Table of Contents

Go Back

Exploring Structure-Behavior Relations: Eigenvalues and Eigenvectors versus Loop Polarities

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Abstract

The main motivation behind this study is to clarify the distinction between the loop polarities and the eigenvalues/vectors of a system in the context of system behavior. To this end, the phase plane analysis is utilized to emphasize the need for the system dynamics practitioners to use more of the already available analytical tools in studying structure-behavior relations. The main advantage of phase planes is that one can observe the motion of system state on a space defined by system structure. Particularly the eigenvectors characterize the system structure on this space and create trajectories for the system state to follow depending on the initial conditions just like magnetic fields created by a magnet. It is also shown how investigating phase plane clarifies issues such as positive loops giving rise to goal-seeking or oscillating behaviors. The analysis is accompanied with the corresponding system stories. The main disadvantage of the phase plane approach is that at most three states can be represented at the same time on a phase plane.

Background

The fundamental axiom of system dynamics has been that structure drives behavior. The concept of feedback stems from this notion of endogenous sources for the creation of system behavior. In spite of this guiding principle, the accurate depiction of the relation between a model's structure and its behavior has mostly remained a mystery partly because there has been little study in the system dynamics field on this crucial subject. "Understanding model behavior" has claimed the first rank in a list of eight problem areas put forward as deserving the attention of system dynamics practitioners now and in the future (Richardson 1996). It should be noted though that the number of related studies seems to be increasing recently (Graham 1977; Forrester 1983; Richardson 1995; Modjahedzadeh 1996; Davidsen 1991; Ford 1999; Saleh and Davidsen 2000, 2001). As a result, there have been some improvements in revealing the structure-behavior relation through analytical and empirical means; however, the findings are still not enough to uncover most of the hidden and complicated dynamics that operate within –especially large–models.

It has been known that positive loops can generate goal-seeking or oscillating behavior; they can even exhibit shift from goal-seeking to exponential growth(decay). We are all too familiar with the single positive feedback loop that can give rise to exponential decay as well as exponential growth. It is obvious that even linear systems are not that trivial (Graham 1977; Richardson 1995; Saleh and Davidsen 2001). What kinds of feedbacks give rise to what kinds of behavior under which conditions is still not well-understood. This important research problem has been ignored probably as the examples giving rise to these strange behaviors are overlooked as uncommon. Nevertheless, we live in a mostly nonlinear world; the analysis of transients in linear systems, apart from being interesting by themselves, may also lead to better understanding of nonlinear systems.

Some of the material presented in this work has been previously addressed in some form or another (Graham 1977; Richardson 1995; Sterman 2000). It is the author's belief, however, that most of this material is not well known to the wider system dynamics community. Therefore, even though the observations in this paper may have some significance for the practice of system dynamics the paper's major purpose is to clarify the distinction between loop polarities and eigenvalues/vectors of a system in the context of system behavior.

One of the objectives of this study is to show that there are already well-developed analytical tools in the literature that can be utilized more in the system dynamics field. To make the case, the state space and phase plane methods are introduced to analyze structure-behavior relation in simple system dynamics models. The state space methods and phase plane analysis are important tools in the analysis of differential equations. While state space methods provide an automated approach to the solution of differential equation systems, phase plane analysis provides a proper way of understanding how the system behavior unfolds through comparison of the behavior of system states. Although state space representation has found some use, it is interesting that phase plane analysis have rarely been applied in the analysis of system dynamics models (Graham 1977; Aracil 1981, 1986; Özveren and Sterman 1989; Sice *et al.* 2000). This probably has been the case because the state space representation is suited better for linear models while the phase plane analysis is not very suitable for high-order models. In this paper, it is shown how both methods can be used together to bring about a renewed understanding to the unfolding of behavior of linear systems. This will be complemented with narrative explanations to demonstrate the prospects of using analytical and empirical/narrative methods concurrently. Along the way, there will be new light on unexpected behaviors of systems.

Methodology

The idea of state-space comes from the state-variable method of describing differential equations. In this method, the differential equations representing the system structure are organized as a set of first-order differential equations in the vector-valued state of the system, and the solution can be visualized as a trajectory of this state-vector in state space (Franklin *et al.* 2002). The ordinary differential equations (ODEs) do not have to be linear and/or time-invariant for the state-space method to be implemented although the examples in this paper are linear and time-invariant. Having them in state-variable form gives a compact, standard form to study. The general state-variable representation is:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

where $\mathbf{x}(t)$, $\mathbf{u}(t)$, and $\mathbf{y}(t)$ are n-dimensional state, m-dimensional input, and k-dimensional output vectors; and $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$, and $\mathbf{D}(t)$ are coefficient matrices with $n*n$, $n*m$, $k*n$, and $k*m$ dimensions, respectively.

The above representation is generic. The coefficient matrices may or may not be time-dependent. Furthermore, for all practical purposes, only the first equation will be considered in the subsequent examples and $u(t)$ is taken zero (i.e. there is no exogenous forcing function on the system in question). Once put into state-space form, the analysis may continue on the phase plane. As mentioned above state-space approach has been utilized time to time in the system dynamics literature (Özveren and Sterman 1989).

