

Butterflies and buffers

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**Abstract:** This paper presents a brief discourse on chaotic behaviour and provides an implementation of a classical example from existing chaos theory; the Lorenz strange attractor. The implementation is carried out using standard System Dynamics software and techniques. Chaotic behaviour is differentiated from random behaviour. The paper then goes on to describe a classical buffered feedback system; the Ktesibios clock. The implementation of this feedback system is again carried out using standard System Dynamics software and techniques. Both systems are then amalgamated to unite the butterfly effect of the Lorenz attractor with the buffered nature of the water clock. It is postulated that the resultant taming of the chaotic behaviour generated by the Lorenz attractor through the feedback buffer of the Ktesibios machine is common to many systems; brief examples are given. It is concluded that, in some cases at least, it is the overwhelming of the buffer that leads to a tipping point returning the whole system to a state of chaotic behaviour. Implementing a model of a natural buffered system with chaotic input is identified as an area for further work.

**Keywords:** Chaos, System Dynamics, Systems Thinking, Lorenz, Attractor, Ktesibios, Simulation, Butterfly effect

## **Introduction and background**

Chaos as an area of academic enquiry has been with us for some time now. A French applied mathematician, Jules Henri Poincare 1854-1912, came across complex interactions in celestial mechanics that were a direct precursor to theories of chaotic behaviour if not a direct description of it.

In more recent times contributions by both Edward Lorenz and Robert May established, with some resistance, the ideas of chaos and chaotic dynamics as part of mainstream academic thought.

The System Dynamics Review devoted a special issue to Chaos, also expressed as non linearity, in dynamic systems (Volume 4, Number 1-2, 1988). Recognition of complex non-linear behaviour and transforming it into a more uniform dynamic system has been a problem that has been tackled by humans since ancient times and by nature since natural systems existed.

The references to Chaos made in the System Dynamics literature are often mirrored in the literature on Chaos theory. Consider the following abridged quote “populations of plants and animals ...tend to increase after dropping to unusually low densities (at which point, the conditions become most suitable for maximum growth) and, after reaching unusually high densities, they tend to decrease again” (Robert May, in (Hall, 1992, pp 82)). Describes one, high level, way of interpreting System Dynamics models such as are found in predator-prey simulations.

### **Chaos is not chaotic**

Various routes have been followed in the development of a theory of Chaos but where exactly did they lead? At this point it might be appropriate to introduce some accepted definitions of Chaos and chaotic behaviour. From the System Dynamics literature, Sterman describes chaos as an oscillatory pattern that “does not have a regular amplitude, periodicity, or shape, even though the environment is completely constant and the system is completely free of random shocks” (Sterman, 2000, p133). A further definition is provided on the website of the Society for Chaos Theory in Psychology & Life Sciences:

“‘Chaos Theory’ is one of a set of approaches for studying nonlinear phenomena. Specifically, chaos is a phenomenon that appears locally unpredictable may indeed be globally stable, exhibit clear boundaries, and display sensitivity to initial conditions. Small differences in initial states eventually compound to produce markedly different end states later on in time.”

This latter appears to be as good a definition as any that might be produced and it is a definition that exists outside the world of System Dynamics. However, as with most ‘summary’ definitions it does not necessarily result in an actual definition of the chaos that Chaos theory addresses. Reading the definitions above one might be left with the expectation that chaotic behaviour would be random or near random and be pattern free or so complex in its structure that it was not amenable to analysis. This is not so and this will be demonstrated in the remainder of this paper. It could also be that chaos is simply the opposite of order; which leads directly to the need to define order.

Whilst the preceding attempt at defining chaos may itself be viewed as chaotic, or even somewhat light hearted, it does provoke an important concept. That chaos is insufficiently defined or perhaps a misnomer and any definition of chaos is subjective and open to interpretation.

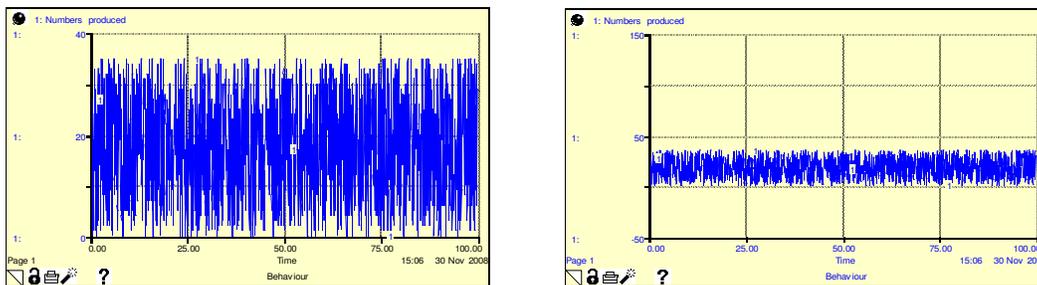
### Chaotic behaviour

In attempting to address chaotic behaviour we run into exactly the same issues that were encountered in defining chaos and Chaos theory. Here chaotic behaviour is defined as:

The set of behaviours' that do not conform to preconceived notions of expected behaviour or are so complex in nature as to defy attempts to easily explain them in terms of expected output given the available knowledge about the input.

