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Using *Digest* to Implement the Pathway Participation Method for Detecting Influential System Structure

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Abstract

This paper briefly defines and describes the Pathway Participation Metric (PPM), a mathematical calculation that can help to identify the linkages between the structure and behavior of a dynamic system. PPM has been implemented in an experimental piece of software called *Digest*, which we then present. *Digest* is not a simulation language, but rather a companion to any commercial system dynamics package, which accepts a text version of a simulation model and performs post formulation analysis of the model. *Digest* detects and displays which feedback loops are most influential in explaining a selected pattern of behavior in a model. Output from a sample *Digest* run is presented and described.

System Stories: Understanding How and Why Patterns of System Behavior Arise from Most Influential System Structure

An important purpose of most system dynamics modeling efforts is to help managers better understand the systems which they manage and in which they live. One key task in this search for insightful, system level understanding is the telling of "system stories" -- coherent, dynamically correct explanations of how influential pieces of system structure give rise to important patterns of system behavior[1].

A key task in creating system stories is accurately detecting exactly what part of the system structure gives rise to (or contributes most importantly to) some pattern of behavior identified in one or more simulation runs. Richardson (1996) has identified this task as one of the key research problems presently facing the field of system dynamics. Over the years, this problem has been examined using a range of approaches (Graham,

[1] Of course, other valid purposes for client-based systems work exist. For example Senge (1990) and many others advocate the use of Systems Thinking approaches that do not rely on formal simulation. However, these approaches may suffer from other conceptual limitations starting with something as simple as proper interpretation of causal-loop diagrams (Richardson, 1986).

1977; Forrester, 1982; Eberlein, 1984; Kampmann, 1996; Davidsen, 1991; Ford, 1999; and Salleh, 2002.

Despite the importance of understanding the linkages between the structure and dynamic behavior in simulation models, tools to accomplish this task are lacking. Most skilled practitioners approach the challenge of identifying the most influential structure with some combination of intuition and analysis coupled to a program of repeated simulations, testing hypotheses about what structure controls what behavior in a controlled way with some of experimental logic.[2] Years of experience with system dynamics models is needed for launching artful hypotheses and testing them via repeated simulation, and no satisfying accounts exist in the published literature prescribing a precise set of steps for completing this key task. Even experienced modelers experience difficulty in testing their hunches about the connection between structure and behavior.

For linear dynamic systems, some mathematical tools exist to make this trial-and-error process more tractable. Indeed, modes of overall system behavior have a clearly defined meaning for linear systems. System behavior is understood to arise from a linear combination of the dynamics associated with the eigenvalues of the linear matrix of system structure. Hence, the calculation of a system's eigenvalues can go a long way toward describing overall behavior modes of a linear system (Eberlein, 1989).

Closely coupled to eigenvalue analysis of dominant modes of system behavior is the notion of "dominant loops." Usually thought of intuitively as the feedback structure "most important" in determining some portion of the dynamics of a system, dominant loops can be seen as a reduced set of closed feedback paths that contribute most to the overall behavior mode of a model. Indeed, for linear systems, one can work out mathematical relationships between a set of such dominant loops and the system's eigenvalues (Forrester, 1983; Kampann, 1996). Nonlinear systems, however, have the capability to shift loop dominance and therefore require more than what eignevalue analysis can provide (Mosekilde, 1996; Forrester, 1987).

The work presented in this paper continues in the line of eigenvalue and dominant loop analysis in that it continues the search for formal analytic approaches to support the detection of which pieces of system structure contribute most to selected patterns of system behavior. However, in contrast to previous attempts to solve this problem, our approach does not focus on eigenvalues or on dominant loops as the key building blocks of influential system structure[3]. Rather pathways, links of causal structure between two

[2] For example standard texts in the field such as Richardson and Pugh (1981)describe an approach to model analysis that relies on repeated simulations, as does Goodman (1974) and Sterman (2000).

[3] Indeed, the PPM method can be related to eigenvalue analysis. Appendix B demonstrates that for a second order linear system, the PPM method produces values that are mathematically related to the two eigenvalues of the system. The same result can be shown to hold for higher order linear systems.

system stocks, are envisioned as the primary building blocks of influential structure.[4] Of course, one or more pathways can define closed feedback loops. Under this new approach, some combinations of pathways (some of which form closed feedback loops) define the most influential system structure. This most influential system structure, explicitly linked to a pattern of behavior identified by the modeler, forms the basis for creating insightful system stories.

This paper briefly reviews the conceptual underpinnings of this new pathway-determined approach and then presents an experimental piece of software, *Digest*, that can be used to implement this approach.

Pathway Participation Metrics (PPM): A Mathematical Algorithm for Detecting Most Influential Structure

Mojtahedzadeh (1996, 1997) has proposed the Pathway Participation Metrics (PPM) as a mathematical tool that could help support modeler intuition in dealing with the task of unraveling relationships between system structure and system behavior. The basic behavioral building block of the PPM is a single phase of behavior for a single variable. A single phase of behavior for a selected variable is a time slice of the simulation where the selected variable maintains the same slope and curvature (first and second time derivatives). Hence, there are seven patterns of behavior that may exist within a single phase: (1) reinforcing growth, (2) linear growth, (3) balancing growth, (4) reinforcing decline, (5) linear decline, (6) balancing decline and finally, (7) equilibrium. Figure 1 depicts these seven patterns of behavior.

