Chaotic Behavior in a Modified Goodwin's *Growth Cycle* Model

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Abstract

Goodwin's "A Growth Cycle" [1967] represents a milestone in the non-linear modeling of economic dynamics. In terms of the two variables "wage share" and "employment rate" and on the basis of few simple assumptions, the Goodwin Model (GM) is formulated exactly as the well-known Lotka-Volterra system, with all the limits of such system, in particular the lacking of structural stability. A number of extensions have been proposed with the aim to make the model more robust. We propose a new extension that: a) removes the limiting hypothesis of "Harrod-neutral" technical progress: b) on the line of Lotka-Volterra models with adaptation, introduces the concept of "memory", which certainly plays a relevant role in the dynamics of economic systems. As a consequence an additional equation appears, the validity of the model is substantially extended and a rich phenomenology is obtained, in particular transition to chaotic behavior via period-doubling bifurcations.

1. Introduction: Goodwin's *Growth Cycle* model

Goodwin's [1967] *Growth Cycle* model (GM) represents a milestone in non-linear economic dynamics, receiving for this reason many extensions and improvements.

The economic system described in the GM is a one-sector non-monetary economy: there are only two social classes, capitalists and workers, which produce a homogeneous commodity, which may be consumed or invested and whose price may be normalized and put equal to 1. The following list presents all the variables, parameters and definitions which describe the economy:

Y Output

K Capital

S Savings

$I \equiv \dot{K} + \eta K$ Gross investment (where η

N Labor supply (expressed in units)

L Employment (expressed in units)

$a = Y/L$ Average labor productivity

w Real wage

$u = wL/Y = w/a$, $0 \le u \le 1$ Wage share (i.e., the share of income going to workers)

$v = L/N$, $0 \le v \le 1$ Employment rate

The main hypotheses which define the structure of the economy are the following:

A1) Capitalists' average propensity to save = 1, while workers' = 0. Capitalists reinvest all their savings (= profits), that with no capital depreciation[1] implies:

$$S = I \equiv \dot{K} = Y - wL$$

while workers consume entirely their earnings.

A2) Capital/output ratio constant: $\frac{K}{Y} = \sigma$;

A3) Population, and then labor supply, grows at a constant rate $n > 0$: $N = N_0 e^{nt}$;

A4) Labor productivity grows at a constant rate $\alpha > 0$: $a = \frac{Y}{L} = a_0 e^{\alpha t}$;

Note that from hp. A4) and A2) it follows that the rate of growth of K/L equals the rate of growth of Y/L (this kind of technical progress is defined as "Harrod-neutral" technical progress).

A5) Real wages change according to a linear "Phillips curve", which represents the *bargaining equation* on the labor market:

$$\frac{\dot{w}}{w} = -\gamma + \rho v \qquad \gamma, \rho > 0$$

On the basis of the above assumptions and definitions, after simple manipulations, the dynamic behavior of the economic system may be described by the following non-linear differential system (GM):

$$\dot{u} = \left[-(\gamma + \alpha) + \rho v\right] u \tag{1.1}$$

$$\dot{v} = \left[\frac{1}{\sigma} - (\alpha + n) - \frac{u}{\sigma}\right] v \tag{1.2}$$

Equations (1.1) and (1.2) exhibit the same formal framework as the well known [LOTKA, 1925] and [VOLTERRA, 1926] *predator-prey* model (LV), sharing the main characteristics and limits of the latter. The system has two fixed points: the trivial one at the origin (a saddle point), and:

$$u^* = 1 - \sigma(\alpha + n)$$
$$v^* = \frac{\alpha + \gamma}{\rho} \tag{1.3}$$

In order to be economically meaningful, (1.3) has to satisfy the conditions: $\alpha + n < 1/\sigma$ and $\alpha + \gamma < \rho$. The dynamical proprieties of LV model are well known: since the eigenvalues of the jacobian matrix of the system are purely imaginary, the fixed point (1.3) is a centre, neutrally stable, and the flow of the system around the point will be described by a family of closed orbits whose "amplitude" depends on initial conditions. Then, the initial values of u and v will determine which of the infinitely closed orbits describes the actual dynamic behavior of the system.

[1] In this Sec., in line with the original Goodwin's contribution, we assume that there is no capital depreciation, i.e., $\eta = 0$. As will be clear, this assumption involves no qualitative modifications in the model. In the next Secs., however, we will consider the more realistic (and usual) assumption $\eta > 0$.

The economic meaning of the model is straightforward: u acts as "predator" and v as "prey". As can be seen from Fig .1, a low wage share (point A) – that is, a high profit share – leads to the increase in the employment rate; the latter, implying an increase in the workers bargaining power (according to hp. A5), pushes the wage share up: the growth rate of profit share begins to decrease and, consequently, also the employment rate decreases (portion BC). At C predators "have eaten too many preys", and their number begins to decrease, allowing a new increase in the profit share. At D, the low u re-establishes the conditions for a new expansionary phase of the cycle.

One of the main criticism to Goodwin's model concerns its *lacking of structural stability* that is a characteristic of the LV model. A definition of structural stability, proposed by [HIRSH & SMALE, 1974] relies on the concept of topological equivalence between two (or more) systems under perturbations. For our purposes the point is that it is easy to find large classes of economically meaningful perturbations which, no matter how small, lead to qualitative changes in the dynamic behavior of GM[2]. A useful criterion is the one proposed by [GUCKENHEIMER & HOLMES, 1983], according to which a dynamical system is structurally unstable when the value of one parameter is at a bifurcation point.

As several authors have shown, however, with further hypotheses GM can be modified in order to become structurally stable, exhibiting limit cycles persistent to parameter perturbations: in this light, all the extensions to GM can be seen as the introduction of perturbations which make the system dissipative. In other words, as already pointed out (see, for example [VENEZIANI, 2001]) GM can be considered as a *particular* case of an extended model, which looses structural stability since a parameter is at a bifurcation value. Among the various two-dimensional extensions to GM, we recall [DESAI,1973] who analyzes a monetary economy in the presence of an "augmented" Phillips curve, [WOLFSTETTER, 1982] who adds government sector inquiring the effect of stabilization policies, [CHOI , 1995] and [MANFREDI & FANTI, 2000] who introduce the efficiency-wage hypothesis[3]. Three-dimensional extensions can be found in [MANFREDI & FANTI, 1999] who besides efficiency-wage hypothesis introduce a gestation lag in investment plan and [BRODY & FARKAS, 1987] and [CHIARELLA, 1990] who consider a time-lag in the Phillips curve, assuming that wages growth rate is an increasing function of the weighted average of all past values of the employment rate.

Another problematic feature intrinsic to GM is represented by the economic meaningfulness of its solutions. In fact, using estimates of actual parameter values – as those provided by [HARVIE, 2000] – it is easy to realize, once numerical simulations are performed, that u and v values are not constrained in the unit square $[0, 1]\times[0, 1]$ as required by their definitions. This is true also for most of the extensions to GM that we have recalled above[4] (clearly, this fact gives no problems in the original LV model, which is formulated in *absolute* values).

In the present paper we will proceed along this line of research, removing some of the most limiting hypotheses which characterize the original Goodwin's contribution, but trying to preserve, at the same time, its main features and peculiarities. Particularly, the main modifications that we are going to carry out concern the assumptions about technical progress and economic agents' behavior. With regard to the former we will remove Goodwin's assumption of "Harrod-neutral" technical progress. With regard to the latter we will consider also the influence of *past* events on agents' current behavior, that is, we will take into account the influence of their *memory* on the dynamics of the

[2] For instance, if we let capitalists' average propensity to save be a decreasing function of u, or if we let capital/labour ratio be an increasing function of real wages, the trace of the jacobian becomes negative (and then the fixed point asimptotically stable).

[3] Other extensions can be found in [Wolfstetter, 1977], [Flaschel, 1984; 2000]. For a survey of GM-types, see Veneziani [2001].

[4] In order to confine u and v solutions in the unit square, [Desai *et al.*, 2004] have modified GM, making the following *ad hoc* assumptions: $\frac{\dot{w}}{w} \to +\infty$ as $v \to 1^-$ and $\frac{\dot{v}}{v} \to -\infty$ as $u \to 1^-$.

economic system. These supplementary assumptions lead to a three-dimensional model which may exhibit, for some parameter values, a complicated behavior: the emergence of different limit cycles and even transitions to chaos with the appearance of aperiodic attractors.