This definition is itself highly subjective and this is a recognised limitation of the language (English) used. It is also worth noting that as mathematics is a branch (subset or superset according to preference) of language it too fares no better when attempting to produce definitions (of anything) including a definition of chaos and chaotic behaviour (Hall, 1992, Chapter 16). Examples of chaotic behaviour in well known systems are provided in (Mosekilde and Larsen, 1988) who deal with the 'Beer game' and in (Allen, 1988) who deals with chaotic evolutionary systems.

Persevering with the definition of chaotic behaviour provided just above consider the time plots shown in figure 1 below.



**Figure 1a, b: Chaotic behaviour?**

The first figure, 1a, illustrates behaviour that appears to be random (chaotic?) in nature. In figure 1b the scale has been changed so that the output now appears in a greater context space and could, arguably, be interpreted as noisy linear behaviour. The data plotted in both examples is identical. It is only the context that has changed. The data is one representation of constrained chaotic or random behaviour. Each data point apparently bears no relation to the previous point but is constrained within a range of 0 to 36 and is further constrained to show only integer values. The behaviour shown is illustrated over 100 time periods in both cases.

The behaviours generated in figure 1 are the output from a single equation in a Stella v 9.1 model:

$$\text{Numbers\_produced} = \text{INT}(\text{RANDOM}(0,37,1))$$

It would be possible to define this equation as a model of the behaviour, in terms of the numbers selected, on a roulette wheel; which is (near) random though constrained. An

additional constraint is that the random function supplied by the software is not random. It produces a uniformly distributed sequence of numbers within its upper and lower bounds (0 and 37). As the random function operates on a uniform distribution and is pre-programmed with rules determined by its writer it is not possible that the sequence generated be truly random. Brief experimentation with this model and analysis of the results reveals that it is indeed not uniformly distributed. The model was analysed over 32002 iterations and was still found to produce a non-uniform sequence of results. Deriving random numbers from natural processes is theoretically possible using for example quantum processes (Hall, 1992, Chapter 15) but the random nature of such processes is yet to be determined and may never be so determined. All of which presents a hypothesis that Chaotic behaviours are only chaotic because the relationship between the input and output, though strictly known, has not been fully determined by the viewer. This could in turn be interpreted as a concrete example of bounded rationality as defined by (Simon, 1978).

A classical definition of a system containing chaotic behaviours and amongst the first to achieve widespread academic acclaim is that devised by Edward Lorenz as part of a weather forecasting simulation exercise. These are given a fuller treatment in (Mosekilde, 1988) but can be summed up as a set of three coupled non-linear differential equations which are reproduced below:

$$\begin{aligned} \frac{dx}{dt} &= \sigma x + \sigma y \\ \frac{dy}{dt} &= xy + \rho x - y \\ \frac{dz}{dt} &= xy - \beta z \end{aligned}$$

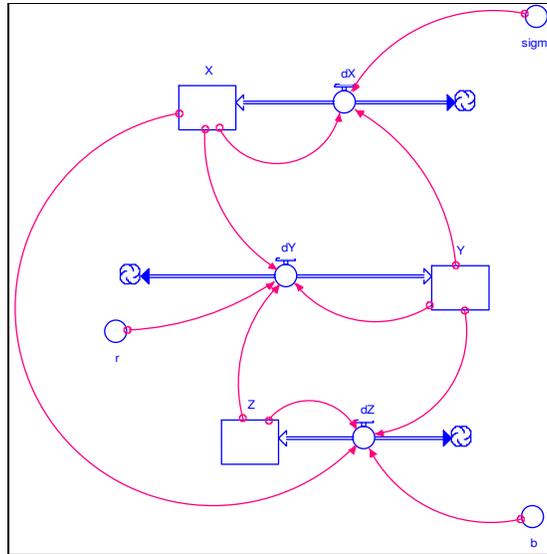
**Figure 2: Lorenz equations**

Translating these equations into a Stella v9.1 model we arrive at the following:

```
X(t) = X(t - dt) + (dX) * dt
INFLOWS:
dX = sigma*(Y-X)
Y(t) = Y(t - dt) + (dY) * dt
INFLOWS:
dY = X*(r-Z)-Y
Z(t) = Z(t - dt) + (dZ) * dt
INFLOWS:
dZ = (X*Y)-(b*Z)
b = 8/3
Initial_X = 1
Initial_Y = 0.5
Initial_Z = 0.1
r = 28
sigma = 10
```