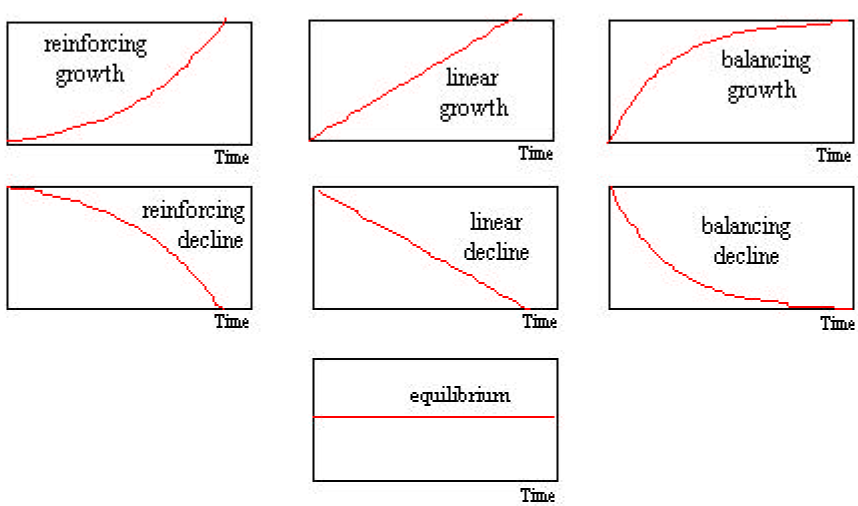

Figure 1: Seven patterns of over time behavior

[4] Actually, while most pathways are from one system state variable to another, some pathways can connect a state variable to an ordinary auxiliary variable found in the system "between" two or more state variables. See Mojtahedzadeh (1996) for a more formal definition of a pathway.

The PPM approach begins when the modeler analyst selects a variable of interest. The PPM approach will detect what structure of the model is most influential in determining the behavior of this selected variable. Figure 2 below shows a typical S-shaped growth for some variable X selected to be studied in a hypothetical system. The Digest software slices the time path for X into discrete patterns representing the seven patterns of overtime behavior. For the example shown in Figure 2, the trajectory of X consists of only two time slices—an initial time slice of reinforcing growth followed by a second time slice of balancing growth. Once the time trajectory for the selected variable has been decomposed into separate patterns, the PPM approaches answers the question, "What structure is most influential in explaining one pattern of over time behavior for the selected variable?" For the example shown in figure 2, these questions reduce to "What structure in the model most influences the initial phase of reinforcing growth in this system?" and then sequentially, "What structure in the model most influences the balancing growth phase of the simulation?"

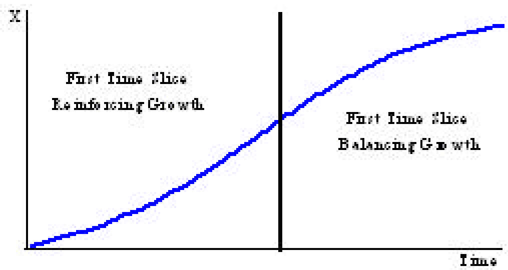

Figure 2: Digest "slices" the hypothetical time trajectory for X into separate patterns of over time behavior according to its slope ($\dot{X}$) and curvature ($\ddot{X}$)

The mathematics of the PPM sets out to determine which pathway from a system state to the variable of interest contributes most to the current behavior pattern of that variable. This apparently simple question requires some mathematics to be answered.

The PPM calculates how much the net-flow ($\dot{X}$ or dX/dt) could change given a small change in the state variable under consideration, ($d\dot{X}/dX$); this is called Total Pathway Participation Metrics. Since $d\dot{X}/dX$ can be transformed into $d\dot{X}/dt$ divided by dX/dt, the Total Participation Metric contains information about both slope and curvature of the variable of interest and is thus an appropriate tool for analyzing behavior[5]. This measure of the Total Participation Metric for state variable X is then partitioned among pathways coming into the net-flow[6]. The most influential pathway is defined as the one whose participation is the largest in magnitude and has the same sign as the total changes in the net-flow x-dot when it is disturbed by a infinitesimal change in the state variable at the

[5] Mohamed Saleh, 2002, also use slope ($\dot{X}$) and curvature ($\ddot{X}$) and calls it BPI (Behavioral Pattern Index) to characterize behavioral patterns.

[6] Richardson (1995) proposed that the net time derivative of a state variable with respect to the state variable itself ($d\dot{X}/dX$) can be an important measure of when a loop shifts dominance. The PPM approach calls $d\dot{X}/dX$ the Total Pathway Participation Measure.

tail of the pathway. For a more complete description of pathway participation metrics see Appendix A).

A simple example might help to clarify what is going on here. Figure 3 shows a hypothetical fourth order system showing only 4 of what might be a much larger number of pathways. Two pathways, P_1 and P_2, lead from state X_2 to change X_1. Only one pathway, P_3, connects the state variable X_3 with X_1. And finally pathway P_4 represents the linkage between X_4 and X_1. If X_1 is the selected variable and is showing reinforcing growth as indicated in Figure 2, the PPM approach asks the specific question, "Which of these four pathways dominates the initial reinforcing growth of X_1 (contributes the most to $d\dot{X}/dX$)?". The PPM approach answers this question by calculating the partial contribution of each of the four pathways to the total pathway measure, $d\dot{X}/dX$, and then selecting that pathway that has the same sign and greatest magnitude as $d\dot{X}/dX$. Let us assume that all these calculations identify P_2 as the most influential of the four pathways shown in Figure 3. We now know that X_2 has the strongest influence on X_2 and furthermore that influence is exerted through the pathway P_2. But what structure now influences X_2 the most? The process of analysis continues.