The rest of the paper is organized as follows. In Sec. 2 we present our model, and in Sec. 3 we investigate the main properties of its equilibrium points. In Sec. 4 we show how the non trivial equilibrium undergoes a Hopf bifurcation with the emergence of a limit cycle, continuing the analysis of the time-dependent behavior in Sec. 5, which is also devoted to the study of transitions to chaos. In both Secs. 4 and 5 the analysis will be carried out also through extensive numerical simulations. Finally, Sec. 6 sums up the main conclusions of this work.

2. The Adaptive Goodwin's Model with Capital/Labor Substitution

As Goodwin himself recognized, the theoretical groundwork of his model was represented by Marx's *general law of capitalist accumulation*. According to this "law", presented in Ch. XXV of [Marx, 1990], real wages are a decreasing function of industrial reserve army (IRA)[5] variation, the latter depending upon the various phases of the business cycle. During the expansionary phase of the cycle, IRA's decrease leads to increasing strains on the labor market, since workers' bargaining power is becoming stronger and stronger: as a consequence, real wages increase and (then) profit share decrease. Profit share reduction, however, leads to the automatic reduction in investments – since also in Marx's description the hypothesis (A1) holds – which, implying a decrease in the employment rate, will reconstitute IRA and the conditions for a new expansionary phase. Therefore GM may be seen as an elegant and simple mathematical formulation of the dynamics depicted above, containing at the same time a further interesting feature: while in Marx's exposition the oscillations concern wage *absolute values*, in GM real wage *growth rates* are concerned, so the latter model being more suitable for the description of both short run (cycle) and long run (growth) dynamics.

Nevertheless, in the Marxian law of capitalist accumulation there is a fundamental element, with regard to capitalists' behavior, which is not present in GM. According to Marx, capitalists, rather than adapt passively to the oscillations of real wages, may vary capital/labor ratio substituting workers with machinery as wages increase, moreover fostering the process of re-constitution of IRA[6]. It is worth noting that this explanation of K/L increase is quite independent from the other Marx's explanation of increasing mechanization, which relies on the idea that the productivity of labor is an increasing function of capital/labor ratio. According to Marx, then, capitalists modify K/L for two main reasons:

a) because Y/L is an increasing function of K/L (long-run effect of technical progress);
b) against real wage increases in the expansionary phase of the cycle (short-run contingent reaction).

Only the former point is present in GM, in the form of "Harrod-neutral" technical progress, while the latter is lacking. Following Marx's suggestion – in line, moreover, with standard neo-classical growth theory – we will assume that capitalists may substitute labor for capital as the wage share increases. Such a modification to GM needs further specifications: *i)* we will assume that only K/L "long run" increase favorably affects productivity of labor, while K/L changes in response to real wages changes – given their contingent nature – have no effect on Y/L; *ii)* finally, if according to point b) we assume that K/L increases as u increases, we have also to consent that capitalists may act in the opposite manner as u decreases, that is, we will assume that K/L is an increasing function of u[7]. As a direct consequence

[5] According to Marx [1990], IRA represents "a surplus labouring population … a mass of human material always ready for exploitation".

[6] This view represents an interesting anticipation of neoclassical growth theory, as far as substitutability between capital and labour is concerned.

[7] See below for a precise definition of the relation between K/L and u.

of this, factor a) acts only in the direction of a continuous increase in K/L, while factor b), depending on u variations, will involve acceleration or deceleration of K/L increase; note, moreover, that assumption b) implies the removal of the assumption of Harrod-neutral technical progress, since now also capital/output ratio may vary.
The other modification in our model concerns agents' behavior. We maintain that both workers' and capitalists' *current* behavior should depend also on their past performances, i.e., what they actually do should be also related to what they have done in the past. Particularly, we will follow the *adaptive hypothesis* discussed in [Lacitignola & Tebaldi, 2003], according to which there is a time-dependent learning-by-doing process, affecting both "predators" and "preys". In this way we want to emphasize the role of *memory* on the current performance of the economic system.
We assume that the system has a non-trivial equilibrium at which $u = u^*$, whose value will be computed in the next paragraph. Among the various variables which can be taken as indicators of the history of the system, we have chosen the weighted average of all past values of the difference $u- u^*$, where $u*$, the equilibrium value, can be seen as the "natural" wage share[8]. $u- u^*$ can therefore also be interpreted as the result of the bargaining. Specifically we assume the following formulation in terms of an exponentially distributed lag (weakly decaying kernel which implies *fade-out memory* hypothesis):

$$y(t) = \frac{1}{T}\int_{-\infty}^{t} e^{\frac{\xi - t}{T}} \left(u(\xi) - u^*\right) d\xi \tag{2.1}$$

where T measures the time scale for adaptation (*memory span*) of the system[9].

As a consequence, both workers' and capitalists' current behavior should depend on $y(t)$ (*memory variable*). As far as workers' behavior is concerned, we assume that (see above, hp. A5):

$$\rho(t) = \rho_1 + \rho_2 y(t), \qquad \rho_1, \rho_2 > 0, \tag{2.2}$$

which leads to the modified Phillips curve:

$$\dot{w} = \left[-\gamma + (\rho_1 + \rho_2 y(t))v\right]w \tag{2.3}$$

According to Eqs. (2.2) and (2.3), the growth rate of real wages is not only a function of current employment rate, but also of the weighted average of past values of $u-u^*$. Eq. (2.3) means that the higher average wage share has been in the past, the higher workers' bargaining power will be at present (*adaptive hypothesis*, which implies workers' learning by doing). Moreover, Eq. (2.3) also implies that, if one consider two economic systems which have experienced the same "amount" of bargaining conflicts, workers' bargaining power will be stronger in the system where these conflicts have been more recent (*fade-out memory hypothesis*).
As regards capitalists' behavior we assume that:

$$\frac{K}{L} = \left[K_0 + \zeta(y)\right]e^{\beta t} \quad K_0 > 0 \tag{2.4}$$

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[8] "Natural" in [FREIDMAN, 1975] sense, since $u*$ corresponds to $v*$, which can be defined as the natural rate of employment.

[9] We point out that our fade-out memory hypothesis differs from the one proposed by [BRODY-FARKAS, 1987] and [CHIARELLA, 1990], since they substitute v in the Phillips equation (1.1) with the weighted average of its values given by $x(t) = \frac{1}{T}\int_{-\infty}^{t} e^{\frac{\xi - t}{T}} v(\xi) d\xi$.

where β is the constant *K/L* growth rate, while *ζ(y)* is the capitalists' reaction function, which specifies the relation between *u* and *K/L*. Once again, we have assumed the adaptive hypothesis, since also capital/labor substitution depends on the weighted average of $u-u^*$ past values (and then on the past reactions of capitalist to *u* increases).
For the productivity of labor we put:

$$a = \frac{Y}{L} = \varphi K_0 e^{\beta t}, \qquad \varphi > 0 \tag{2.5}$$

according to which contingent K/L modifications, due to *u* variations, do not affect the productivity of labor, summing up what we have said about the relation between *K/L* and *Y/L*.
The last matter to discuss concerns the exact specification of *ζ(y)*, which may be defined as the *reaction-substitution function* between labor and capital. For the moment we pose only the (general) restrictions that:1) *ζ(y)* is a monotonically increasing function on ($y \in [-u^*, 1-u^*]$), $\zeta(0) = 0$, and convex for $y > 0$; 2) the derivative of *ζ(y)* is very small in the neighbourhood of the equilibrium point, increasing as $|u-u^*|$ increases. Assumption 1) is obvious, in the light of what we said above about the substitution function; assumption 2) implies that around the equilibrium there is a low incentive to substitute labor for capital, but that this incentive increases as the distance between u and u^* increases. More precise specifications of the reaction-substitution function, and the consequences of these different specifications on the dynamics of the system, will be taken into account in the following paragraphs.
From Eqs. (2.1), (2.2), (2.3) and (2.4), after simple manipulations, we get the following three-dimensional system:

$$\dot{u} = \left[-(\gamma+\beta) + (\rho_1 + \rho_2 y)v\right]u \tag{2.6}$$

$$\dot{v} = \frac{\varphi K_0(1-u) - (\beta+n+\eta)\left[K_0 + \zeta(y)\right] - \zeta'(y)\left(\dfrac{u-u^*-y}{T}\right)}{K_0 + \zeta(y)} v \tag{2.7}$$

$$\dot{y} = \frac{u-u^*-y}{T} \tag{2.8}$$

3.Analysis of the Model

The model has two equilibrium points: $E_0=(0,0,-u^*)$, a saddle point, which corresponds to the trivial GM saddle point in (0,0), and E^*:

$$E^*: \qquad \begin{aligned} u^* &= 1 - \frac{(\beta+n+\eta)}{\varphi} \\ v^* &= \frac{\beta+\gamma}{\rho_1} \\ y^* &= 0 \end{aligned} \tag{3.1}$$

that corresponds to the interesting equilibrium of GM. It can be shown that the uniqueness of the non-trivial equilibrium is guaranteed by the monotonicity of *ζ(y)*. The Jacobian matrix computed in E^* is:

$$J^*=\begin{pmatrix} 0 & \rho_1 u^* & \rho_2 v^* u^* \\ -\varphi v^* - \dfrac{\zeta'(0)v^*}{K_0 T} & 0 & \dfrac{v^*}{K_0}\zeta'(0)\big(1/T-(\beta+n+\eta)\big) \\ \dfrac{1}{T} & 0 & -\dfrac{1}{T} \end{pmatrix}$$