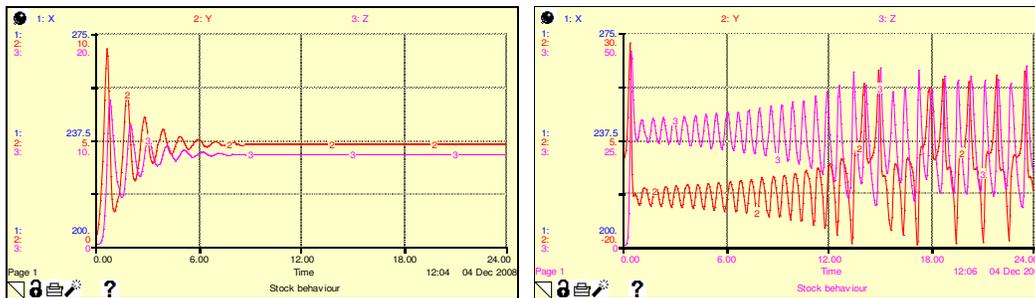
**Figure 3: Translation of Lorenz equations with values into Stella**

Figure 3 is based on one of the models described in (Fiddaman, 2008) and has the following graphical structure:



**Figure 4: Graphical model of structural relationships in the Lorenz equations**

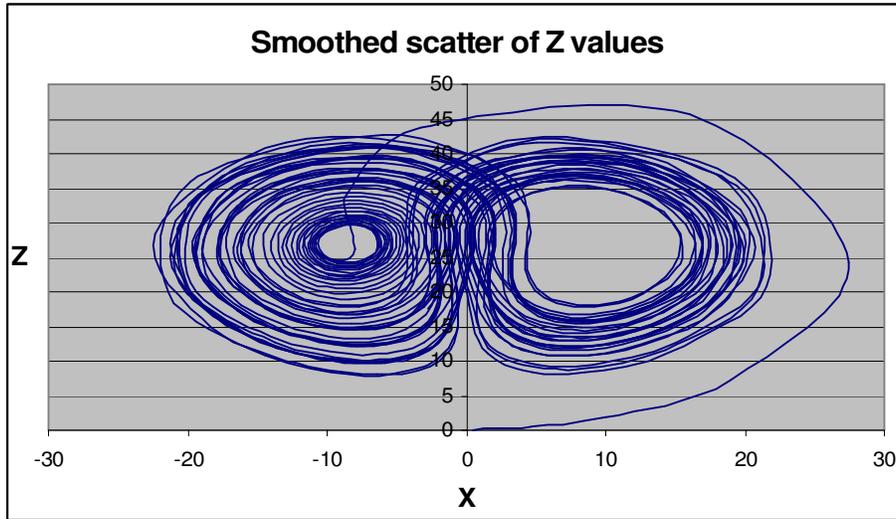
There are two factors that impact directly on the behaviour of this system and lead to the commonly described chaotic behaviour; the time and the value of the  $r$  (rho). The model described in figures 3 and 4 above results in a set of behaviours (over 24 time periods) that are reproduced in figures 5a and 5b below. Figure 5a illustrates the behaviour of the X, Y and Z values of the system with the value of  $r$  set to 9.6 and figure 5b illustrates the behaviour with  $r$  set to 28.



**Figure 5a and b: Output from Lorenz attractor for X, Y and Z**

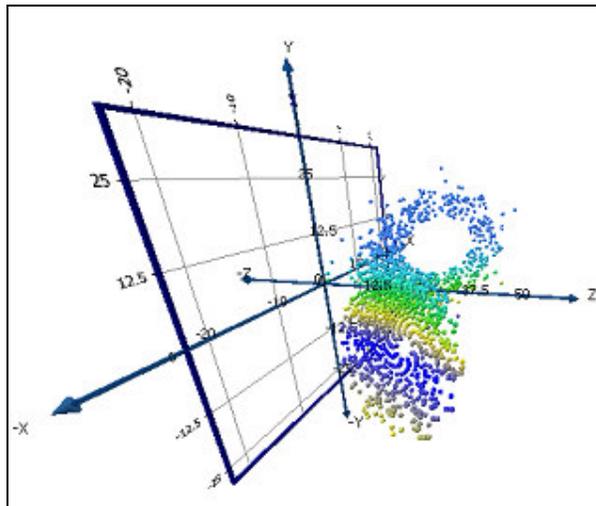
These outputs (figures 5a, b) are not random and their inputs are entirely determined; making this a model of deterministic behaviour. In the case of figure 5a we can see that the output, after initial turbulence, settles down to something approaching equilibrium. It should be noted that the definition of equilibrium used here is limited by the ability of the hardware and software to resolve the equations on which the computations are carried out; in essence the behaviour observed at the end of time period 24 in figure 5a is not actually stable but fluctuates only very slightly.