The phase plane approach is derived from the geometric or qualitative theory of differential equations. This approach enables one to make use of the ideas of geometry in analyzing differential equations. On the phase plane, which can be considered as a subset of state-space, the concepts of distance and of orthogonal and parallel lines, as well as other concepts from geometry can be useful in visualizing the solution of an ODE as a path in the state space (Arnold 1978).

Consider a second order differential equation with states x_1 and x_2. The solution of this differential equation can be viewed as a parametric representation for a curve in the x_1x_2 plane. This curve, called *trajectory*, is specified by the differential equation (or in other words, by the very nature of the system) and characterizes how x_1 vs x_2 behaves. The x_1x_2 plane is called the *phase plane*, and the set of trajectories is referred to as a *phase portrait*. In short, given a dynamic system of equations with n states, along with analyzing the change in the state variables through time (i.e. $x_1(t)$, $x_2(t)$, …,$x_n(t)$) one can also analyze how state variables evolve with respect to each other (e.g., how x_1 vs. x_2 behaves) on the state space (e.g., on the x_1x_2 plane).

The phase plane is defined by the eigenvectors of the system. Their combined effect anywhere on the phase plane determines the direction of the trajectory at that point. The eigenvectors on phase plane act as attraction rays that stretch the space just like magnetic fields. This analogy is not purely imaginary: the trajectories on a phase plane cannot cross each other or themselves (unless they are closed orbits in which case they represent limit cycles), as the magnetic lines cannot. In addition, through any point, there is one and only one trajectory as a result of the existence and uniqueness of the solutions of differential equations.

They are useful in visualizing and analyzing the phase plane trajectories. The contribution of each eigenvector on the state vector at a specific point on the state space is equivalent to the component of state vector on each eigenvector at that point. This phenomenon is observed on Figure 1 for a second-order linear system. In the figure, there are two eigenvectors: *eigenvector 1* associated with the positive eigenvalue and *eigenvector 2* associated with the negative eigenvalue. *State vector* shows the system's direction of change at the point located with black dots on the phase plane on the left and the enlarged image on the right. In addition, *s1* and *s2* are the components of the *state vector* along the *eigenvector 1* and the *eigenvector 2*, respectively. The isoclines are contours of equal slope. They are generally defined for zero slopes. For example, for a two state system where the states are defined as x_1 and x_2, there are two zero-slope isoclines on the phase plane: one where $\partial x_1 / \partial x_2 = 0$ and the other where $\partial x_2 / \partial x_1 = 0$. The phase plane method has not seen wide spread use in the analysis of system dynamics models except some rare work such as Graham (1977), Aracil (1981, 1986), and Sice *et al.* (2000).

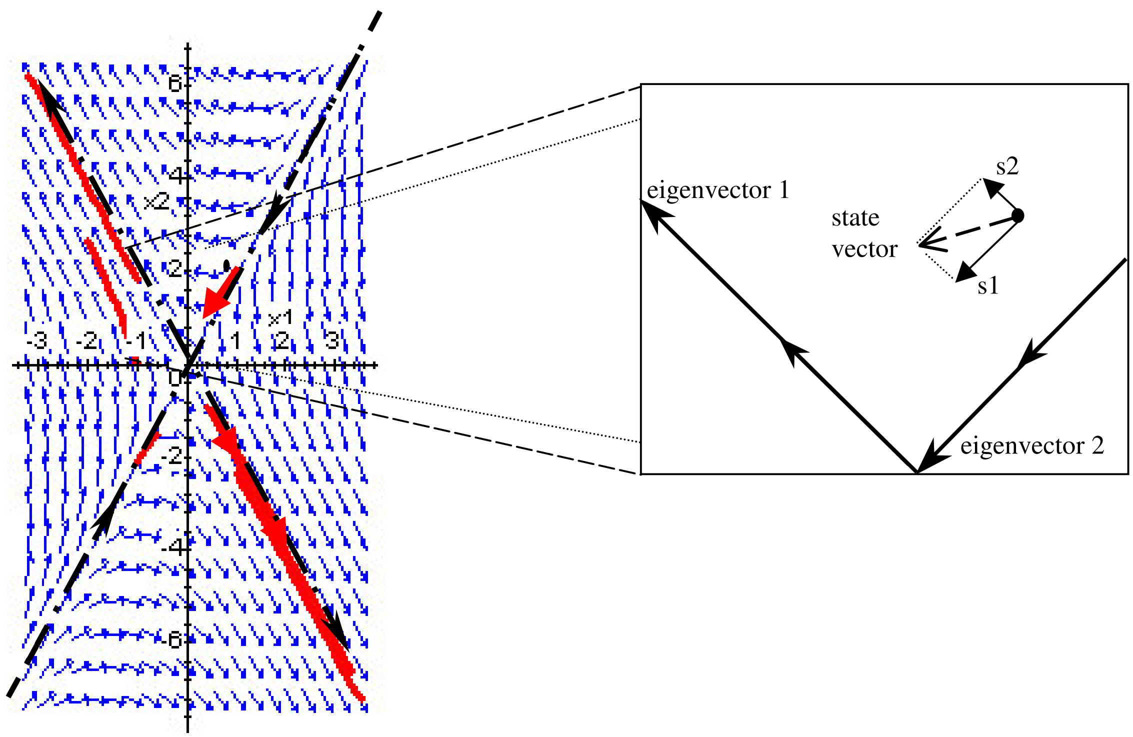

Figure 1. The phase plane and the decomposition of the state vector to its components for a second order system.