The case of interest is $\zeta'(0) = 0$, as will be discussed in the following, and the characteristic equation becomes ($A = \varphi\rho_1 u^* v^* > 0$):

$$\lambda^3 + \frac{\lambda^2}{T} + \lambda A\left[1 - \frac{\rho_2}{\rho_1 \varphi T}\right] + \frac{A}{T} = 0 \qquad (3.2)$$

This additional hypothesis about the ζ function implies that in Eq. (3.2) the elements characterizing the capital-labor substitution disappear, i.e. the substitution mechanism is assumed to be ineffectual at the equilibrium E^*. However, differently from the GM, in Eq. (3.2) there are ρ_2 and T, which measure respectively the effect of the memory variable in the bargaining equation and the memory span of the system. Fade-out memory is then the mechanism which makes the linearization of our system different from that of the GM.

It is easy to verify that, when $\rho_2 = 0$, independently of the value of T, Eq. (3.2) has a pair of pure imaginary roots, and precisely:

$$\lambda_{2/3}(\rho_2 = 0) = \pm i\sqrt{A} \, .$$

Furthermore, as $T \to +\infty$ the two eigenvalues tend to become purely imaginary, as for the GM non-trivial equilibrium. In the numerical study of the roots of Eq. (3.3) we have fixed the parameters using values which are consistent with Harvie's article, that are:

$\beta = .04$ (correspondent with α in the original GM)

$n = .03$

$\eta = .05$

$\rho_1 = 22$ (correspondent with ρ in the original GM)

$\gamma = 18$

$\varphi = .33$ (correspondent with $1/\sigma$ in the original GM)

$K_0 = 2$

With this set of parameter values (which has been used for all the simulations) E^* becomes:

$u^* = .63$

$v^* = .82$

$y^* = 0$

From a numerical analysis we find that the absolute value of the real negative eigenvalue λ_1 grows with ρ_2 and decreases with T. Independently of T, the value of ρ_2 characterizes the stability of E^*: when $\rho_2 < 0$ the equilibrium is stable; at $\rho_2 = 0$ the real part of the complex conjugate eigenvalues becomes zero, so we re-obtain a centre as in GM; a value $\rho_2 > 0$ destabilizes E^* which turns into an unstable focus.

Moreover, at $\rho_2 = 0$ $\frac{d\,\mathrm{Re}(\lambda_{2/3})}{d\rho_2}(0) > 0$ holds, and we are in the hypotheses of the Hopf Bifurcation Theorem: around E^* either a stable limit cycle emerges when $\rho_2>0$ (supercritical Hopf bifurcation) or an unstable cycle is present when $\rho_2<0$ (subcritical Hopf bifurcation). According to the theorem the cycle arises with the period $P = 2\pi/\sqrt{A}$, i.e., for the parameter values chosen, $P \approx 3.20$.

Finally, about the dependence on the memory span, we can note that the real part of the complex eigenvalues becomes smaller as T increases. For larger values of ρ_2 the complex conjugate eigenvalues become real and distinct, and the unstable focus turns into an unstable node.

Before to proceed with numerical simulation, we have to choose an explicit form for the ζ function satisfying the assumptions posed above. We assume the form:

$$\zeta(y)=\zeta_0 y^3 e^{my} \quad \zeta_0 > 0,\ m \geq 0 \tag{3.3}$$

As is clear from Eq. (3.3), ζ_0 and m characterize the substitution function. ζ_0 is a proportionality constant in the ζ function, controlling the scale and the efficiency of the *K-L* substitution, while m affects the shape of the reaction function. When $m = 0$, ζ is an anti-symmetric function, while for $m > 0$, as m increases the function becomes steeper in the region where $y>0$ and flatter when u is below its natural value. From Eqs. (2.4), (2.5) it is easy to verify that capitalists' reaction involves the variation of *K/Y* ratio: consequently, $m = 0$ indicates an equal possibility of increasing (when $y > 0$) or decreasing (when $y < 0$) K/Y, while $m > 0$ implies that it is easier to substitute capital for labor, through *K/Y* increase, than vice versa. In the following we will consider both cases, even if we think more consistent with our model the second case ($m > 0$), since it is less likelihood that *K/Y* may decrease as u falls below its natural value u^*.

4. Numerical Analysis of the Limit Cycle

In order to study the time-dependent behavior induced by destabilization of E^*, we have to rely on numerical simulations, performed using Matlab v. 6.5 and Ermentrout [2002] XPPAUT (Ver. 5.9) software package.

Taking initial conditions close to the equilibrium, for $\rho_2 > 0$ the solutions spiral away towards a limit cycle which arises at the bifurcation with period $P = 2\pi/\sqrt{A}$. The Hopf bifurcation therefore is found to be supercritical and structural stability is regained for $\rho_2 > 0$. In comparison with the GM, we note that the set of different persistent oscillations is replaced, in our model, by a stable limit cycle obtained for any initial condition, but E^*, chosen.

Next step is the investigation of the limit cycle behavior with respect to variations of the parameter values characterizing the model: ρ_2, T, ζ_0 and m. We point out that the study of a system of ordinary differential equations dependent on four parameters – even three-dimensional – requires considerable computational efforts. Therefore our analysis aims to obtain a general, rather than an exhaustive, picture. We anticipate that, while variations of ρ_2, T and m induce the main qualitative changes (bifurcations or subcycles), the role of ζ_0 is crucial in keeping the limit cycles on the *uv* plane inside the unit square, preserving the economic relevance of the model. This fact appears to be very relevant by comparison with other extensions of the GM (see, for example, [CHIARELLA, 1990] and [MANFREDI & FANTI, 2000]). In those models although a continuously distributed lag for one of the variables is able to destabilize the equilibrium point through a Hopf bifurcation, no mechanism constrains the cycle inside the meaningful domain.

In our model this mechanism is provided by capitalists' reaction, described by the function $\zeta(y)$. In fact, as appears in Eq. (2.7), when the solution enters the subdomain $u>u^*$, $v>v^*$, the ζ'-depending term induces a decrease in v. In this part of the cycle the factor $u-u^*-y$ is positive: in fact, as the numerical study confirm, in this phase both $u-u^*$ and y are growing (respectively from 0 and negative values), but y conserves memory of the past values of u-u^*, and so remains smaller than u-u^*.

The reaction-substitution function ζ is therefore the tool able to limit too high increases (or decreases) of the wage share u, mitigating distributive conflicts. On the contrary, as we will show, an increase of T and/or ρ_2 leads to wider oscillations of u. We now turn to a more detailed description of the role of the various parameters on the limit cycle.

4.1. Effect of ρ_2

The equilibrium point, which is a stable focus for $\rho_2 < 0$, destabilizes at $\rho_2 = 0$, with the appearance of a limit cycle whose size grows as ρ_2 increases, as predicted by the Hopf theorem. In agreement with the theorem, as the limit cycles approach E* the period tends to $2\pi/\sqrt{A} \approx 3.20$, also increasing with ρ_2.

This behavior is also confirmed for relatively larger values of ρ_2 (Fig. 2 $m = 0$ and Fig. 3 $m = 15$): the limit cycle looses its "elliptical" shape, as expects, the oscillation amplitude becomes larger and the rate of convergence of the solutions increases. The oscillations of the variables, approximately sinusoidal close to bifurcation, develop short and very fast growing (or diminishing) phases, alternated to longer ones characterized by much smaller values of the time derivatives. Absolute maxima (minima) of u and minima (maxima) of v tend to synchronize, becoming more and more peaked.

