + t - 1} - \dfrac{w_t}{n_w + n_b + t - 1}}{(n_w + n_b + t)} \quad (9).$$

Since $X_t = \dfrac{w_t}{n_w + n_b + t - 1}$, then $X_{t+1} = X_t + \dfrac{\xi_t(X_t) - X_t}{(n_w + n_b + t)}$. The *expected value* for the increment in X_t is given by the relation

$$E\left\{\frac{\xi_t(X_t) - X_t}{(n_w + n_b + t)}\right\} = \frac{[1 \cdot f_t(X_t) + 0 \cdot \{1 - f_t(X_t)\}] - X_t}{(n_w + n_b + t)} = \frac{f_t(X_t) - X_t}{(n_w + n_b + t)} \quad (10).$$

Eventually the fluctuations in ΔX_t diminish to zero, and X_t reaches a *steady state*, so that $f_t(X_t) - X_t = 0$. It is said that X_t converges to the roots of $f_t(X_t) - X_t = 0$ as t → ∞ with zero or positive probability; and for an isolated root Φ, the fastness of the convergence of $f_t(X_t) - X_t = 0$ in a neighborhood around Φ, depends on the smoothness of $f_t(X_t)$ at Φ. Other useful way of describing the previous properties of this urn scheme is by defining a function $f(X_t)$ such, that $f_t(X_t) = f(X_t) + \delta_t(X_t)$ and in the limit t → ∞, $\delta_t(X_t)$ approaches 0, and $f_t(X_t)$ approaches $f(X_t)$. This simple urn scheme displays *positive feedback* and, two patterns of evolution reaching a steady state. The behavior of X_t over time describes trajectories with random walks, approaching a limit that can take on any value from the sub-set of real numbers [0, 1].

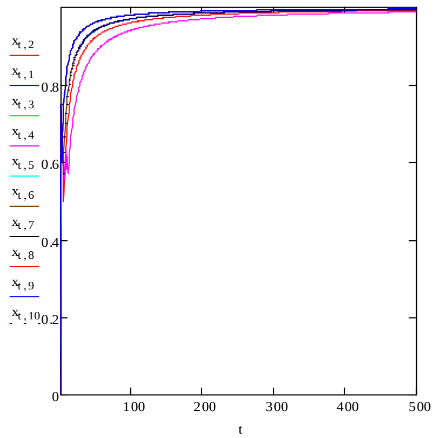
Let us consider a market with two competing technologies *A* (with $n_A \geq 1$ units) and *B* (with $n_B \geq 1$ units) such that a new consumer enters the market at time *t* = 1, 2, …, n. The pattern of the evolution of adoption of both technologies in the market is clearly modeled by the previous urn scheme, where the function $f(X_t)$ is constructed according to the decision rule that the new consumer uses to make his choice. As an example, Arthur *et al.* (1987) considered the following basic rule:

A new consumer asks an odd number *p* of users which technology they bought, and if at least $\frac{p+1}{2}$ of them use *A*, he will choose *A*, otherwise *B*. The function $f(X_t)$ that represents the probability of the new consumer choosing *A*, depends on the current proportion X_t of product *A* in the market,

$$f\ (x_t) = \sum_{pm=\frac{p+1}{2}}^{p} \frac{p!}{(p-pm)!pm!} {x_t}^{pm}(1-x_t)^{p-pm} \tag{11}$$

We are interested in the solution of $f\ (X_t) - X_t = 0$ on $X_t \in [0,1]$. There are three roots on the sub-set of real numbers [0, 1]: 0, $\frac{1}{2}$, and 1; however, there is no possible market structure corresponding to the root $X_t = \frac{1}{2}$, i. e., X_t converges to this root with zero probability as $t \rightarrow \infty$. In the other hand, roots 0 and 1 correspond to possible market structures, i.e. X_t converges to each of them with positive probability. $X_t \rightarrow 1$ corresponds to the proportion for *A* to dominate if the initial number of units n_A of the technology *A* is greater than the initial number of units of technology *B*. If we run a simulation with these conditions and applying formula 11, we can find out that different outcomes will result, depending on n_A, $n_{B,}$ and p. Let us start by considering a situation which $n_A = 2$, $n_B = 1$ and p = 21. Then the results of the simulations are as shown in Figure 2.

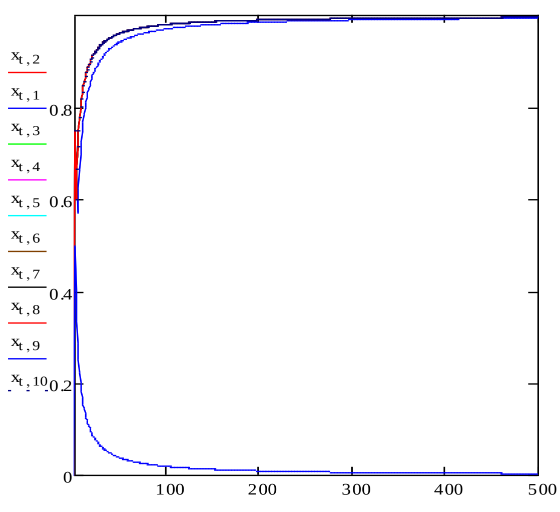
Figure. 2 Arthur's scheme $n_A = 2, n_B = 1$ and p = 21



As we can see in this figure, dominance by technology A ($x_t \to 1$) takes place with high probability. Therefore there are few chances that technology B monopolizes the market.