The next section, the phase plane analysis will be employed to understand how structure drives behavior. In the process, it will shed light why and how controversial behavior is generated in some specific situations.

Application on two homogeneous second order linear systems

The system composed of positive loops only

The first example is a simple second order linear system consisting of three positive feedback loops. The causal loop diagram where stocks and flows are represented explicitly is given in Figure 2.

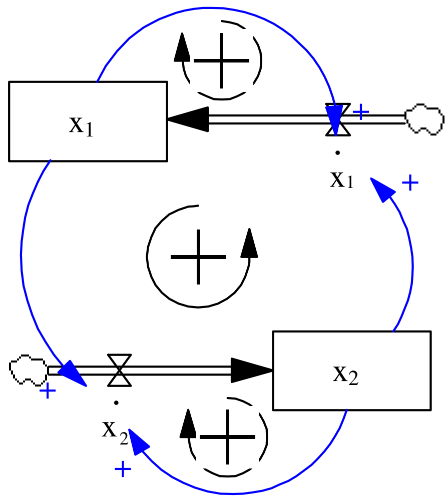

Figure 2. The casual loop diagram.

The state space representation of this system is

$$\dot{x}_1(t) = 1 * x_1(t) + 1 * x_2(t) \tag{1a}$$

$$\dot{x}_2(t) = 4 * x_1(t) + 1 * x_2(t) \tag{1b}$$

The matrix notation of the same system is

$$\dot{\mathbf{X}}(t) = \mathbf{AX}(t)$$

where the gain matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.

The eigenvalues and eigenvectors of the system, respectively, are

$$\lambda_1 = 3 \text{ and } \lambda_2 = -1$$

$$\xi_1 = \begin{bmatrix} 0.4472136 \\ 0.8944272 \end{bmatrix} \quad \text{and} \quad \xi_2 = \begin{bmatrix} -0.559017 \\ 1.118034 \end{bmatrix}$$

Now we know that positive loops typically generate run-away behavior such as exponential growth or decay. A system purely composed of positive loops is more than likely to generate the same type of behavior. In fact, for a range of initial conditions the system's behavior is just as expected. An example is provided for $x_1(0) = 1$ and $x_2(0) = 2$ in Figure 3.

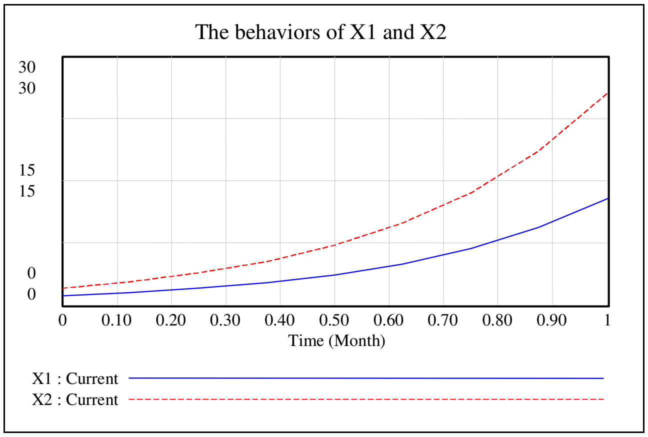

Figure 3. The behaviors of x_1 and x_2 for $x_1(0) = 1$ and $x_2(0) = 2$.

But how about for a different set of initial conditions like $x_1(0) = 0.6$ and $x_2(0) = -1$? While x_2 is still exhibiting exponential growth x_1 does something strange: first a foray towards the unstable equilibrium point (0,0) of the system then it follows the exponential growth pattern (Figure 4).

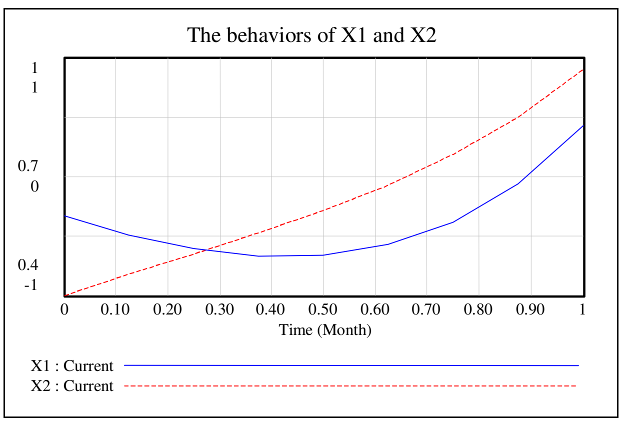

Figure 4. The behaviors of x_1 and x_2 for $x_1(0) = 0.6$ and $x_2(0) = -1$.