As the maximum (minimum) of u and minimum (maximum) of v are reached, we can see the appearance of local phenomena – which can be interpreted as small-scale subcycles – related to local loss of monotonicity in time, as a consequence of some kind of "overshooting". As ρ_2 is increased, the overshooting strengthens: the number of subcycles increases, as the appearance of relative extreme points in the time plots shows (Figs. 2-3). This is an interesting result, since subcycles of the same kind were found in the analysis of actual time series in [HARVIE, 2000].

Economic interpretation of all this phenomenology will be discussed after the analysis of the role of ζ_0 and m. However we notice the loss of symmetry of the limit cycle and the strong effects on the subcycles for the case $m > 0$.

4.2. Effect of ζ_0 and m

ζ_0 and m characterize the substitution function. Subcycles due to the mechanism of overshooting are a consequence of the interaction of the two key elements characterizing our model: adaptation (controlled by ρ_2 and T), and capitalists' response against wage claims (controlled by $\zeta(y)$). When these mechanisms are strong enough, employment rate drops to such level that a new expansionary phase of the cycle, on a smaller scale, can take place.

An increment of ζ_0 reduces the size of the limit cycle, with no relevant effects on its shape (Fig. 4). For this reason the value of ζ_0 is crucial to maintain phase trajectories inside the unit square, maintaining meaningful the relevant changes of the limit cycles when ρ_2 is increased.

If the substitution function is anti-symmetric (i.e. $m = 0$), overshooting phenomena occur both on the region of the cycle where $u > u^*$ and $u < u^*$, even if they are more relevant in the former case, and other smaller-scale subcycles may appear.

As m increases (Fig. 5), the overshooting effect for $u < u^*$ tends to disappear. Moreover the shape of the cycle for $u > u^*$ is substantially changed: capitalists' reaction against wage increases leads to a dramatic reduction of the employment rate on a short time-scale, which in turn determines a new and fast expansionary phase of the small-scale cycle. As m increases further, both the number of subcycles and their size also increase.

4.3.Effect of T

Finally we analyze how a variation of T affects the system, and particularly the limit cycle. We recall that T represents the time-scale related to the memory span of the system. Starting from $T = 0$ (no memory) we have considered values of T of the order up to 1/10 of the period of the cycle, which corresponds to the characteristic time-scale of subcycles. When T increases to a value comparable with the characteristic time of a subcycle, the rate of change of the variables becomes smaller around their relative maxima and minima, i.e. the maximum and minimum peaks correspondent to the subcycles flatten (Fig. 6). For further increment of T, sub-cycles disappear. This behavior can be related to a lower weight given to very recent bargaining dynamics by both workers and capitalists, since a longer memory allows the system to avoid changes in its behavior due to contingent or relatively temporary factors.
Further increment of T causes a period-doubling bifurcation followed, for even higher values, by a period halving which re-establishes a more regular dynamics, i.e. "sinusoidal" oscillations of u and v.
For different parameter values the period-doubling bifurcation can be the first in an infinite sequence leading to chaos and this case will be discussed in detail in the next section.

5. Bifurcations of the Limit Cycle and Transitions to Chaos

The presence of a period doubling bifurcation for the limit cycle suggests to seek – through continuation analysis – for possible sequences of period doublings, which can lead to chaotic behavior ([FRANCESCHINI & TEBALDI, 1979]) according to Feigenbaum scenario for one-dimensional maps ([FEIGENBAUM, 1978]) . Bifurcations for all values of the parameters ζ_0 and m tested have been found, but it is remarkable that only when the ζ-function is sufficiently steep (i.e. the K-L substitution is strong enough) the doubled orbits do lie in the domain. We keep m=15, ζ_0=100 as before, and we present the phenomenology of the system occurring varying T and ρ_2: bifurcation analysis was performed with the software package XPPAUT which includes the bifurcation analysis tool AUTO.
At first a sequence of bifurcations occurring for T=.4 when ρ_2 is varied is analyzed (see Fig.7). At $\rho_2(1)\approx 3.386$ the limit cycle looses stability and a stable closed orbit with period doubled appears (period..), first in the sequence, while at $\rho_2(1')\approx 5.303$ a period-halving bifurcation (with period at the bifurcation P≈3.876), last in the sequence (see Tab.1), restores an orbit with period . For higher values of ρ_2 a complex phenomenology appears: particularly, there are also tangent bifurcations which give rise, as we will see, to multiple stable cycles in some regions of parameter space, leading to further flip bifurcations. We have restricted our analysis to the first sequence of period-doubling/period-halving bifurcations, since a complete investigation of a bifurcation sequence like that depicted in Fig.7 requires a study by its own (and is under consideration). One-parameter continuation shows a sequence, likely to be infinite, of period-doubling and period-halving bifurcations: the first 7 of them have been detected and in Tab.1 the critical values $\rho_2(i)$ are reported, together with the periods (P_i) at bifurcation: at $\rho_2 = \rho_{2C} \approx 4.4174$, an aperiodic attractor is found. We show in Fig. 8 the stable closed orbits for $\rho_2 = 2.91$ (P = 3.28), $\rho_2 = 3.45$ (P = 6.78), $\rho_2 = 4.16$ (P = 15.43), and $\rho_2 = 4.40$ (P = 61.25) and in Fig. 9 an aperiodic attractor found at $\rho_2 = 4.5$.
The transition we are presenting occurs according to the universality prediction of Feigenbaum Scenario for one-dimensional maps ([FEIGENBAUM, 1978]), which in the case of a system of ordinary differential equations has been found for the first time in [FRANCESCHINI & TEBALDI, 1979].

In Tab. 1 we also show the computed values of $\delta_{i-1} = \dfrac{\rho_2(i) - \rho_2(i-1)}{\rho_2(i+1) - \rho_2(i)}$ which at the limit $i \to \infty$ converges to the Feigenbaum constant $\delta \approx 4.6692$. It is remarkable that 7 successive bifurcations have been detected in a system of ordinary differential equations and that the value δ_5 is already very close to the universal asymptotical value.

As far as interpretation, when ρ_2 – the parameter characterizing adaptation in the bargaining equation – is "large enough", the increasing weight of past bargaining struggles leads the system to loose strict periodicity and therefore precise predictability. Different initial conditions, even very close to each other, lead to different time behavior of the system, which, however, is on the whole defined by the strange attractor. As ρ_2 is further increased, clear evidence of the typical phenomenology of alternation between chaotic and periodic behavior is observed. An interesting aspect is represented by the tangent bifurcation at $\rho_2 \approx 4.809$ (see Fig.7) by which a new attracting orbit appears: as is evident comparing Fig.7 and Tab.1, this orbit is present together with chaotic regime. This latter phenomenology may be seen as the presence of bistability in the model: increases of ρ_2 lead, as described, to a sequence of period-halving bifurcations ending at $\rho_2(1)\approx 5.303$; starting from different initial conditions, however, the system may be attracted by an orbit of low period, and no period-doubling cascade starts, since for this orbit has only a period-doubling at $\rho_2 = 5.73$. We point out that for greater T values continuation analysis reveals a even more complex phenomenology, since after the first sequence of period-doubling/period-halving points, tangent bifurcations may lead to new cascades of bifurcations: once more, however, attracting orbits of low period appear inside the chaotic region, implying the same phenomenon of bistability described above. In Figs.10 aspects of this rich and complex phenomenology, for T = .44 is shown. Particularly the presence of tangent bifurcations lead to multiple cascades of period-doubling/period-halving bifurcations, even if these cascades have tangent bifurcations taking place in between.
The phenomenology described above seems to be representative also of a wide range of ζ_0 and *m*.
One-parameter continuation shows an analogous phenomenology for variation of *T*. The first period-doubling, for ζ_0= 100, m = 15, $\rho_2 = 4.5$, takes place at $T \approx .359$, with P = 7.36 (followed at $T \approx 0.63$ by a period-halving bifurcation, with period at the bifurcation P≈6.56): from this point a sequence (likely to be infinite) of period-doubling/period-halving occurs, leading to an aperiodic attractor. Furthermore, the appearance of different orbits by tangent bifurcation lead to further period-doubling/period-halving sequence for higher values of T. Also in this case we have restricted our analysis to the first sequence of bifurcations up to an orbit of period P ≈ 244, and the results are reported in Tab. 2: once more the values of δ_i in the sequence rapidly approach the Feigenbaum constant governing the appearance of the aperiodic attractor. The analogy in the phenomenology observed for variations of T and ρ_2 is not surprising since both parameters, even if with different roles, characterize the adaptation level of capitalists and workers.
Investigations in order to have a more complete picture are still being performed. The task is heavy, in particular for the number of parameters involved: AUTO succeeded in continuing the first period-doubling/period-halving , but failed in continuing points of higher period, probably because of the complexity of the phenomenology.