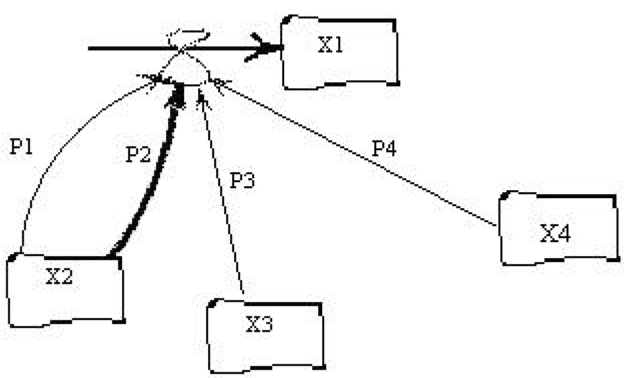

Figure 3: The PPM approach selects pathway P_2 as most influential in the behavior of X_1

Figure 4 gives a more complete look at the structure of our hypothetical system indicating 10 pathways and numerous closed loops. The second iteration of the PPM approaches now seeks to identify which pathway (P_5 or P_6?) contributes most to the behavior of X_2. The calculations are the same as in the first iteration. The PPM approach computes dX2dot/dX2 and the relative contributions of pathways P_5 and P_6 to that total. Let us assume that pathway P6 is selected as the most influential at this stage. We now know that X_1 is most strongly influenced by X_2 through the pathway P_2 and that X_2 is most strongly influenced by X_3 through the pathway P_6. The next iteration of the PPM approach would ask "Which pathway, P_8 or P_{10}, most strongly influences X_3. If the answer in our hypothetical example were to be P_8, then we have identified a closed loop that begins with and ends with X_1.

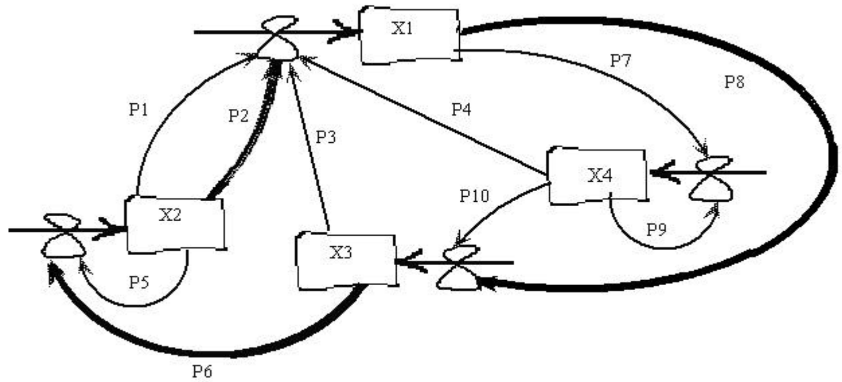

Figure 4: Schema of major and minor loops in hypothetical fourth order system

That closed loop is isolated and displayed in Figure 5. The interpretation of this figure, similar to figures generated by the Digest software, is that the reinforcing feedback loop involving X_1, P_8, X_3, P_6, X_2, and finally P_2 is the loop most influential in determining the initial reinforcing growth of X_1. Figure 4 contains a large number of major as well as minor loops that could have contributed to the initial reinforcing growth in the selected variable X_1. What the PPM has done is to select three pathways that are connected into a single loop and has identified that loop as the most influential of all other possible loops in determining the initial growth in the system. The PPM approach does not always identify a single major loop. Sometimes the most influential structure may be a minor loop or in some cases the system's dynamics may be mainly influenced by an exogenous time series. Frequently a pathway will lead from the selected variable of interest to another minor or major loop located far from it in the overall causal structure of the model. However, it can be shown that repeated application of the PPM mathematics at each step does converge on a unique piece of structure identified as most influential for a given behavior pattern of the variable of interest.

The PPM approach concludes by moving on to the next time slice that differs from the previous one in slope or curvature. The analysis for each time slice is similar. Note that the most influential structure identified for each slice of time may vary.

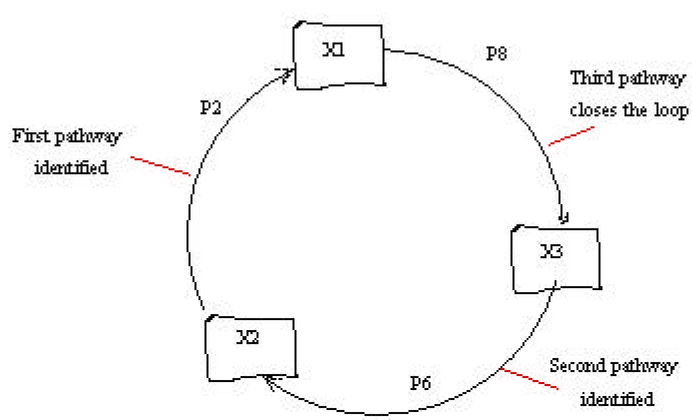

Figure 5: Schema depicting dominant major loop from hypothetical fourth order system shown in Figure 4.

This very brief explanation of how the PPM approach works skips over all of the interesting mathematical details of how the contributions of each pathway are actually computed. An overview of the mathematics underpinning the PPM calculations is provided in Appendix A and full details are provided in Mojtahedzadeh 1996. An example using a model characterized by system overshoot is presented below.

One problem with PPM as an algorithm is that it is cumbersome and difficult to compute and no existing commercial simulation software packages support these calculations. *Digest* is a piece of experimental software that automatically computes the PPM and then uses information from the PPM to automatically detect and display influential pathways and feedback loops.

Digest is not a simulation package such as iThink, Stella, Vensim, or Powersim. *Digest* cannot support most of the simulation functions that these languages can. *Digest* is designed to be used after the model has been constructed to detect and display influential structure. Of course, at some point in the future, the relevant and most useful features of *Digest* could be integrated into any of the commercial simulation packages.

Digest accepts model equations from any commercial simulation package in text form. In its present version, some hand editing of the text equations may be necessary if the model uses macros or functions that are not yet parsed by the *Digest* equation translator. Once a text version of the model equations has been edited and accepted by *Digest*, the software leads the modeler through a series of step-by-step procedures that uses the PPM calculation to first detect and then display model structure[7].