Here although the steady-state behavior is exponential growth, the system exhibits goal-seeking behavior during the transient phase. This not only shows that positive loops are capable of generating goal-seeking behavior (a phenomenon that we already know) but also that dominance shifts are possible even for linear systems (a fact not acknowledged frequent enough). But why one set of initial conditions generates pure exponential growth while the other generates transient goal-seeking behavior with a dominance shift? Additional experimentation shows that the transient goal-seeking behavior emerges only for certain initial conditions. What's more, finding these initial conditions through trialerror turns out to be prohibitively difficult. However, the exact expression of the system is formed through its eigenvalues and eigenvectors. Hence, it is possible to reveal the regions of state space that give rise to transient behavior on the phase plane. The phase plane analysis where the coordinates are defined by the system states tells us that not only the structure, but also the initial conditions of the system plays a role in how the system behavior evolves over time. This may be regarded as trivial for linear models but has important implications for nonlinear models.

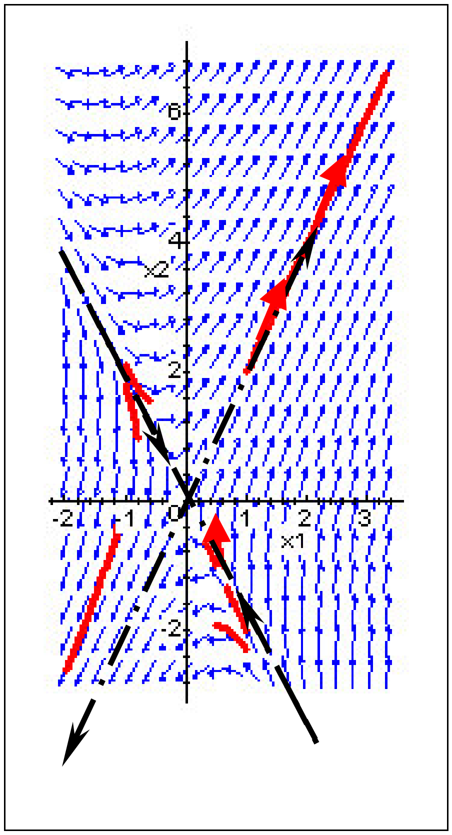

Figure 5. The phase plane characterization of the first system (arrows indicate the direction of each trajectory).

The phase plane with the eigenvectors, isoclines, and trajectories is given in Figure 5. The individual trajectories of several runs with different initial conditions are also shown. The particular initial states we used above are shown with a different color. Now we can see that as long as the initial state is not exactly on the eigenvector associated with the negative eigenvalue, the system explodes (i.e. goes to infinity). But during the transient phase, the system may exhibit goal-seeking behavior depending on the location of the initial state of the system on the phase plane. Another word of caution is that although, almost no matter where the system starts, it eventually falls under the influence of the eigenvector associated with the positive eigenvalue it would be wrong to assume that the run-away behaviors are all identical. As a result, once again the mystery around the positive loops giving rise to goal seeking behavior disappears if we try to understand the structure through analytical means. Then we are able to see the real mechanism beyond the loop polarities.

The system with coupled positive and negative loops

Yet another example for a non-trivial linear system is a second order system with a central positive loop and two coupled minor negative loops (Figure 6).

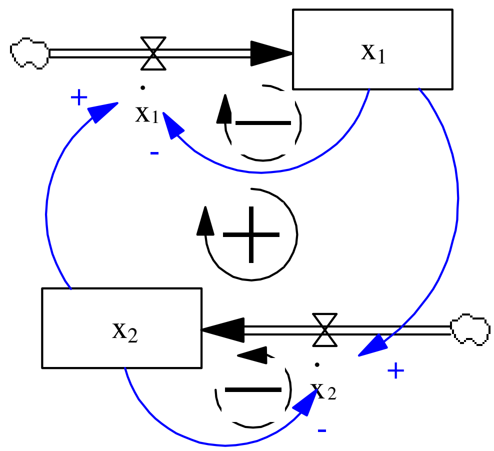

Figure 6. The casual loop diagram.

The state space representation of the system is

$$\dot{x}_1(t) = -0.5 * x_1(t) + 0.1 * x_2(t) \tag{2a}$$

$$\dot{x}_2(t) = 0.7 * x_1(t) - 0.3 * x_2(t) \tag{2b}$$

The matrix notation of the same system is

$$\dot{\mathbf{X}}(t) = \mathbf{AX}(t)$$

where the gain matrix $\mathbf{A} = \begin{bmatrix} -0.5 & 0.1 \\ 0.7 & -0.3 \end{bmatrix}$.