Figure 5b presents more complex behaviour and is consistent with chaotic behaviour as defined above. Although there are patterns in the data they are not easily discerned nor is any overall coherent behaviour apparent. If we now re-present some of the data as a smoothed scatter of Z against X we arrive at the description modelled in figure 6 below. Figure 6 shows the behaviour of figure 5b continued over 50 time periods; a traditional and expected butterfly patterning. That there are patterns within the data is now apparent and what was perhaps chaotic now has a definite structure and even an orderliness.



**Figure 6: Smoothed scatter of Z values (against X)**

The orderliness is a feature of the strange attractor described by the Lorenz formulation. It could be described as an attraction/repulsion on the points  $Z = 27.5$  and  $x = (+ \text{ or } -) 10$ . These are approximate values and are the foci of the Lorenz attractor.



**Figure 7: Phase space model of behaviour**

To further illustrate the points that there is order in chaos the model of behaviour presented in figure 7 shows distinct data patterning. As with figure 6 the twin focal values of the strange attractor are again apparent but in figure 7 we see additional patterning around the point  $(X = -10, Y = -8 \text{ and } Z = 27.5)$ . It is suggested that this spiral patterning is reminiscent of naturally occurring phenomenon such as a spiral galaxy. In the case of the patterning that surrounds the periphery of the attractor focus at  $(X = 10, Y = 8 \text{ and } Z = 27.5)$  we see evidence of behaviour that is, prima facie, related to a Mandelbrot fractal, derived from an iteration of the Mandelbrot set.

## Dealing with chaotic behaviour

Using our definition of chaotic behaviour i.e. “the set of behaviours’ that do not conform to preconceived notions of expected behaviour or are so complex in nature as to defy attempts to easily explain them in terms of expected output given the available knowledge about the input”, we will look at an early example of an attempt at constraining it.

We will use an interpretation of the water clock of Ktesibios (285 – 222 BC). The design is thought to be the earliest known example of a mechanical feedback mechanism (Richardson, 1991). The Ktesibios machine can be described as follows:

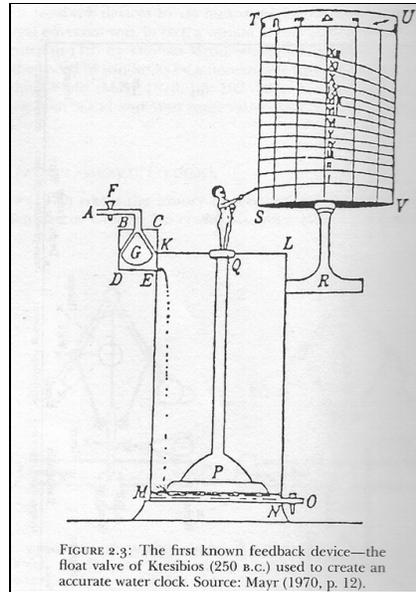
The Ktesibios (285 - 222 BC) machine is an example of a water clock. The example given here illustrates one possible interpretation of the Ktesibios' water clock as a System Dynamics model. This clock addressed two problems with water clocks:

- The source of water which flowed into the clock (input) could vary therefore producing erratic timing (output).
- Where a float (or header) tank was used to regulate the inflow, the amount of water in the float tank could vary (more water gives higher pressure and increases the flow), again giving erratic timing.

Ktesibios’ clock offered a mechanism for regulating water flow into the main system by means of an automatic float valve that provided a steadier water pressure in the float tank producing a more reliable means of time measurement.

The model consists of two stocks of water held in the float tank and the main tank. The float tank provides water to the main tank and it is this float tank that contains the mechanical feedback mechanism. It is also apparent that the clock as a whole is a feedback mechanism providing information to its observer on the time of day. Within the clock feedback is provided by means of a valve which, when the float tank becomes full, rises to the surface of the tank thereby blocking the input flow. When the water level falls in the float tank the valve falls and re-opens the input flow.

By means of the float valve the float tank maintains a certain level (within limits) of water and therefore water pressure, ensuring a steady flow of water into the main tank. The diagram below (Figure 8) is reproduced from (Richardson, 1991); the float valve is marked ‘G’.