Using Digest to Analyze a Simple Overshoot Model:

This section analyzes the behavior of one variable in a simple overshoot model using *Digest*. In doing that we need the equations of a "simulatable" model saved in a text file format. The model used as an example is a classic structure that illustrates how Industrial Structures in a particular region grow over time until all the resources needed to support the growth of Industrial Structures are depleted. Figure 6 depicts the structure of the model. (A list of the equations of the model is provided in Appendix C).

[7] *Digest* is currently available in a Beta test version. Readers interested in experimenting their own with this Beta version are encouraged to contact the authors for a copy of *Digest*.

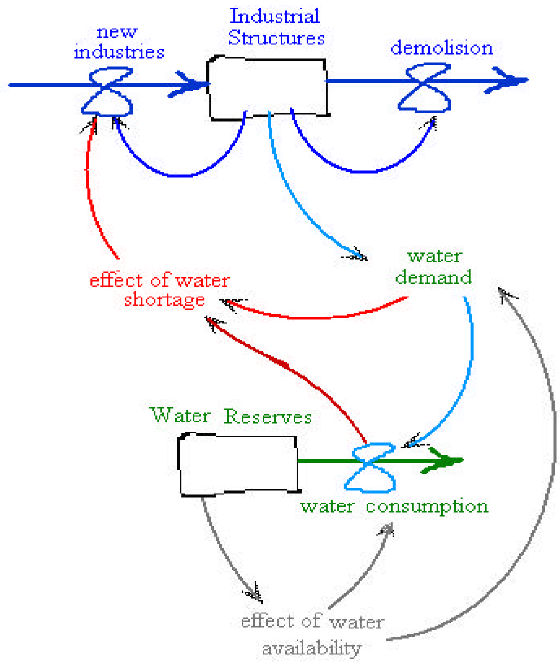

Figure 6: A simple model for the growth of Industrial Structures

The model captures three real-world processes:

1. Industrial Structures grow with new industries through a reinforcing loop and demolish by a balancing loop;
2. Industrial Structures consume water, which decreases water reserves;
3. Water shortage (defined as the ratio of water consumption to water demand) affects new industries indirectly;
4. Water availability (defined as the ratio of water reserves to water demand) controls water consumption.

For an appropriate set of parameters and initial values, the model generates an overshoot in the behavior of Industrial Structures, while Water Reserves follows an S-shape decline. In explaining the behavior of the model the question is what feedback loops are more influential in generating the behavior of any variable of interest. For example, what is making Water Reserves to decline rapidly and what controls it? What is driving Industrial Structures to grow rapidly in the first few years? What part of the structure is responsible for the decline of Industrial Structures followed by its growth? For modelers who have worked with this sort of model, it is not difficult to explain the growth phase and the declining phase of the behavior of this simple structure. However, it may not be as easy to distinguish what part of the structure contributes most to the behavior of Industrial Structures in the transition from reinforcing growth to a balancing decline. Using *Digest* one can identify the most influential structure as the behavior of the model unfolds.

The outputs of Digest:

Once a model is loaded in Digest environment, using the information embedded in the equations for the model, it could produce four different outputs. These outputs are:

1. A list of variables of the model by which the user could select the variable of interest.

2. Digest automatically identifies pathways associated with the user-selected variable of interest

Once the variable of interest is chosen, the causal route associated with the behavior of interest will appear in the second window. This diagram reveals how the variable of interest is determined by other variables in the model. For the Industrial Structure as the variable of interest, Figure 7 shows the causal route diagram that is associated with Industrial Structures. Arrows with a plus sign indicate a direct (positive) impact of the variable at the tail of the arrow at the dependent variable and an arrow with a negative sign refers to an indirect impact of the cause on the effect (a negative or indirect relationship).

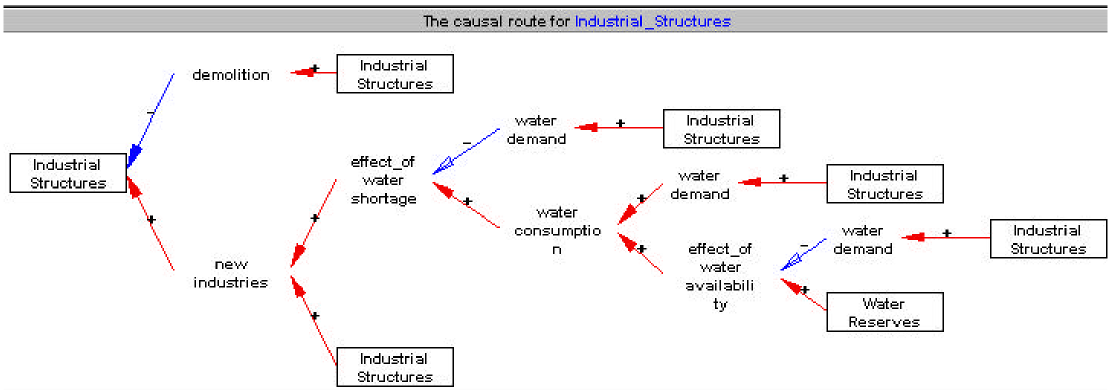

Figure 7: Causal route diagram for Industrial Structures

3. Digest identifies distinct phases in the behavior pattern of user-selected variable of interest

Digest produces the overtime behavior of the variable of interest and identifies the shifts in the pattern of behavior. Figure 8 shows the behavior phases of the variable of interest, Industrial Structures.