The eigenvalues of the system are

$$\lambda_1 = -0.6828427 \text{ and } \lambda_2 = -0.1171573$$

$$\xi_1 = \begin{bmatrix} -0.4798415 \\ 0.8773552 \end{bmatrix} \quad \text{and} \quad \xi_2 = \begin{bmatrix} -0.3684064 \\ -1.4104172 \end{bmatrix}$$

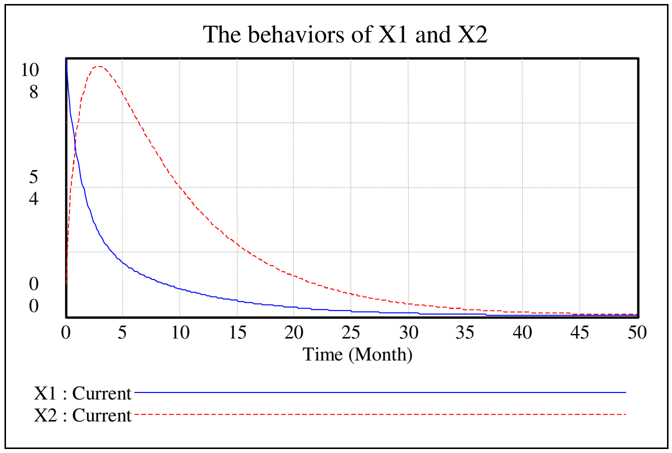

Figure 7. The behaviors of x_1 and x_2 for $x_1(0) = 10$ and $x_2(0) = 1$.

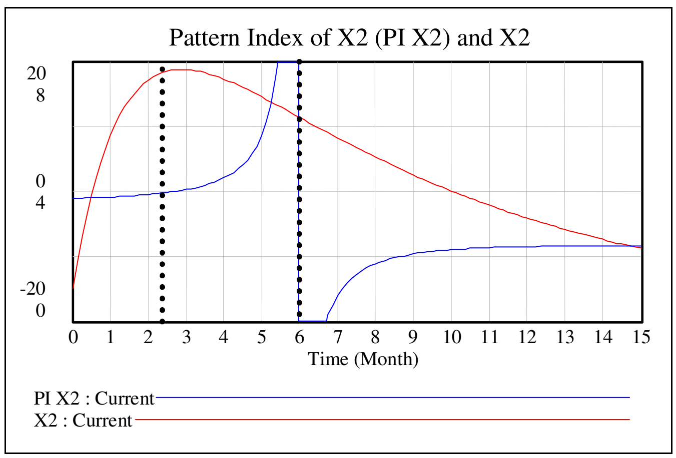

Figure 8. The pattern index of x_2 (PI x_2) and x_2 for $x_1(0) = 10$ and $x_2(0) = 1$.

The system's both eigenvalues are negative, so naturally we expect the system to exhibit goal-seeking behavior. The behavior of x_1 is as such, however, the behavior of x_2 does not conform to our expectations (Figure 7). The pattern index graphs introduced by Saleh and Davidsen (2000) also confirms that for a certain time interval (in between the dashed lines in Figure 8), x_2 exhibits run-away behavior.

The phase plane with the eigenvectors, isoclines, and trajectories for this system is given in Figure 9. Once again, the initial system state together with the eigenvectors and corresponding eigenvalues (i.e. the structure) of the system determine the trajectory (or path) to be followed.

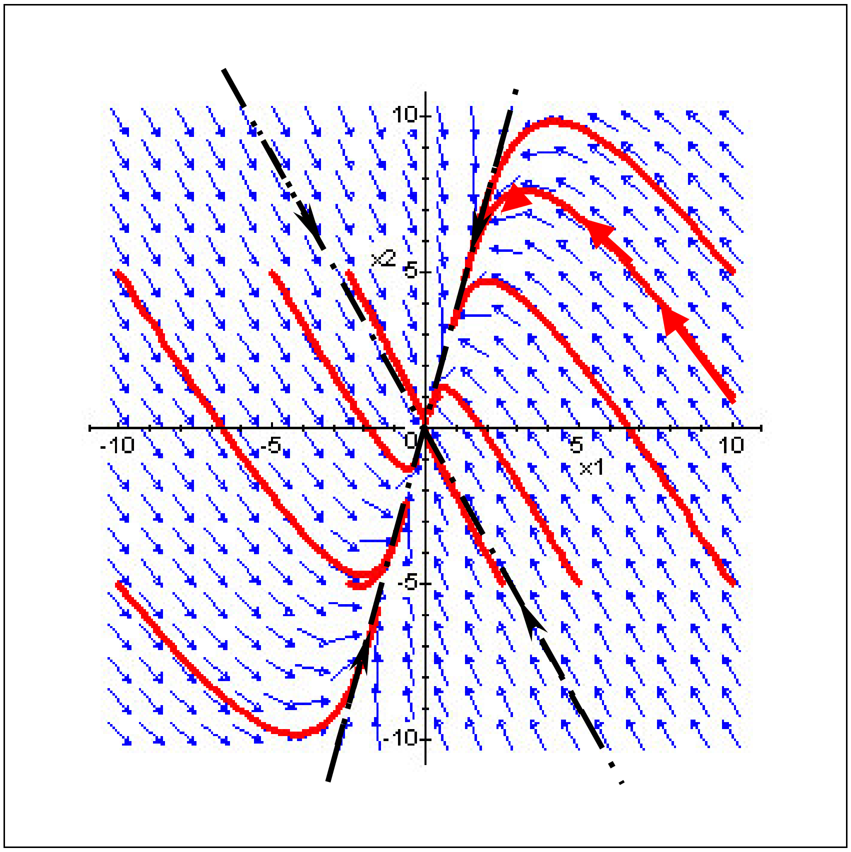

Figure 9. The phase plane characterization of the second system (arrows indicate the direction of each trajectory).