Conclusions

The first general conclusion which should be drawn from our study is that Lotka Volterra-type models remain a good basis for the study of dynamical phenomena exhibiting cyclic behavior. As we have shown, in fact, through two supplementary (simple) assumptions – the removal of Harrod-neutral technical progress in order to consent capital/labor substitution, and the introduction of adaptive hypothesis concerning agents' behavior – the validity of GM is substantially extended, since trajectories lie into the meaningful domain, and structural stability is re-established, leading to a richer and more interesting phenomenology.
As an example the "overshooting effect", i.e., the emergence of small scale-cycles in the presence of a limit cycle, after destabilization of the Goodwin equilibrium point, can be considered interesting, since this phenomenology seems to be characteristic of actual time series of many industrialized countries [Harvie, 2000]. This effect can be considered a direct consequence of our hypothesis on capitalists' response to wage share increases, described by the reaction-

substitution function. Indeed, as we have shown, the relevance of overshooting increases as the ζ function (controlled by ζ_0 and m) becomes steeper.

Furthermore the model can exhibit a dynamics that is by far more complex because of a number of transitions to chaos characterized, strictly speaking, by aperiodic behavior. However all the transitions observed occur according to Feigenbaum scenario, which preserves a basic "periodicity", differently from other possible transition scenarios. A relevant result is that also this complex dynamics takes place in the meaningful domain of the model, with the presence there of strange attractors. The phenomenology, which seems to be (more) consistent with the actual time-series of economic variables under consideration ([Brody-Farkas, 1987]), can be ascribed to the joint effect of our two assumptions, in particular the adaptive one. In fact, as we have seen, fade-out memory affects the competitors skill in the bargaining conflict: for some combination of the parameters controlling the memory span (T) and its effect in the bargaining process (ρ_2, ζ_0 and m), the system shows sensitive dependence on initial conditions.

Another relevant feature is represented by the presence of multiple stable/unstable orbits for some regions of the parameters space: particularly, as we have seen, a number of attracting orbits –

entered in the system by tangent bifurcations – appear, which lie, in part, inside the chaotic region. This occurrence of bistability is very interesting, since it may imply the presence of hysteresis in the system. As we have shown in Figs. 7-10, further increases of workers' bargaining power (ρ_2) lead to the sequence: period-doubling- aperiodic behaviour-period-halving, eventually restoring a "regular" dynamics. If we then take the final point as starting condition, and come back decreasing (ρ_2), the system may be attracted by an orbit of low period appeared by tangent bifurcation, and no period-doubling cascade starts (see Fig. 10).

Furthermore, as we have seen, when the memory span of the system T is "large enough" a richer dynamics appears, given the presence of tangent bifurcations leading to multiple cascades of period-doubling/period-halving bifurcations. Even if the original GM is based on starkly simplified hypotheses, it was able to capture for the first time the cyclical nature of the distributive conflict. Our extension allows to describe further relevant mechanisms in the economic cycle such as subcycles, hysteresis and sensitive dependence on initial conditions, consenting at the same time to forecast employment "boom" even in the presence of aperiodic behavior.

On this line, further modifications of the basic Goodwin's assumptions, kept in our model, could be implemented, for instance changing the hypotheses on capitalists' saving behavior and/or specifying in monetary terms the main variables. Such modifications are likely to lead to more realistic results, and are under consideration by the authors.

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Figures and Tables

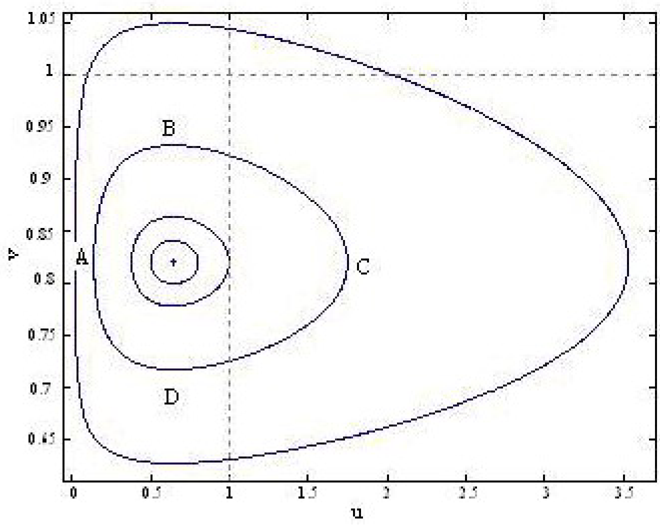

Figure 1
Original Goodwin model cycles. Orbits exceeding the meaningful domain (whose contour is represented by the dotted lines) are also shown . In this and in the next Figs., v = employment rate, u = wage share (a cross denotes the fixed point).

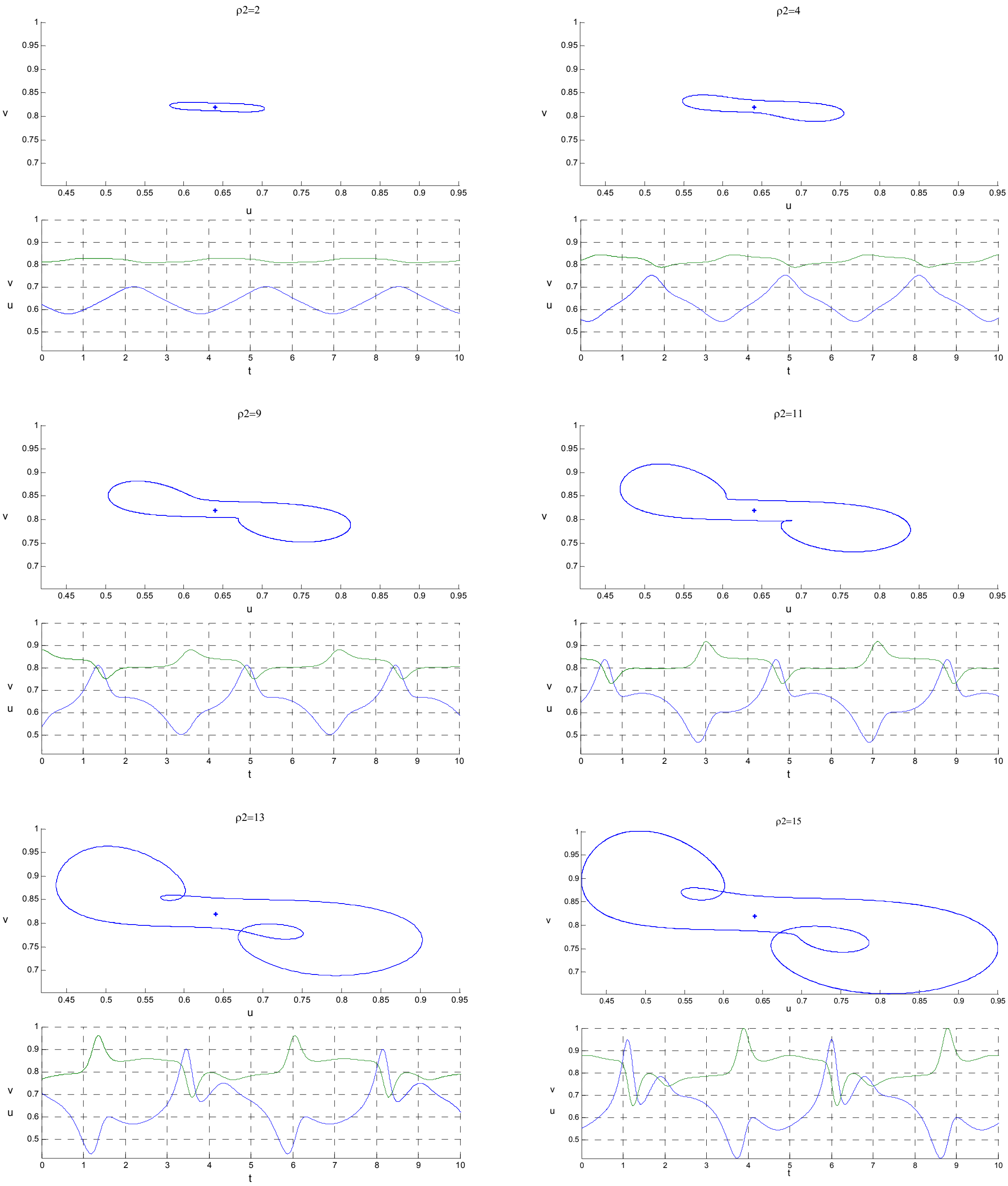

Figure 2

Changes of the limit cycles obtained for different values of ρ_2 (T = .3, ζ_0 = 100) with an anti-symmetric substitution function (m=0): symmetry of $\zeta(y)$ induces symmetry of the closed orbits. It can be observed that as ρ_2 increases the "overshooting" strengthens, until sub-cycles appear.