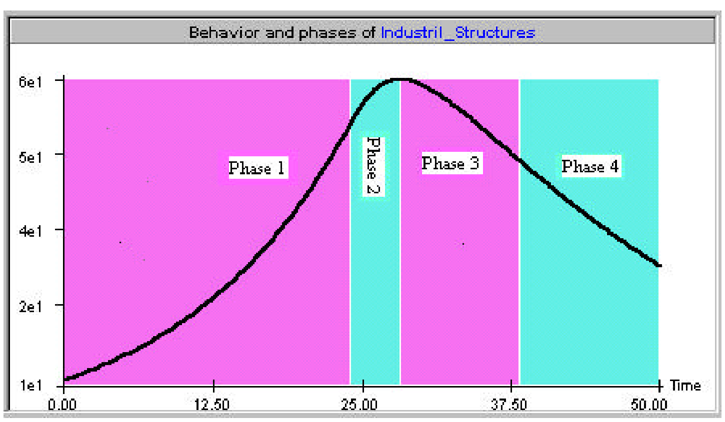

Figure 8: The behavior of Industrial Structures and its four phases

The first phase of Industrial Structures is a reinforcing growth. The reinforcing growth lasts for 24 years. During the first 24 years of simulation time, both slope (first time derivative) and curvature[8] (second time derivative) of the variable of interest, Industrial Structures, remain positive. The next distinct phase in the behavior of Industrial Structures, identified by *Digest*, is balancing growth. In this phase the slope and curvature of the variable of interest have opposite signs. The third distinct behavior phase in Industrial Structure is reinforcing decline. And finally, in its fourth phase, the variable of interest experiences a balancing decline in its over time behavior.

4. Digest detects and displays most influential structure contributing to behavior pattern in each phase

Corresponding to the first phase of the behavior of Industrial structures, there is a reinforcing feedback loop that, according to *Digest*, is the most influential feedback loop in generating the reinforcing growth in Industrial Structures. The reinforcing feedback loop is shown in Figure 9. Based on this feedback loop a higher level of Industrial Structures attracts more new industries, which in turn increases Industrial Structures. By inspecting the structure of the models in Figure 6, one could identify about 6 feedback loops. Using pathway participation metrics, *Digest* automatically selects the reinforcing feedback among all the other loops in the model without intervention by the model builder or analyst.

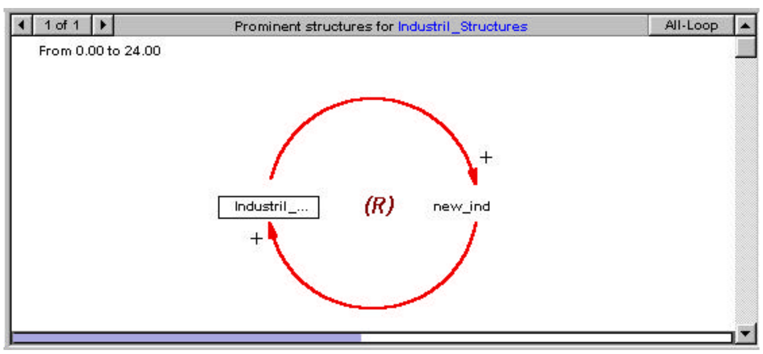

Figure 9: The most influential structure in creating the first phase of the behavior of Industrial Structures

The most influential structure in creating the second phase of the behavior of the variable of interest shifts from the reinforcing feedback loop to a balancing feedback loop associated with Water Reserves. Figure 10 depicts the balancing loop that controls water consumption as Water Reserve continues to fall, along with a pathway that carries the effect of the balancing feedback loop to the variable of interest, Industrial Structures.

[8] Actually *Digest* calculates neither the first nor second time derivative of the variable of interest; it merely determines $d\dot{X}/dX$ at any time. This derivative is related to first and second time derivatives of the variable of interest. A positive sign of the derivative indicates that both slope and the curvature of the variable of interest have the same signs. (See Mojtahedzadeh 1996 for details).

This structure remains most influential in the third phase of the behavior of the variable of interest.

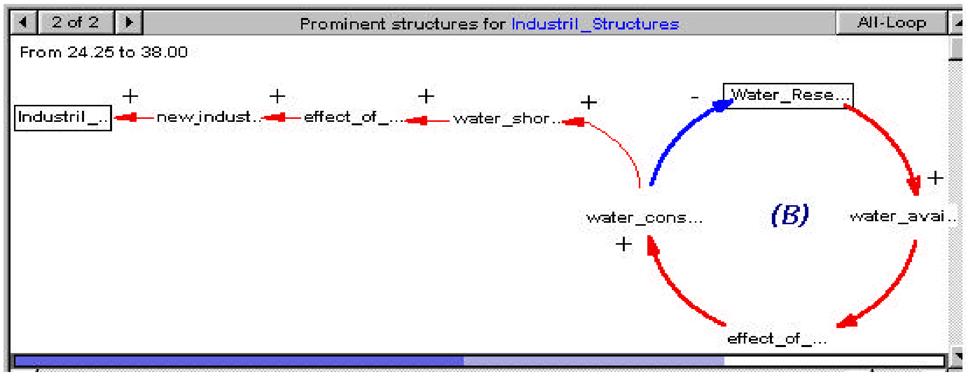

Figure 10: The most influential structure in the second and third phases of the behavior of Industrial Structures

It may not be difficult for the modelers to see the role of the balancing feedback loop that controls water consumption, when striving to explain why Industrial Structures is generating a balancing growth in its second phase. Water availability is dropping and, therefore, new industries are restricted. The subtlety in explaining the behavior of the variable of interest is the subsequent reinforcing decline in the behavior of Industrial Structures in phase four. Some novices may even look for a reinforcing feedback loop to explain the reinforcing decline. *Digest* reveals that what forces Industrial Structures to fall faster and faster is exactly the same process that controls it. The balancing loop that controls water consumption continuously lowers new industries and once new industries falls behind industrial demolition, the Industrial Structures generates a reinforcing decline.

The last phase of the behavior of the variable of interest, Industrial Structures, is influenced the most by the balancing feedback loop associated with demolition, as shown in Figure 11.