Discussion

The phase planes reveal why the system exhibits certain behavior modes at certain stages. This is because the phase plane is completely portrayed by the system structure through its eigenvalues and eigenvectors. One can trace the path of the system state on the phase plane beginning from any initial condition. The direction of the state vector at any point on the phase plane is dictated by the relative strength of each eigenvector at that point. In other words, system behavior arises from a linear combination of the dynamics associated with the eigenvalues of the linear matrix of system structure. It is, however, more fruitful if we combine this analytical analysis with “system stories” (Modjahedzadeh and Andersen 2001).

System stories are systematic verbal accounts of how model structure generates observed behavior patterns. They help to communicate the system’s dynamics to the less mathematically advanced in a simple language. For that reason, in order to make more sense out of the elementary phase plane analysis presented, the narrative explanations of the behavior patterns on Figures 4 and 7 are given in the following.

In the first system, although the link gains associated with $\dot{x}_1$ are the same, since the magnitude of the initial value of x_2 is greater than that of x_1, x_2 causes x_1 to get less positive through its rate $\dot{x}_1$ (Eq. 1a). x_1 decreases but it is still positive and its respective link gain associated with $\dot{x}_2$ is sufficiently greater than that of x_2 so that $\dot{x}_2$ is positive and hence x_2 becomes less negative. As x_2 approaches zero the decrease in x_1 also gets smaller until $\dot{x}_1=0$ (i.e. $-x_1 = x_2$) (Figure 5). Before this point the contribution of the eigenvector associated with the negative eigenvalue was larger than that of the eigenvector associated with the positive eigenvalue. After this point x_2 continues to become less negative and since $|x_1|>|x_2|$, x_1 begins to grow. This eventually leads to exponential growth in both states. In other words, the contribution of the eigenvector associated with the positive eigenvalue gets larger than that of the eigenvector associated with the negative eigenvalue.

In the second system, x_1 exhibits exponential decay throughout the simulation. However, during initial phases, its value is much greater than that of x_2. Therefore, the strength of the central loop practically eliminates the effect of the negative loop composed of x_2 and $\dot{x}_2$ and x_2 increases rapidly until x_1 falls to a level that the change in x_2 is temporarily zero (i.e. $\dot{x}_2 = 0$). This point is marked on the phase plane (Figure 9). Until this point, both states exhibit goal-seeking behavior but x_2 seems to be fooled by x_1 and approaches a non-existing positive equilibrium point. When x_1 falls further, x_2 once again fooled this time to exponential decay because x_1 falls faster than x_2, which causes $\dot{x}_2$ to become larger in subsequent time steps creating an artificial positive loop effect. However, x_1 decreases with a decreasing rate while x_2 falls with an increasing rate. At some point – which is time 6 in Figures 7 and 8– the superficial effect caused by relative decrease speeds of both states vanishes and the negative loop composed of x_2 and $\dot{x}_2$ becomes dominant and drives both states smoothly toward the single equilibrium point (0,0) of the system.

The exponential decay of x_2 can be observed on the phase plane although it is not associated with any of the system eigenvalues. The reason for the occurrence of such behavior is due to the extreme difference between the magnitudes of system eigenvalues. Namely, the first eigenvector is so powerful that the second one cannot exert its influence until the state vector comes very close to it. This leads to the strange behavior pattern exhibited by at least one of the states in the model. In this case, it is always x_2 because of the particular system structure (Eq. 2; Figure 9).

The system narratives given above complement the analytical results drawn from the phase plane analyses. In the end, the coupling of the two approaches hopefully leads to a more complete understanding of the systems' dynamics.

In linear systems that have at least one positive eigenvalue, no matter how relatively small a magnitude it has, the positive eigenvalue(s) eventually dominate(s) the system behavior and the system explodes except for certain initial conditions (Franklin *et al.* 2002). These initial conditions are the ones located exactly on the eigenvectors associated with negative eigenvalues of the system. However, the negative eigenvalue(s) may still dominate(s) the system at the transient phase depending on the initial state. The transient goal-seeking response of the first system can be clearly seen on its phase plot (Figure 4). This shows us that looking at the link gains alone may fail to reveal the behavior pattern exhibited by the system. The traditional definition of loop polarity serves as an indicator to get a "feel" for the system structure and behavior. Although we made use of the loop polarities in the system narratives the phase plane approach gives a clearer and more analytic picture than using loop polarities to explain model behavior. Nevertheless, considering the communication challenges, it seems the best way is to complement the two approaches in an analysis.