This means that in Arthurs configuration there is an alarming bias towards technology A to dominate the market. In fact, the occurrence of dominance by technology B has very low probability. Figure 3 shows a simulation of 10 realizations, in which B dominates the market. But this result occurs very few times in many simulations.

Figure. 3 Atypical Arthur's scheme$n_A = 2, n_B = 1$ and p = 21



This result is due to the fact, that when$n_A = 2$, $n_B = 1$ and p = 21, the $f(x_t) =$ 0.944, and there is a low probability (but it exists), that the result is 0/100, with technology B tipping the market.

$$f\ (x_t) = \sum_{pm=\frac{p+1}{2}}^{p} \frac{p!}{(p-pm)!pm!} x_t{}^{pm} (1-x_t)^{p-pm} = 0.944 \tag{12}$$

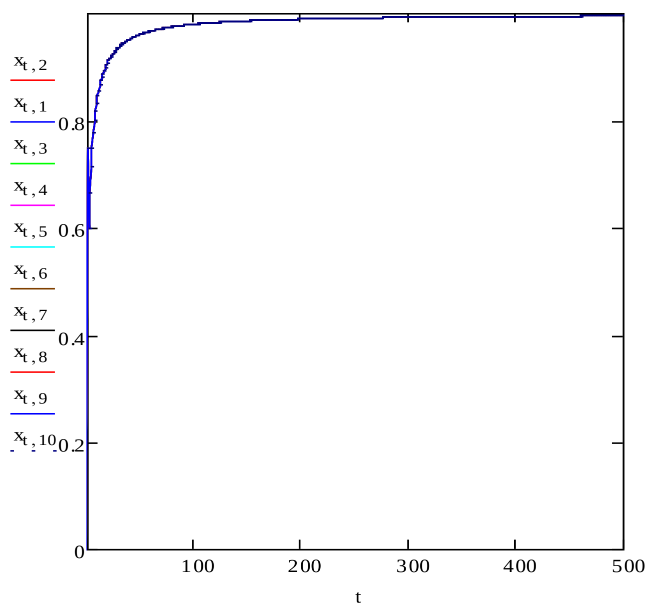
We run 500 iterations, and we get 16 out of these 500 iterations, which confirm what we said above concerning the fact that with $n_A = 2$, $n_B = 1$ and p = 21, it is practically impossible for technology B to tip the market. Obviously, there exist the possibility for technology B to monopolize the market, but the probability is very low. In fact, in our exercise of running 500 simulations, the probability of this to happen is about 3%.

If p is increased, for example, p = 101,

$$f\ (x_t) = \sum_{pm=\frac{p+1}{2}}^{p} \frac{p!}{(p-pm)!pm!} x_t{}^{pm} (1-x_t)^{p-pm} \to 1 \tag{13}$$

Obviously this process is very restrictive because there is a tendency within the process to choose always technology A. Here there is no possibility for technology B to tip the market.

Figure. 4 Arthur's Scheme $n_A = 2, n_B = 1$ and p = 101

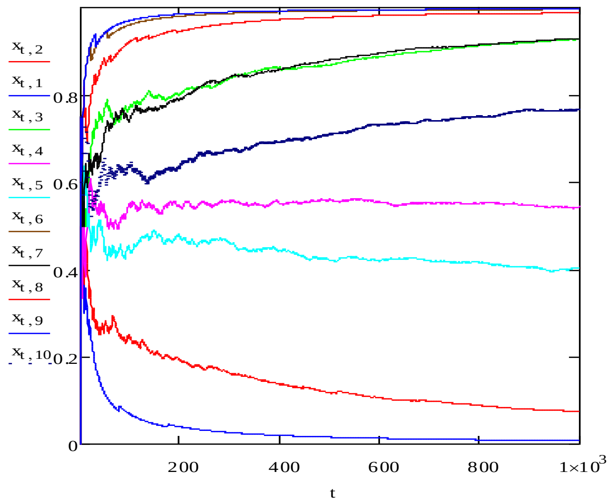


Now let us consider the case where$n_A = 2$, $n_B = 1$ and p = 3. In this situation the results are as shown in Figure 5. As we can see in this figure, with a low value of p, then what Arthur would foresee cannot take place, and the process becomes a kind of conventional Polya-process, so the roots x_t could be of the sub-set of the real numbers [0,1].