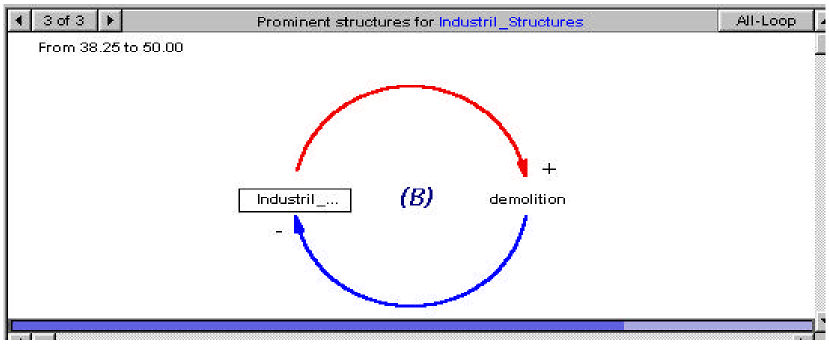

Figure 11: The most influential structure in the fourth phases of the behavior of Industrial Structures

Digest could redraw distinct phases in the behavior pattern of user-selected variable of interest based on shifts in the most influential structure, as shown in Figure 12.

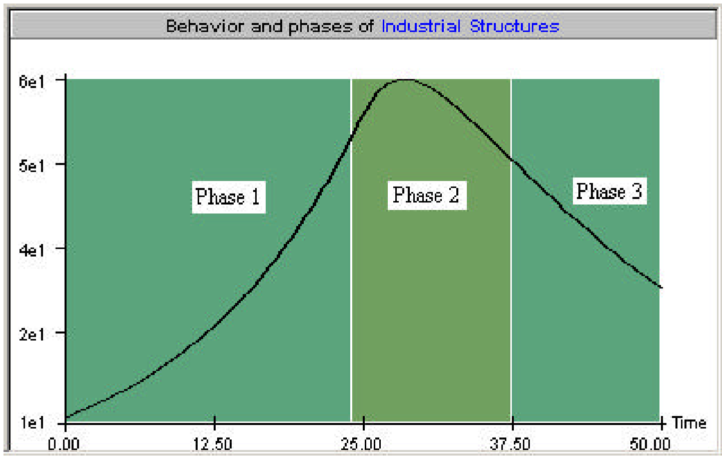

Figure 12: The essential structure for explaining Industrial Structures growth model

Conclusion

Model analysis, the process of understanding and then describing how the structure of a complex dynamic system gives rise to over time behavior, is still in its relative infancy. Well developed mathematical techniques exist for linear systems as well as for some regimes of complex non-linear dynamics such as deterministic chaos (Andersen, 1982; Mosekilde, 1996). However, in common practice with client-based modeling, skilled modelers create dynamic insights using carefully crafted simulation experiments to formulate explanations about what pieces of model structure drive the overall system behavior. But this intuition is difficult to codify and develops slowly over a career of practice. Modern software promises to help. For example, the latest release of Vensim allows for real time visualizations of structural sensitivities using brute force computing power and speed to create these visualizations.

This paper explores a promising additional approach. The Pathway Participation Metric, described in overview form in this paper, relies on the analysis of individual linkages or pathways between nodes of a model as the basic building blocks of structure. The approach leads to dominant feedback structure, if that's appropriate, but does not begin with the feedback loop as the basic building block. Using a recursive heuristic systematic analysis, the PPM calculations always yield a reduced structure of a key feedback loop plus one or more pathways that contribute most to a given mode of behavior for a selected model variable.

Important questions remain about this approach. Do the automatically identified "most influential structures" yield important insights for clients working on real world problems? Do clients and modelers alike have a strong enough intuition about the PPM to “trust” the structure that it identifies?[9] How can a set of loops, each of which is connected to a single phase of behavior, be combined into a fuller explanation of the complete dynamic trajectory of a single variable? How can analyses for two or more variables be merged into a coherent story of the system taken as a whole?

Digest is an experimental piece of software that can help us begin to answer these questions. Because *Digest* automatically and quickly analyzes and displays the results from the PPM calculations, we now have a tool that will allow us to experiment with yet another approach to the critical question of how to quickly and reliable relate system structure to system behavior.

In the near future, all system dynamics simulation software packages will contain new functions that support automatic model analysis[10]. We view *Digest* as an early experimental tool to move the field toward this future. We hope to encourage a vigorous experimental program to move questions and results in this critical area of inquiry forward.

[9] Mojtahedzadeh (1996) has begun an investigation of these last two questions by working with a number of models, such as the simplified Urban Dynamics model presented by Alfeld and Graham (1976). However, this work needs to be extended and deepened.

[10] Automation of model analysis functions within standard software packages is essential for their uptake in practice. For example, the “Reality Check” feature advocated by Peterson and Eberlein (1994) was made possible as a practical tool by being integrated into Vensim.

Appendix A: A Mathematical Definition of Pathway Participation Metrics

This appendix introduces the mathematics of pathway participation metrics (PPM). Consider the following n-order non-linear system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x};\mathbf{p}) \qquad [1]$$

Where $\mathbf{x}$ is the vector of state variable $\dot{\mathbf{x}}$ is the vector of derivative of $\mathbf{x}$ with respect to time, and $\mathbf{p}$ is the vector of the parameters of the system. The equation of the k^{th} state variable as the variable of interest may look like:

$$\dot{x}_k = f_k(x_1, x_2, ..., x_n; \mathbf{p}) \qquad [2]$$

Taking the derivative of the net changes in the state variable of interest, $\dot{x}_k$, with respect to the state variable of interest, x_k, yields:

$$\frac{d\dot{x}_k}{dx_k} = \frac{\partial f_k}{\partial x_1}\frac{dx_1}{dx_k} + \frac{\partial f_k}{\partial x_2}\frac{dx_2}{dx_k} + ... + \frac{\partial f_k}{\partial x_k}\frac{dx_k}{dx_k} + ... + \frac{\partial f_k}{\partial x_n}\frac{dx_n}{dx_k} \qquad [3]$$