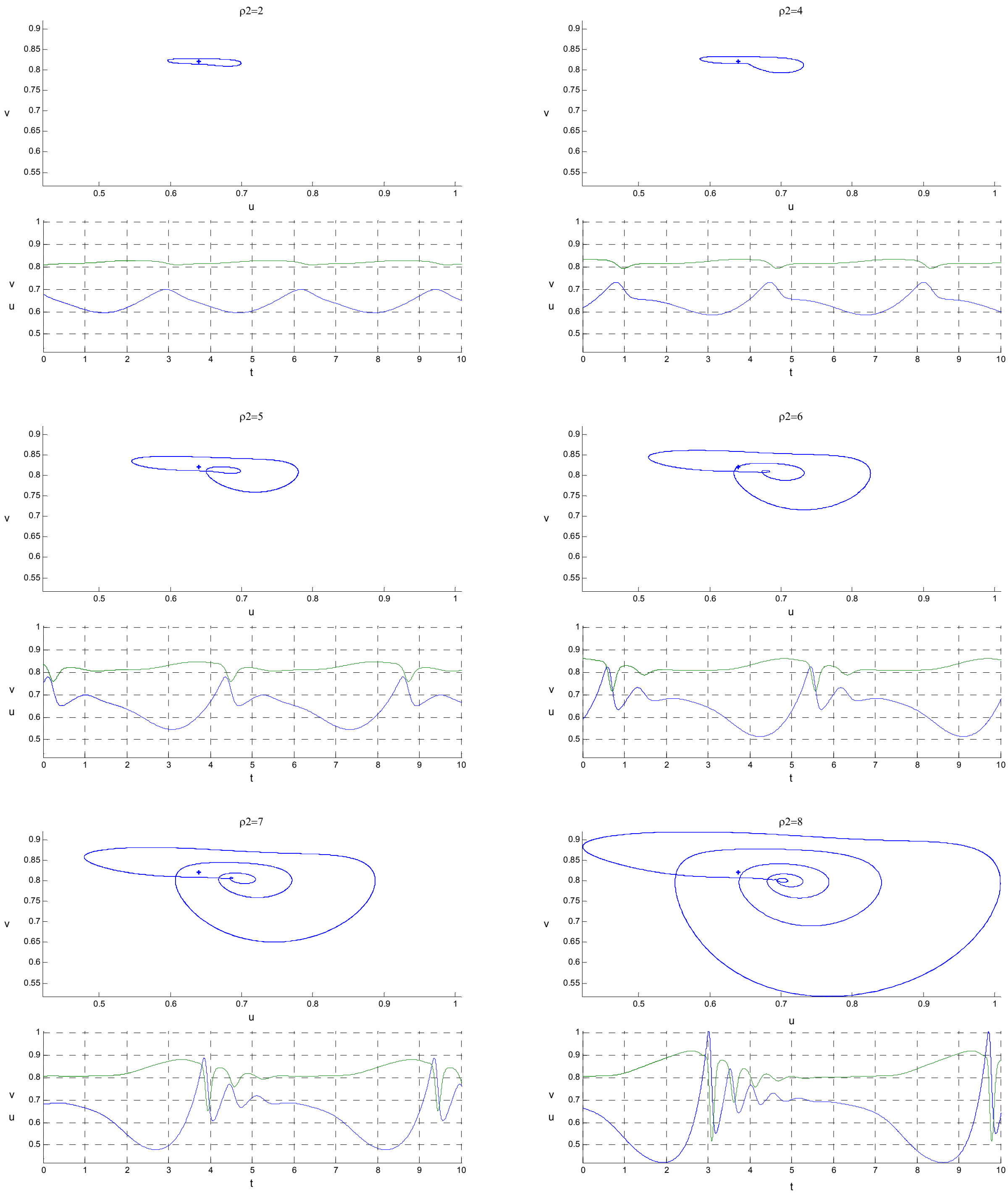

Figure 3

Changes of the limit cycles for variation of ρ_2 with an asymmetric ζ-function ($T = .3$, $\zeta_0 = 100$, $m = 15$).In comparison with the case $m = 0$ the overshooting is strengthened (the same increases of ρ_2 lead to more subcycles), even if it occurs only for low values of the employment rate.

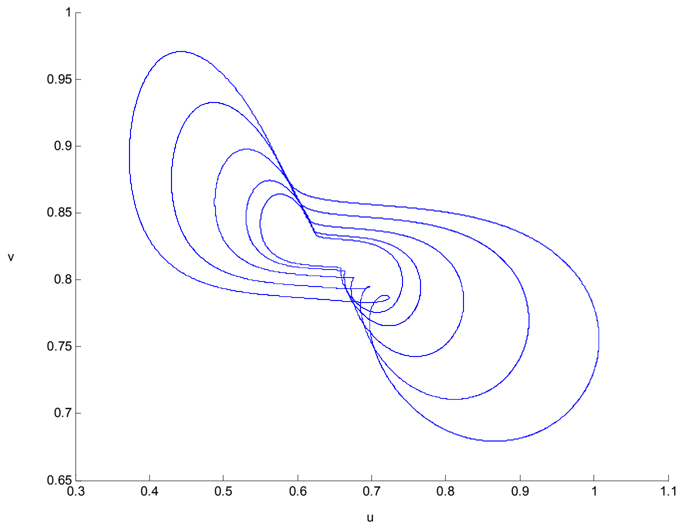

Figure 4
Limit cycles for $T = .3$ $\rho_2 = 10$ m = 0 and different values of $\zeta_0 = 30, 50, 100, 200, 300$. As ζ_0 increases stable orbits contract, without changing their shape. It also can be seen that the larger is the value of ζ_0, the smaller is the contraction effect.