In this case, therefore, when p = 3, then:

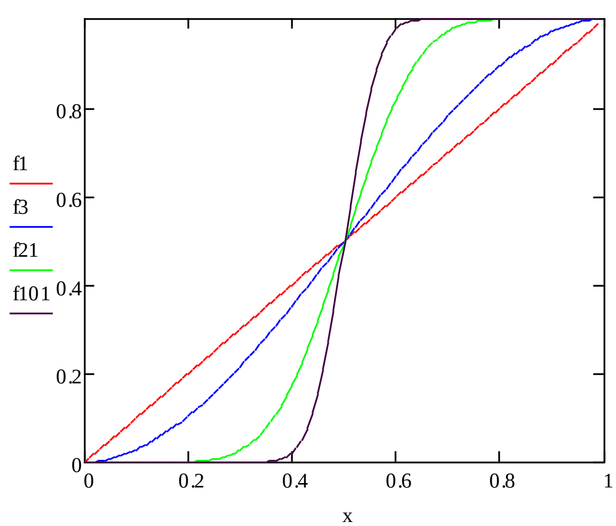
$$f\ (x_t) = \sum_{pm=\frac{p+1}{2}}^{p} \frac{p!}{(p-pm)!pm!} {x_t}^{pm} (1-x_t)^{p-pm} = 0.741 \qquad (14)$$

Figure. 5 Arthur's Scheme $n_A = 2, n_B = 1$ and p = 3



What we just said, can be visualized by relating x_t to $f(x_t)$, where x_t is the proportion of technology A and $f(x_t)$ is the probability of adding a white ball, which represents a new adoption of technology A by a new consumer. If we establish a relationship between $f(x_t)$, with different values of p (1, 3, 21, and 101), we obtain what Figure 6 shows. Specifically, with larger values of p, such as 21 and 101, we obtain probability functions $f(x_t)$, which tend rapidly to radical results such as 100/0. When p = 101, and $x_t > 0.5$, then the probability of adding a new black ball (technology B) is practically cero. By the contrary, when p = 1, the process becomes a conventional Polya-

Figure. 6 Relationship between f(x_t) and x_t in Arthur's Scheme



process. This confirms in some way what we stated before.

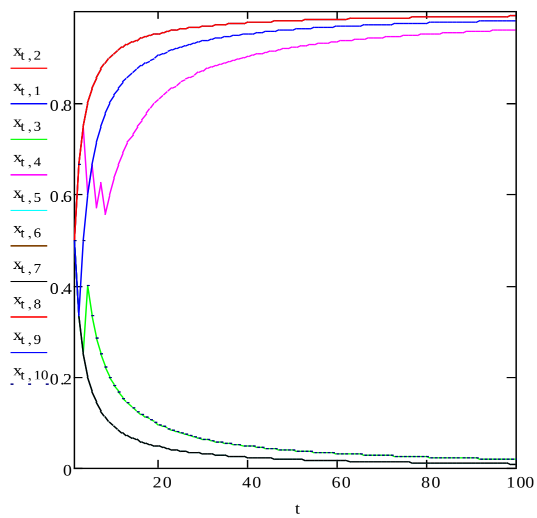
Here the problem then is to try to assure that the Polya process will end up with a complete dominance of one of the technologies. Usually tipping markets are associated to *strong* network externalities. In order to model network externalities, suitable for a Polya process, we consider the following function.

$$f(x_t) = \frac{w_t!}{w_t! + b_t!} \qquad (15)$$

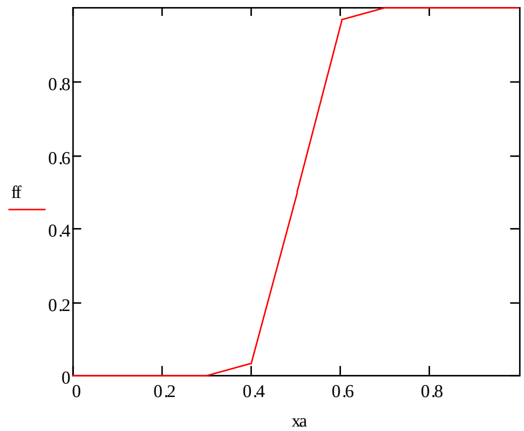
Function 15 describes the kind of situation where a consumer who has to choose between two technologies (A and B), and has to decide based on what he has observed previous consumers have done. This is very important, because the benefits of his decision depend on the size of the network he is going to be connected to. Let us assume that tree consumers have already adopted technology A. This implies that in that network there are 6 potential connections. If a new consumer joins this network, then the number of potential connections will be 24. So we can express this process as a factorial: n! Fluctuations in ΔX_t diminish to zero, and X_t reaches a *steady state*, so that $f_t(X_t) - X_t = 0$

With this we obtain Figure. 7, when $n_A = 1$ and $n_B = 1$, the steady state could be a market share of 100/0 or 0/100. This depends on the early random movements. We have to underline, though, that function 8 does not correspond to the proportion of individuals, but to the number of connections.

Figure. 7 Network externalities $n_A = 1, n_B = 1$



The plot of the relationship of x_t and $f(x_t)$, when we apply equation 8, is as shown in Figure. 8. As we can see the process takes up high values of $f(x_t)$, when $x_t > 0.5$. By the contrary, when $x_t < 0.5$, then $f(x_t)$ tends to cero.


0.8
0.6
ff
0.4
0.2
0
0
0.2
0.4
0.6
0.8
xa

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