Or simply,

$$\frac{d\dot{x}_k}{dx_k} = \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i}\frac{\dot{x}_i}{\dot{x}_k} \qquad \text{(for} \quad \dot{x}_k \neq 0 \text{)} \qquad [4]$$

Each term in equation [4] represents all minor feedback loops and pathways leaving i^{th} state variable and coming into the variable of interest, x_k. We can decompose the effect of each minor feedback and pathway coming into the state variable x_k.

$$\frac{d\dot{x}_k}{dx_k} = \sum_{i=1}^{n} \sum_{j=1}^{m(i)} \frac{\partial f_k^{\,j}}{\partial x_i}\frac{\dot{x}_i}{\dot{x}_k} \qquad [5]$$

Where *m(i)* is number of minor loops and pathways that leave a i^{th} state variable and come into the k^{th} state variable, and $\partial f_k^{\,j} / \partial x_i$ is the polarity of the pathway or minor feedback loop. The ratio $\dot{x}_i / \dot{x}_k$ represents the net changes in the i^{th} state variable and the net changes in k^{th} state variable. The total effect infinitesimal change in x_k of the net rate of x_k is the not only driven by the polarity of the feedback loops and pathways but also the ratio of net changes in the two state variables.

The effect of each pathway can be normalized in such a way that it varies between –1 and 1. Thus for each pathway coming into variable of interest we could have a metric that measure the impact of that pathway (or minor feedback loop) in creating the behavior of the variable of interest. This metric is called pathway participation metrics (PPM).

$$PPM(i,j) = \frac{\dfrac{\partial f_k^{\,j}}{\partial x_i}\dfrac{\dot{x}_i}{\dot{x}_k}}{\sum_{i=1}^{n}\sum_{j=1}^{n(i)}\left|\dfrac{\partial f_k^{\,j}}{\partial x_i}\dfrac{\dot{x}_i}{\dot{x}_k}\right|} \qquad [6]$$

The most influential pathway (or minor feedback loop) is defined as the one whose participation metrics (PPM) is the largest and has the same sign as $d\dot{x}_k / x_k$. For $i \neq k$ the same calculation is done until there is a feedback loop.

If the variable of interest is a non-state variable, we need to determine the net changes the variable of interest and the follow the same procedure. Suppose **a** presents the vector of non-state variables and it is related to state variables through **g** and a vector of parameters **q**. Thus we have,

$$\mathbf{a} = \mathbf{g}(\mathbf{x};\mathbf{q})$$

If the a_k is the variable of interest, that is a non state variable, the net changes in a_k over the period of dt will be,

$$\dot{a}_k = \sum_{i=1}^{n}\frac{\partial g_k}{\partial x_i}\dot{x}_i \qquad [7]$$

Taking the derivative of the net changes in the variable of interest, $\dot{a}_k$, with respect to the state variable of interest, a_k, yields:

$$\frac{d\dot{a}_k}{da_k} = \sum_{i=1}^{n}\left(\frac{\partial^2 g_k}{\partial x_i \partial a_k}\dot{x}_i + \frac{\partial g_k}{\partial x_i}\frac{d\dot{x}_i}{da_k}\right) \qquad [8]$$

Which can be rearranged as:

$$\frac{d\dot{a}_k}{da_k} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(\frac{\partial^2 g_k}{\partial x_i \partial x_j}\frac{\dot{x}_j}{\dot{a}_k}\right)\dot{x}_i + \frac{\partial g_k}{\partial x_i}\frac{d\dot{x}_i}{dx_i}\frac{\dot{x}_i}{\dot{a}_k}\right) \qquad [9]$$

The pathway participation metrics can be determined after decomposing the impact of each pathway leaving a particular state variable and coming into the variable of interest.

Appendix B: Pathway Participation Metrics and Eigenvalues

In linear systems there is a close relationship between the pathway participation metrics and eigenvalue of the system. In fact we show that

- *In the steady state condition, the total participation metric is equal to the largest eigenvalue of the system.*

In doing so, we use a second order linear system and derive pathway participation metrics for a state variable. Then, we show that the sum of the pathway participation metrics, or total participations metrics, for any state variable equal the largest eigenvalue of the system.

Consider the following second order system:

$$\dot{x} = ax + by \qquad [1]$$

$$\dot{y} = cx + dy \qquad [2]$$

The pathway participation metrics for state variable x is:

$$\frac{d\dot{x}}{dx} = a + b\frac{\dot{y}}{\dot{x}} \qquad [3]$$

There are two pathways coming to the state variable x whose participation metrics are:

Participation metrics for Pathway 1: a

Participation metrics for Pathway 2: $b\dfrac{\dot{y}}{\dot{x}}$

The pattern of behavior of x is determined by the total participation metrics, which is the sum of participation metrics for these two pathways. If total participation metrics is positive the state variable x experiences a reinforcing growth and if it is negative, x shows a balancing behavior. The most influential pathway then is the one whose participation metrics is the largest in magnitude and has the same sign as the total participation metrics.