**Figure 8: Ktesibios' water clock**

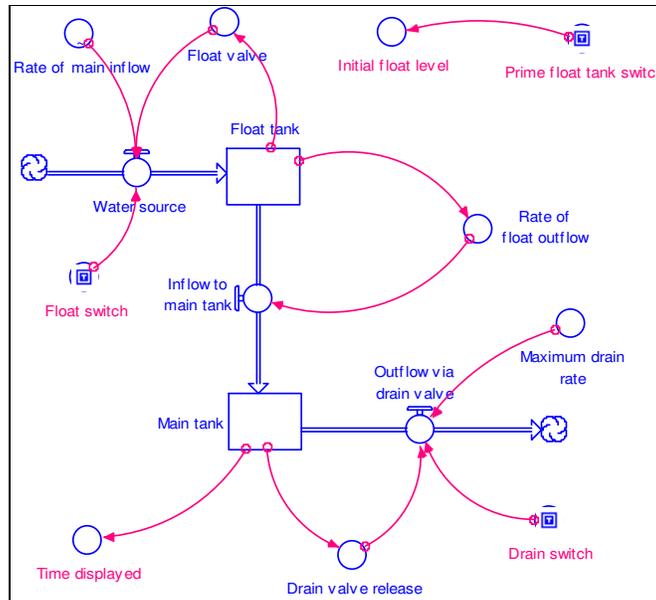
The model described in figure 8 is translated to the set of equations reproduced below in figure 9 and the graphical representation of that translation is shown in figure 10.

```

Float_tank(t) = Float_tank(t - dt) + (Water_source - Inflow_to__main_tank) * dt
INIT Float_tank = Initial_float_level
INFLOWS:
Water_source = if Float_switch = 1 then
Rate_of_main_inflow * Float_valve
else
Rate_of_main_inflow
OUTFLOWS:
Inflow_to__main_tank = Rate_of__float_outflow
Main_tank(t) = Main_tank(t - dt) + (Inflow_to__main_tank - Outflow_via__drain_valve) * dt
INIT Main_tank = 0
INFLOWS:
Inflow_to__main_tank = Rate_of__float_outflow
OUTFLOWS:
Outflow_via__drain_valve = if Drain_switch = 1 then
Drain_valve_release*Maximum_drain__rate
else
0
Drain_switch = 1
Drain_valve_release = if Main_tank > 100 then 1 else 0
Float_switch = 1
Float_valve = if Float_tank > 50 then 0 else 1
Initial_float_level = Prime_float_tank_switch*50
Maximum_drain__rate = 10
Prime_float_tank_switch = 1
Rate_of__float_outflow = Float_tank/10
Time_displayed = Main_tank/5
Rate_of_main_inflow = GRAPH(time)
(0.00, 13.5), (2.40, 16.4), (4.80, 18.0), (7.20, 23.8), (9.60, 18.6), (12.0, 13.1), (14.4, 4.75), (16.8, 11.5),
(19.2, 16.4), (21.6, 17.9), (24.0, 8.38)

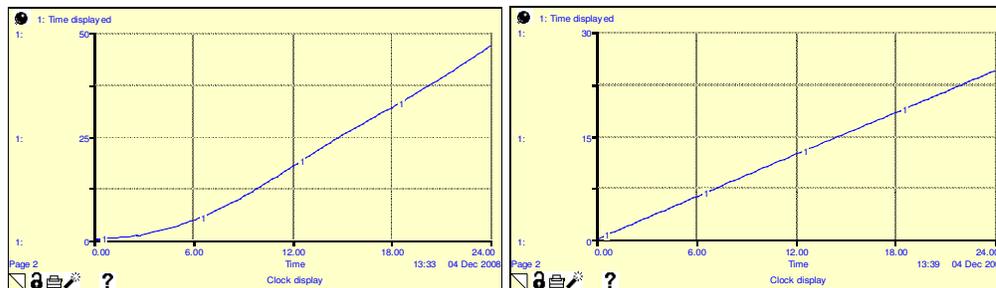
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**Figure 9: Ktesibios' machine expressed as a set of equations with values**



**Figure 10: Ktesibios' machine expressed as a stock and flow model**

Given this model the following behaviours are apparent when the main inflow is variable, (figure 11a) the float tank is not primed, the float valve is inactive and (figure 11b) the float tank is primed and the float valve is operational. In the first instance the output of the clock is somewhat erratic and non-linear and in the latter instance the output (more closely approximates) a linear flow. The latter display is desirable and addresses the issues that the machine was designed to cope with.



**Figure 11a and b, Output from Ktesibios' clock**

More complex behaviours can be obtained by varying the main inflow from zero to beyond the design tolerance of the machine or by varying the period that the machine is in operation.

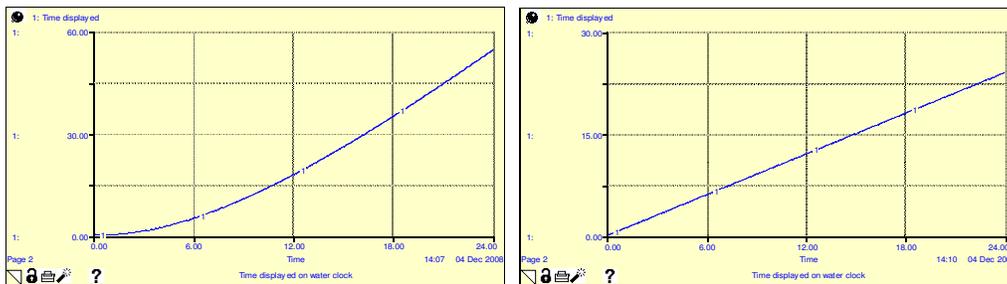
Additionally failing to prime the float tank will cause an initial instability. Given the relationship between initial states and chaotic behaviours described above priming the tank is an important consideration. Failing to consider priming is overlooked at the peril of the reader and would be doing an injustice to the, possibly intuitive, understanding of the importance of initial conditions that the original designers possessed.

### Constraining chaos

I have described above two of the fundamental systems that define both chaotic behaviour, the Lorenz model, and a buffered feedback system, the Ktesibios machine. Combining the two provides an insight into how chaos is handled in the real world.