The following example clarifies the distinction between eigenvalues and eigenvectors of a system. The model given below has a different structure than our first example (Figure 10). The difference is that this model has two negative links but the loop polarities are the same as before. Although both models have the same eigenvalues their eigenvectors, hence the phase planes defined by the two systems are different (Figures 5 and 11). Therefore, looking only at eigenvalues is not sufficient too to understand system behavior. They only characterize certain behavior modes but how those behavior modes unfold is dictated together with the system eigenvectors.

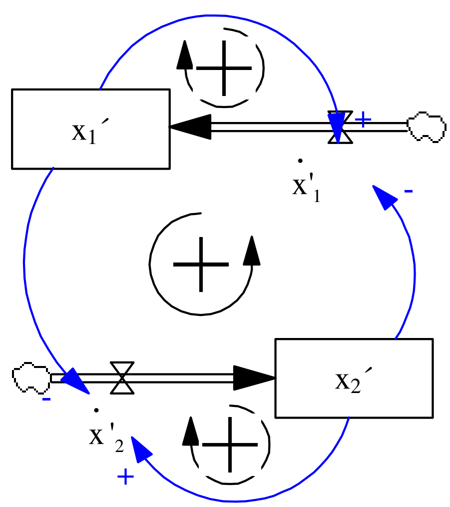

Figure 10. The structurally altered version of the first example.

The use of analytical methods is essential in understanding how structure drives behavior in system dynamics models. In this respect, determining dominant structure responsible for dominant behavior modes becomes important too. Forrester (1983) introduced eigenvalue analysis to investigate dominant structure. More recently, Saleh and Davidsen (2001) improved this approach. Richardson (1986) gives qualitative explanations on the pitfalls of causal loop diagrams. Richardson (1995) investigates loop dominance through analytical means. He refers to positive loops coupled with negative loops and concludes that minor negative loops are responsible for the generation of goal seeking behavior when the gain of positive loop is less than one. Saleh and Davidsen (2001) using system eigenvectors show that it is possible to observe goal seeking behavior even in a system with a single positive loop and that behavior mode shift is possible even for that system.

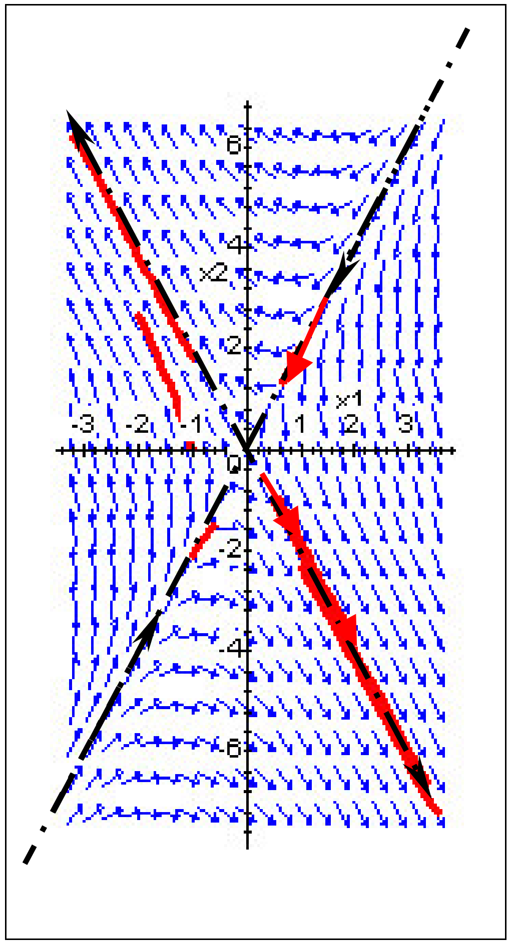

Figure 11. The phase plane of the altered system (arrows indicate the direction of each trajectory).

This study, on the other hand, brings another explanation to this phenomenon using state space and phase plane techniques. The mathematical/analytical elaboration here is intended to serve as a complement to Richardson (1986) where he gives more qualitative explanations and points out “predicting behavior from loop polarity alone without regard for distinction between rate-to-level links and information links is impossible”. In reality, the dominance shift does not take place between different loops but different eigenvalues of the system. That is why it is possible to observe dominant behavior shift in a second order system with only a single positive loop. The system’s behavior is dictated by the initial conditions not alone by the structure and the structure is exactly defined by the system eigenvalues and eigenvectors. It is easy to see that if initially the component of the state vector of the model on the eigenvector associated with the negative eigenvalues is greater than that on the eigenvector associated with the positive eigenvalues the system exhibits transient goal seeking behavior (Saleh and Davidsen 2001). Thus, it is true that even linear systems with order higher than one are capable of exhibiting behavior mode shifts.