Now we calculate $\dot{y}/\dot{x}$ through the response of state variables x and y. We can rewrite the second order linear system presented in [1] and [2] as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad [4]$$

The above system has two eigenvalues, l_1 and l_2. For each eigenvalue we have:

$$ar_{i1} + br_{i2} = \boldsymbol{l}_i r_{i1} \qquad [5]$$
$$cr_{i1} + dr_{i2} = \boldsymbol{l}_i r_{i2} \qquad [6]$$

Where r_{i1} and r_{i2} are the elements of the right eigenvector associated with $\boldsymbol{l}_i$. The time response of the state variables is:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \boldsymbol{j}(t) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad [7]$$

Where x_0 and y_0 are the initial values of the state variables and $\boldsymbol{j}(t)$ with the dimension of 2*2 is the transition matrix of the system which can be calculated as:

$$\boldsymbol{j}(t) = \sum_{i=1}^{2} e^{t l^i} \begin{bmatrix} r_{i1} \\ r_{i2} \end{bmatrix} \begin{bmatrix} f_{i1} & f_{i2} \end{bmatrix} \qquad [8]$$

Where f_{i1} and f_{i2} are the elements of the left eigenvector associated with $\boldsymbol{l}_i$. Substituting [8] in [7] and expanding it yields:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = e^{t l_1} \begin{bmatrix} r_{11} f_{11} & r_{11} f_{12} \\ r_{12} f_{11} & r_{12} f_{12} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + e^{t l_2} \begin{bmatrix} r_{21} f_{21} & r_{21} f_{22} \\ r_{22} f_{21} & r_{22} f_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad [9]$$

The value of x and y at any time is:

$$x_t = (r_{11} f_{11} x_0 + r_{11} f_{12} y_0) e^{t l_1} + (r_{21} f_{21} x_0 + r_{21} f_{22} y_0) e^{t l_2} \qquad [10]$$
$$y_t = (r_{12} f_{11} x_0 + r_{12} f_{12} y_0) e^{t l_1} + (r_{22} f_{21} x_0 + r_{22} f_{22} y_0) e^{t l_2} \qquad [11]$$

We can calculate $\dot{x}$ and $\dot{y}$ by taking derivatives of [10] and [11] with respect to time.

$$\dot{x} = \boldsymbol{l}_1 (r_{11} f_{11} x_0 + r_{11} f_{12} y_0) e^{t l_1} + \boldsymbol{l}_2 (r_{21} f_{21} x_0 + r_{21} f_{22} y_0) e^{t l_2} \qquad [12]$$
$$\dot{y} = \boldsymbol{l}_1 (r_{12} f_{11} x_0 + r_{12} f_{12} y_0) e^{t l_1} + \boldsymbol{l}_2 (r_{22} f_{21} x_0 + r_{22} f_{22} y_0) e^{t l_2} \qquad [13]$$

Using [12] and [13], we calculate the ratio of $\dot{y} / \dot{x}$.

$$\frac{\dot{y}}{\dot{x}} = \frac{\boldsymbol{l}_1 (r_{12} f_{11} x_0 + r_{12} f_{12} y_0) e^{t l_1} + \boldsymbol{l}_2 (r_{22} f_{21} x_0 + r_{22} f_{22} y_0) e^{t l_2}}{\boldsymbol{l}_1 (r_{11} f_{11} x_0 + r_{11} f_{12} y_0) e^{t l_1} + \boldsymbol{l}_2 (r_{21} f_{21} x_0 + r_{21} f_{22} y_0) e^{t l_2}} \qquad [14]$$

Or,

$$\frac{\dot{y}}{\dot{x}} = \frac{\boldsymbol{l}_1 (r_{12} f_{11} x_0 + r_{12} f_{12} y_0) e^{t(l_1 - l_2)} + \boldsymbol{l}_2 (r_{22} f_{21} x_0 + r_{22} f_{22} y_0)}{\boldsymbol{l}_1 (r_{11} f_{11} x_0 + r_{11} f_{12} y_0) e^{t(l_1 - l_2)} + \boldsymbol{l}_2 (r_{21} f_{21} x_0 + r_{21} f_{22} y_0)} \qquad [15]$$

Assuming l_2 is the largest eigenvalue, when time approach infinity terms $l_1(r_{12}f_{11}x_0 + r_{12}f_{12}y_0)e^{t(l_1-l_2)}$ and $l_1(r_{11}f_{11}x_0 + r_{11}f_{12}y_0)e^{t(l_1-l_2)}$ in [15] approaches zero. Thus, for $\dot{y}/\dot{x}$ we have

$$\frac{\dot{y}}{\dot{x}} = \frac{l_2(r_{22}f_{21}x_0 + r_{22}f_{22}y_0)}{l_2(r_{21}f_{21}x_0 + r_{21}f_{22}y_0)} \qquad [16]$$

The above equation can be rewritten as:

$$\frac{\dot{y}}{\dot{x}} = \frac{r_{22}}{r_{21}} \qquad [17]$$

Now we can substitute [17] in [3],

$$\frac{d\dot{x}}{dx} = a + b\frac{r_{22}}{r_{21}} \qquad [18]$$

Equation [18] according to [5] is equal to l_2.

$$\frac{d\dot{x}}{dx} = l_2 \qquad [19]$$

It can be easily shown that the above proposition is true for an n-order system.

Appendix C: A list of the equations of the Industrial Structures Growth Model (iThink version)

Industrial_Structures(t) = Industrial_Structures(t - dt) + (new_industries - demolition) * dt
INIT Industrial_Structures = 10
new_industries = Industrial_Structures*effect_of_water_shortage*normal_growth
demolition = Industrial_Structures*dem_frc
Water_Reserves(t) = Water_Reserves(t - dt) + (- water_consumption) * dt
INIT Water_Reserves = 10000
water_consumption = effect_of_water_availability*water_demand
dem_frc = .05
normal_growth = .12
water_demand = Industrial_Structures*water_demand_per_industry
water_demand_per_industry = 10
effect_of_water_availability = GRAPH(0.1*Water_Reserves/water_demand)
(0.00, 0.00), (0.1, 0.06), (0.2, 0.14), (0.3, 0.255), (0.4, 0.395), (0.5, 0.535), (0.6, 0.685), (0.7, 0.825), (0.8, 0.92), (0.9, 0.975), (1, 1.00)
effect_of_water_shortage = GRAPH(water_consumption/water_demand)
(0.00, 0.00), (0.1, 0.06), (0.2, 0.14), (0.3, 0.255), (0.4, 0.395), (0.5, 0.535), (0.6, 0.685), (0.7, 0.825), (0.8, 0.92), (0.9, 0.975), (1, 1.00)

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