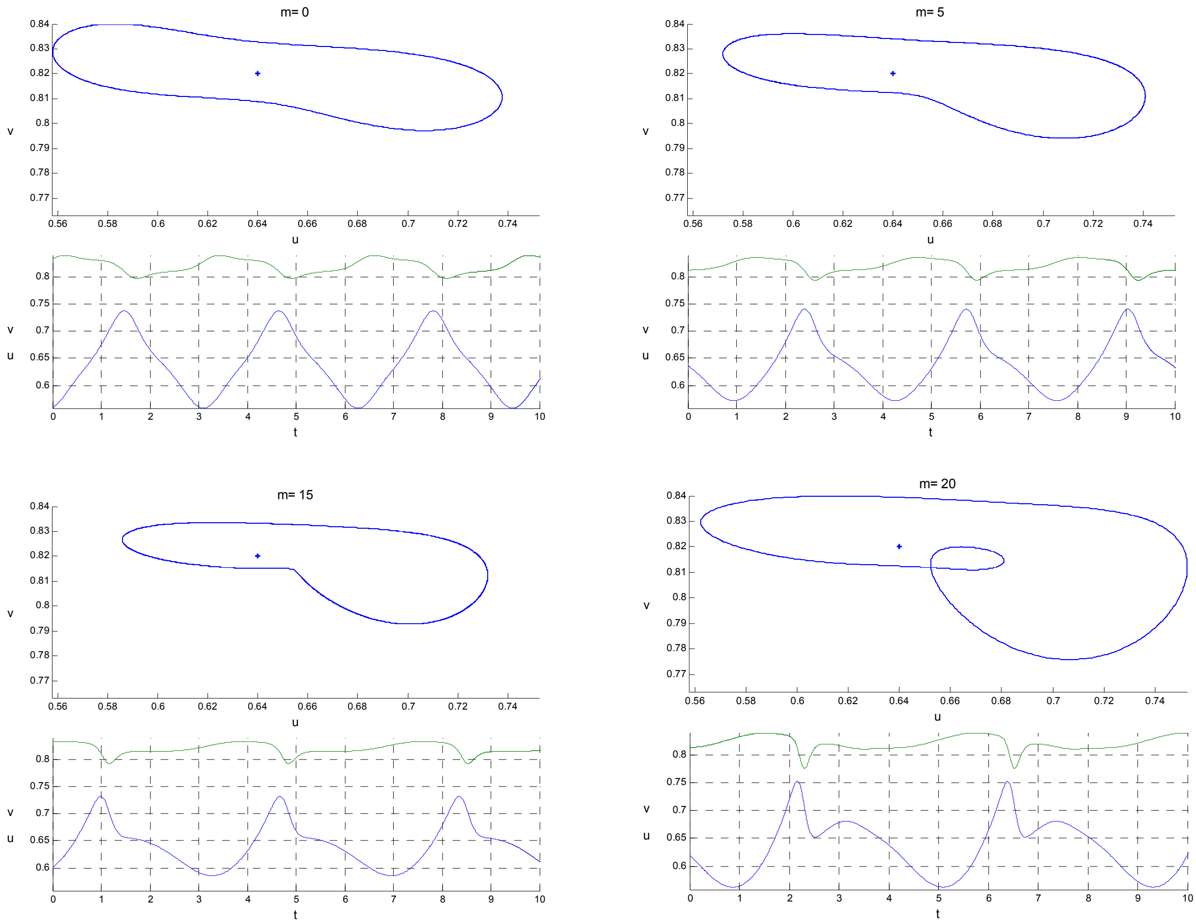

Figure 5
Limit cycles for increasing *m* (T = .3, $\rho_2 = 4$, $\zeta_0 = 100$). An increase in *m* affects the shape of the cycles, which loose their symmetry. As the efficiency of the K-L substitution is strengthened, subcycles increase both in size and in number.

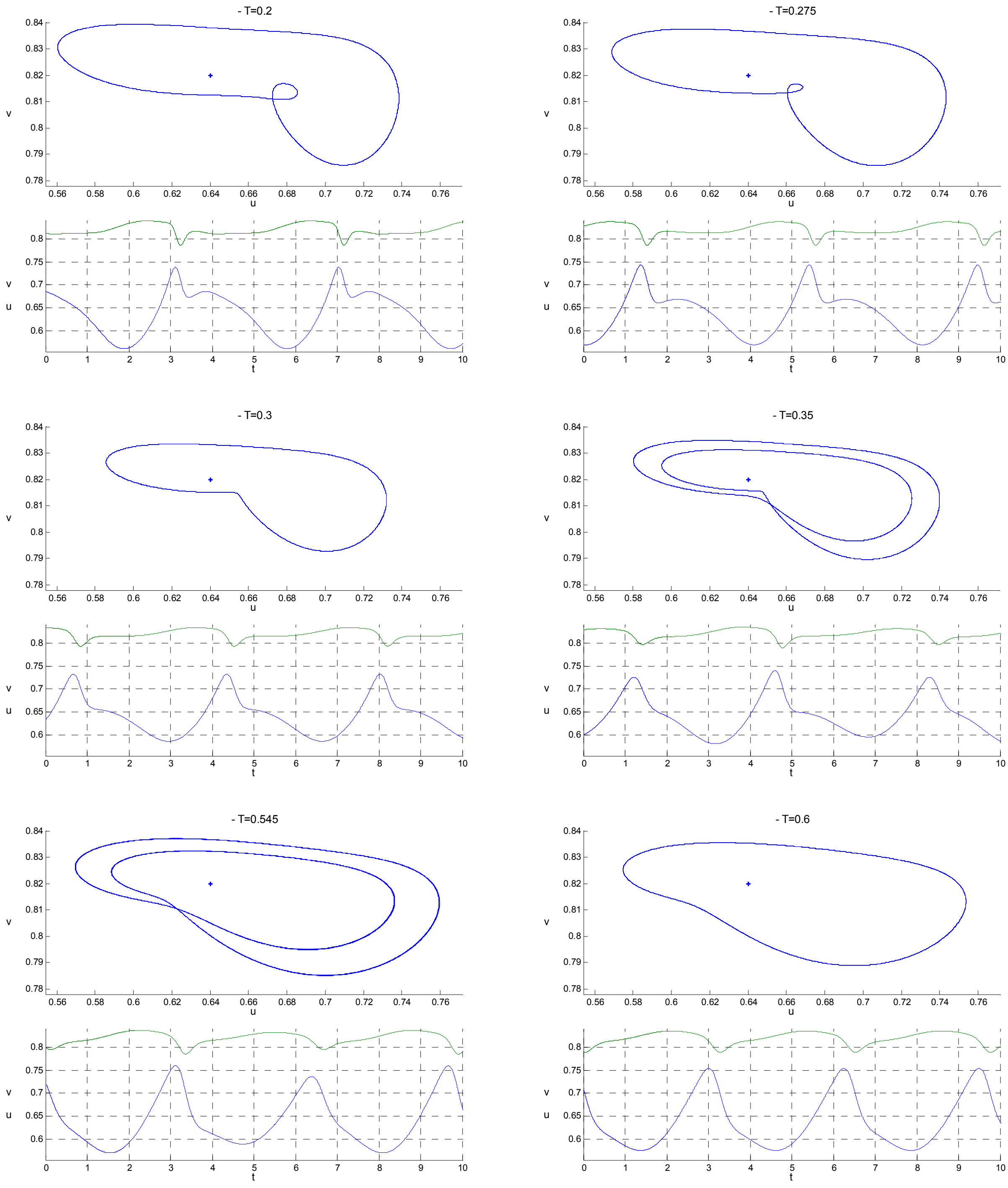

Figure 6

Effect on the limit cycles of increasing T (for ρ_2 = 4, ζ_0 = 100, m = 15). As is evident from the figures, as T increases sub-cycles disappear and further increments of T causes a period-doubling bifurcation followed by a period-halving bifurcation that re-establishes a more “regular” dynamics.

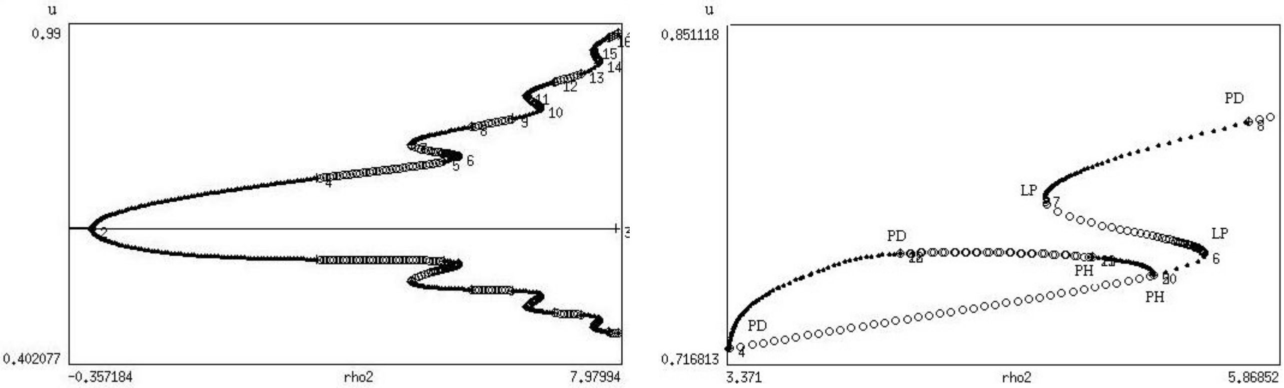

Figure 7
One-parameter bifurcation diagram (on the left), and a magnification of the same diagram after having continued the first period-doubling point (on the right), $T = .4$, $\zeta_0 = 100$, $m = 15$. Filled circles represent stable orbits, empty circles unstable ones; PD, PH, LP, stand respectively for period-doubling, period-halving and limit points (i.e., tangent bifurcations). Note the tangent bifurcation at $\rho_2 = 4.809$ (LP lab. 7) by which a stable orbit of low period appears (P = 4.63 at the bifurcation and increases up to 5.729 at the PD lab. 8): as will be evident in the following, this orbit is present together with chaotic regime. Furthermore, continuation analysis shows that PD lab.8 leads only to a doubled stable orbit with no further doublings.