Finally, it is useful to think of eigenvectors on state space as attraction rays that stretch the space just like magnetic fields. During the transient phase for linear models and in general for nonlinear models, it matters where you start on this plane. Hence, this may be a useful concept in explaining ‘strange’ behavior patterns in the transient phase. It is also worth mentioning that eigenvalue and loop analysis are valid for long-term steady-state behavior unless the analysis does not take into account the phase planes.

Conclusion

Understanding the relation between structure and behavior is at the core of the system dynamics field. Nevertheless, we have come a limited way in achieving this objective. In this paper, analytical tools that are already available are presented with a different perspective for the analysis of structure-behavior relation in linear system dynamics models, which would also be useful in the analysis of nonlinear systems. The state space representation and phase plane analysis enabled to stress the importance of the system eigenvalues and to make a distinction between them and loop polarities in terms of behavior. However, it should be noted that at most three states can be represented at the same time on a phase plane, which may be regarded as a disadvantage of the method.

The methodology is demonstrated on two second-order linear models. The first is composed only of positive loops whereas the second has both positive and negative loops. It is shown that it does not matter for a system what type of loops it is composed of. What matters for potential behavior modes it is capable of generating is the signs of its eigenvalues (and whether they are real or imaginary). Hence, concentrating on the signs of loops is potentially misleading.

The paper’s major purpose is to clarify the distinction between the loop polarities and the eigenvalues/vectors of a system in the context of system behavior. Another motive of this study is to emphasize the need for the system dynamics practitioners to utilize more of the analytical tools in analyzing structure-behavior relationships. What's more, the hardcore mathematical nature of analytical techniques should not be regarded as a discouraging factor in their use for model analysis. The tools and ideas borrowed from engineering control systems may bring so much to the field of system dynamics. In the end, it is the basis that gave rise to this field. This is, of course, does not mean that attempts to form better qualitative analysis tools are futile or of lesser value. To the contrary, these tools as demonstrated in this paper, are to complement each other.

The software in use in system dynamics should incorporate more technical support for understanding the connections between model structure and behavior. There are promising improvements in this direction such as the Digest software developed by Modjahedzadeh (1996). Nevertheless, it would be very nice if system dynamics software were to be able to provide phase plane plots when needed. The availability of such tools will certainly enhance the rate and level of understanding connections between model structure and behavior.

References

Aracil J. 1981. Structural Stability of Low-Order System Dynamics Models. *International Journal of Systems Science* 12:423-441.

Aracil J. 1986. Bifurcations and Structural Stability in the Dynamical Systems Modeling Process. *Systems Research* 3(4):243-252.

Arnold VI. 1978. *Ordinary Differential Equations*. The MIT Press. Cambridge, MA.

Davidsen P. 1991. The Structure-Behavior Graph. The System Dynamics Group, MIT. Cambridge.

Ford D. 1999. A Behavioral Approach to Feedback Loop Dominance Analysis. *System Dynamics Review* 15(1):3-36.

Forrester N. 1983. Eigenvalue Analysis of Dominant Feedback Loops. The 1983 International System Dynamics Conference, Plenary Session Papers: 178-202.

Franklin GF, Powell JD, Emami-Naeini A. 2002. *Feedback Control of Dynamic Systems*. Fourth Edition. Prentice Hall Inc., NJ.

Graham AK. 1977. *Principles of the Relationship Between Structure and Behavior of Dynamic Systems*. Ph.D. Thesis, M.I.T., Cambridge, MA.

Modjahedzadeh M. 1996. *A Path Taken: Computer-Assisted Heuristics for Understanding Dynamic Systems*. Ph.D. Thesis, Rockefeller College of Public Affairs and Policy. Albany, NY.

Modjahedzadeh M, Andersen D. 2001. Digest: A New Tool for Creating Insightful System Stories. *Proceedings of the 2001 International System Dynamics Conference*. Atlanta.

Özveren C, Sterman JD. 1989. Control theory heuristics for improving the behavior of economic models. *System Dynamics Review* 5(2):130-147.

Richardson GP. 1986. Problems with Causal Loop Diagrams. *System Dynamics Review* 2(2):158-170.

Richardson GP. 1995. Loop Polarity, Loop Dominance, and the Concept of Dominant Polarity. *System Dynamics Review* 11(1):67-88.

Richardson GP. 1996. Problems for the Future of System Dynamics. *System Dynamics Review* 12(2):141-157.

Saleh M, Davidsen P. 2000. An Eigenvalue Approach to Feedback Loop Dominance analysis in Nonlinear Dynamic Models. *Proceedings of the 2000 International System Dynamics Conference*. Bergen, Norway.

Saleh M, Davidsen P. 2001. The Origins of Business Cycles. *Proceedings of the 2001 International System Dynamics Conference*. Atlanta.

Sice P, Mosekilde E, Moscardini A, Lawler K, French I. 2000. Using system dynamics to analyse interactions in duopoly competition. *System Dynamics Review* 16(2):113-133.

Sterman JD. 2000. *Business Dynamics: Systems Thinking and Modeling for a Complex World*. New York: McGraw-Hill.

Back to the Top