If we ascribe the value of the main inflow to the Ktesibios machine as chaotic, as per the Lorenz system, the two systems can be coupled together by summing the X, Y and Z values obtained from the Lorenz system. Given that the Ktesibios machine is fully operational and providing that the flow rate produced by the output of the Lorenz system is positive and within the design tolerances we end up with output from the Ktesibios machine approximating that shown in figure 11b.

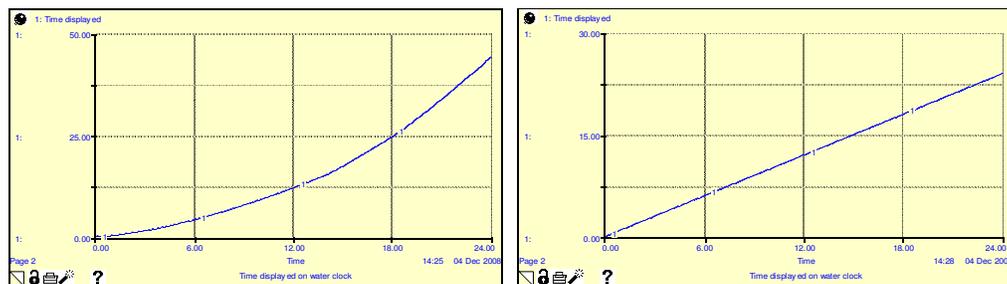
With the float valve inoperative, the float tank not primed and chaotic (but positive) inflow (with an  $r$  value of 10) we observe the behaviour shown in figure 12a. When the float tank is primed and the float valve is in operation, all else as set up for figure 11b, we observe the behaviour shown in figure 12b.



**Figure 12a and b: Output from Ktesibios' machine variable inputs,  $r = 10$**

The terminal value for time displayed in figures 12a and 12b respectively are 54.62 and 24.05.

With the float valve inoperative, the float tank not primed and chaotic (but positive) inflow (with an  $r$  value of 28) we observe the behaviour shown in figure 13a. When the float tank is primed and the float valve is in operation, all else as before, we observe the behaviour shown in figure 13b.



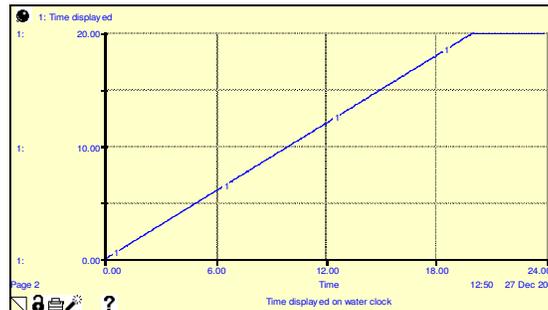
**Figure 13a and b: Output from Ktesibios' machine variable inputs,  $r = 28$**

The terminal value for time displayed in figures 13a and 13b respectively are 44.20 and 24.00. The equations, interface and graphical description of the hybrid Lorenz/Ktesibios model are reproduced in appendix A.

### Tipping points

A tipping point is here understood to be a point at which a system expresses, in its output, a significant change in the nature of its behaviour which may or may not be expected. We have seen from the previous discussion that the Ktesibios machine can cope well with a range of chaotic behaviours however it should be noted that the system has a variety of tipping points.

One of these is illustrated below in figure 14 where the drain switch is activated at time = 12 and the output changes significantly at time = 19.86.



**Figure 14: Output from Ktesibios' machine variable inputs,  $r = 28$**

There are also tipping points when the inflow value becomes too great for the float valve to cope or the input falls to zero. This is acknowledged but not illustrated here.

### **Philosophical implications**

On an esoteric note the presence of buffers, whilst not incompatible with the philosophy of determinism, is somewhat at odds with our strict interpretation of determinism as encapsulated in many System Dynamics or other mathematical simulations. Immediate state transitions and direct strictly logical communication of information exhibit a linear and unswerving cause and effect relationship that is not often seen to be present outside our models. For example it is possible, and a straightforward task, to model a rod that when subject to a force (cause) at one end immediately translates that force into an effect at the other end. If such a rod were to exist it would present an immediate refutation of the speed of light as a limiting constant as any force would be immediately transmitted from one point to another. We can imagine the rod as the flapping of the butterfly's wings both of which, it is postulated, are constrained by naturally occurring buffers.