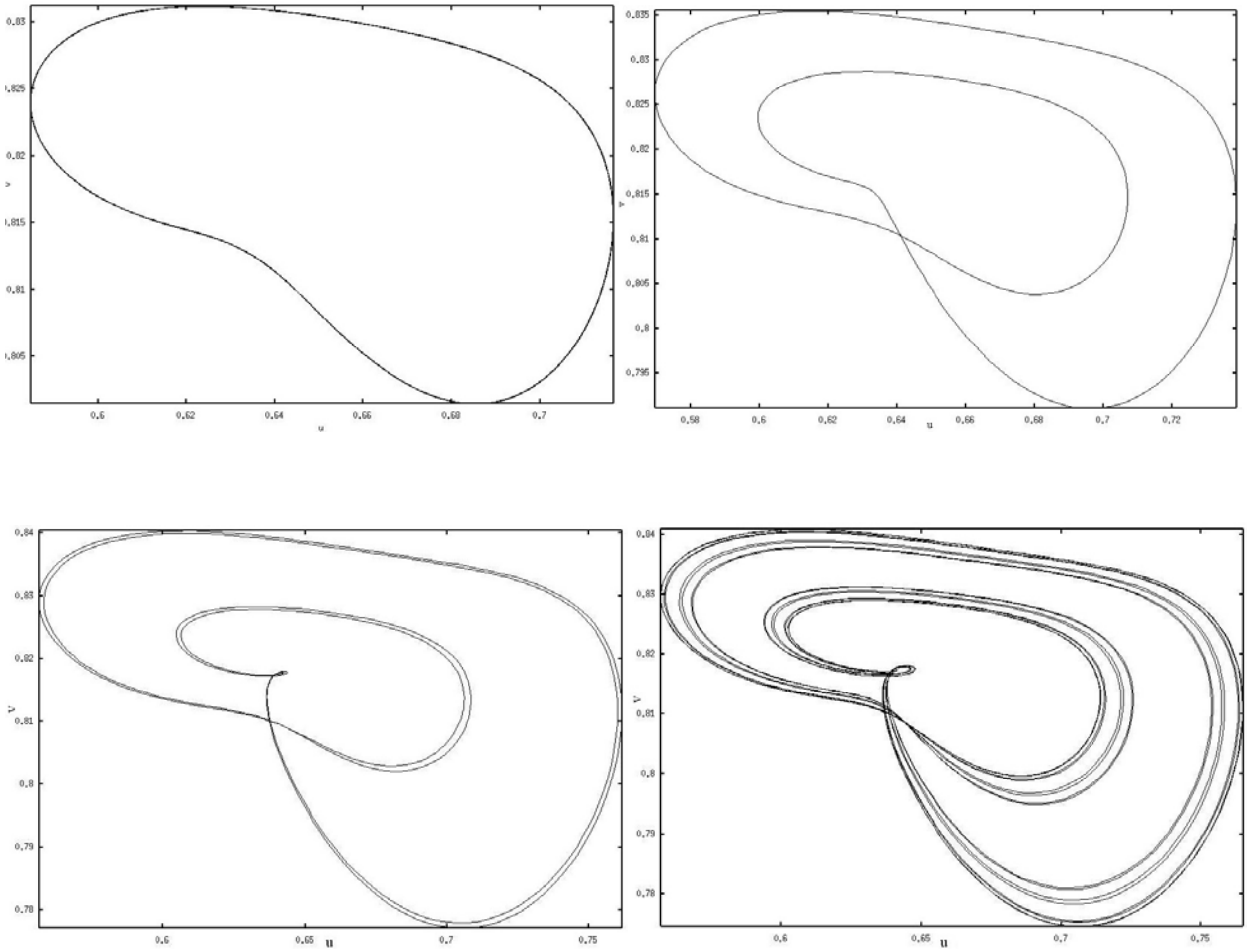

Figure 8
Sequence of period-doubling bifurcations as ρ_2 increases ($\rho_2 = 2.91$, 3.45, 4.16 , 4.4, with period respectively: P = 3.28, 6.78, 15.43, 61.25); $T = .4$, $\zeta_0 = 100$, $m = 15$.

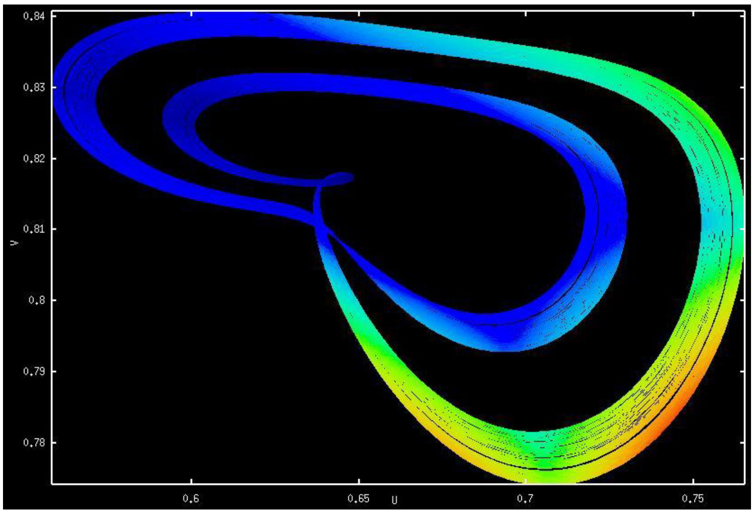

Figure 9

Aperiodic attractor found, after the period-doubling sequence depicted in Fig. 7, at $\rho_2 = 4.5$ ($T = .4$, $\zeta_0 = 100$, $m = 15$). Orbits are colored according to velocity: the minimum velocity is blue, the maximum velocity is red. The computed maximal Lyapunov exponent is: L≈ .037.

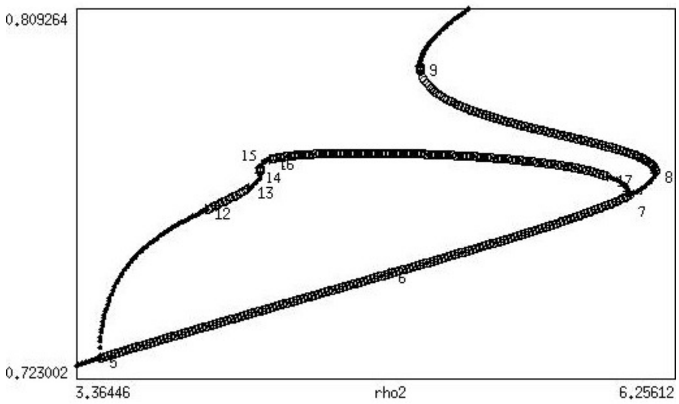

Figure 10

Magnification of the bifurcation diagram for T = .44 (ζ_0= 100, m = 15) after having continued the first PD point (lab. 5, ρ_2=3.478, whit period at the bifurcation P = 6.63). After the PD and PH point (ρ_2=3.992, lab. 12, and ρ_2=4.197, lab.13, with period at the bifurcation respectively: P = 14.51 and P = 7.671), the presence of two LP (lab. 14 and 15) leads to two other PD/PH points (ρ_2=4.3, lab. 16, and ρ_2=5.93, lab. 17, with respectively P = 16.51 and P = 8.11). Continuation analysis shows that from PD lab. 12 only a doubled stable orbit appears coalescing at PH lab. 13, while PD lab. 16 leads to two further PD/PH points (ρ_2=4.328, P = 32.84 and ρ_2=4.367, P = 16.23) followed by two LP and then by two other PD/PH points (ρ_2=4.393, P = 31.42 and ρ_2=5.905, P = 16.18). From then on there is a sequence, likely to be infinite, of PD'-PH-LP-LP-PD-PH-PD'-PH-LP-LP….where PD' represent bifurcations giving rise to no further doublings. Note the LP lab. 9 at ρ_2=5.027, by which an attracting orbit of low period, which is present together with chaotic regime, appears (P = 4.8 at the bifurcation).

i	$\rho_2(i)$	P_i	δ_{i-1}	$T(i)$	P_i	δ_{i-1}
1	3.386	6.68		0.3596	7.36	
2	4.156	15.31	4,245	0.38735	15.24	3.43
3	4.337	30.06	2.960	0.39544	30.54	4.004
4	4.398	61.26	4.066	0.39746	61.2	4.488
5	4.413	122.50	4.504	0.39791	122.4	4.591
6	4.416	243.00	4.642	0.398008	244.82	
7	4.417	490.00		…	…	
…	…	…	…			
7’	4.827	492.6				
6’	4.828	246.40	4.5			
5’	4.830	123.22	4.497			
4’	4.843	61.62	3.870	0.4531	63.2	
3’	4.890	30,84	2.885	0.4567	30.54	4.555
2’	5.026	15.51	2.032	0.4731	17.4	9.579
1’	5.303	7.752		0.6302	6.5	

Table 1-2

Bifurcation values ρ_2(i) and the sequence of δ_i approaching the Feigenbaum constant (T = .4, ζ_0 = 100, m = 15) (left) and sequence of bifurcation values of T (ρ_2 = 4.5, ζ_0 = 100, m = 15) (right). It was more difficult to find the sequence of period-halving bifurcations, given the presence of tangent bifurcation points right after the period-halving points.