### **Conclusions**

The principle conclusion that flows from this paper is that even in those cases where chaotic behaviour arises and threatens the integrity of an existing system there are often buffers, natural or artificial, that intervene to constrain that chaos. The examples shown here can be interpreted as a metaphor for the numerous examples of tamed chaotic behaviour that exist throughout our experience.

In addition the following conclusions flow from the analysis presented above:

That chaotic and/or random behaviour is heavily predicated on perspective and context or the ability of the observer to interpret and contextualise it.

That chaotic behaviour is not as amenable to definition as it might at first appear and the contrast between the formal definition and the vernacular definitions are misleading.

That chaotic behaviour and random behaviour are not necessarily the same thing. That truly random behaviour is not possible within a deterministic world view and hence not possible in a System Dynamics simulation.

That both chaotic behaviour and random behaviour (if such exists) may be the same thing but the means of understanding the underlying determinants of such behaviours are often currently unknown or sufficiently opaque or complex to render them unknown.

That the chaotic nature of systems was recognised from ancient times and attempts had been made to deal with it.

That at least one of those attempts, the Ktesibios machine, is able to cope, within limits, with chaotic behaviour. The point at which the machine reverts to chaotic output is a tipping point.

That there will be examples from naturally occurring systems where buffers that act in the same way as the float valve in the Ktesibios machine act to constrain chaotic behaviour. Examples include cell walls in organic systems, atmospheres in planetary dynamics and mental filters in humans which are used to isolate and discard extraneous sensory input.

It is not inconceivable that a minor, insignificant seeming, variation in conditions at or about the tipping point may cause a significant variation in system behaviour. In existing literature on chaos theory these tipping points are often referred to as initial conditions and indeed the tipping point may be most apparent at the initiation of the system.

#### **Areas identified for further work**

That given the examples above chaos may be more widespread than is currently apparent but that buffers, either natural or artificial, serve to silently constrain that chaos.

That the implementation of a naturally buffered system that constrains otherwise chaotic behaviour would be desirable to support the assertion made in the last conclusion. Specific examples of such systems are presented in (Hall, 1992).

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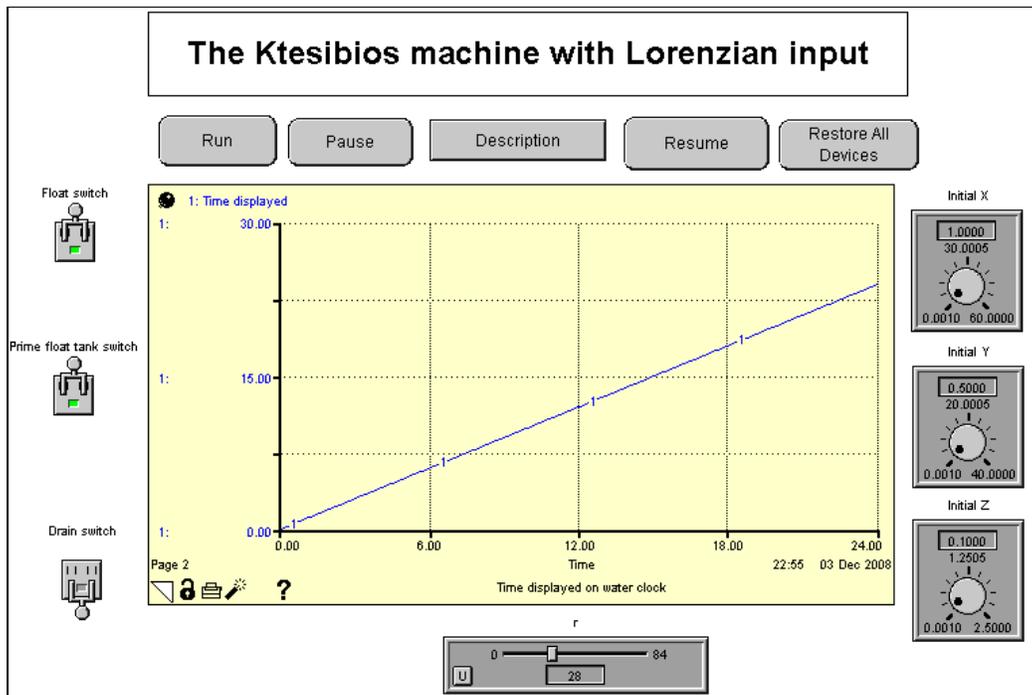
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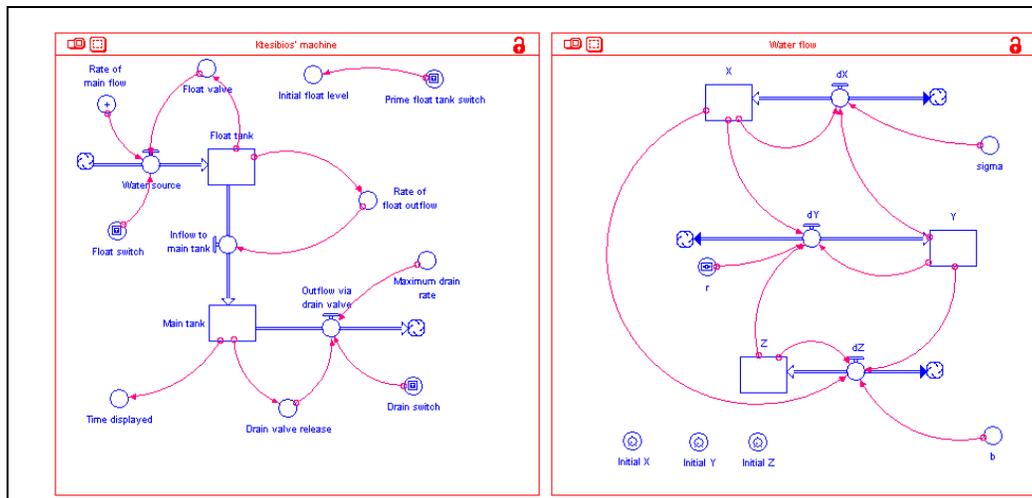
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# Appendix A



Interface to the hybrid system



Graphical structure of the hybrid system

```

Float_tank(t) = Float_tank(t - dt) + (Water_source - Inflow_to__main_tank) * dt
INIT Float_tank = Initial_float_level
INFLOWS:
Water_source = if Float_switch = 1 then
Rate_of__main_flow* Float_valve
else
Rate_of__main_flow
OUTFLOWS:
Inflow_to__main_tank = Rate_of__float_outflow
Main_tank(t) = Main_tank(t - dt) + (Inflow_to__main_tank - Outflow_via__drain_valve) * dt
INIT Main_tank = 0
INFLOWS:
Inflow_to__main_tank = Rate_of__float_outflow
OUTFLOWS:
Outflow_via__drain_valve = if Drain_switch = 1 then
Drain_valve_release*Maximum_drain__rate
else
0
X(t) = X(t - dt) + (dX) * dt
INIT X = Initial_X
INFLOWS:
dX = sigma*(Y-X)
Y(t) = Y(t - dt) + (dY) * dt
INIT Y = Initial_Y
INFLOWS:
dY = X*(r-Z)-Y
Z(t) = Z(t - dt) + (dZ) * dt
INIT Z = Initial_Z
INFLOWS:
dZ = (X*Y)-(b*Z)
b = 8/3
Drain_switch = 1
Drain_valve_release = if Main_tank > 100 then 1 else 0
Float_switch = 1
Float_valve = if Float_tank > 50 then 0 else 1
Initial_float_level = Prime_float_tank_switch*50
Initial_X = 1
Initial_Y = 0.5
Initial_Z = 0.1
Maximum_drain__rate = 10
Prime_float_tank_switch = 1
r = 28
Rate_of__float_outflow = Float_tank/10
Rate_of__main_flow = X+ Y + Z
sigma = 10
Time_displayed = Main_tank/5

```

### Equations for the hybrid system