On Mathematical Structures for Systems Archetypes

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Abstract

A proposal on mathematical structures for systems archetypes is presented. The mathematical structures are based on systems of differential equations and the concept of state variable representation. Vensim is used to construct block diagrams and execute simulations. Finally, benefits are discussed by using both representations as conventional System Dynamics models and as differential equation systems.

Key Words: mathematical models, systems archetypes, differential equation models.

I. Introduction

The main motivations for this work are the search and learning of generic structures that allow designing policies to improve problematic situations in human organizations. It is known that growth, decline, goal seeking, and oscillations are consequences of feedback loop dynamics (Forrester, 1994, 54.) System Dynamics models and differential equations are two effective representations to express changes of things through time.

System Dynamics uses symbolic and graphical representations as well as computer simulation models to represent and understand dynamics of a situation. This latter approach is also found in disciplines in disciplines such as ecology, electrical engineering, chemical engineering, among others but these disciplines use differential equations as their representation tool (Zill, 1997), (Lomen and Lovelock, 1999). Thus it seems to be useful to understand how to pass from one representation to another.

This work has been organized in five sections. The first section is the introduction. The second deals with concepts of state-variable description. The third is the introduction of the mathematical representation for system archetypes using the state-variable description. The fourth section is conclusions. The fifth section presents the references for this work.

II. State-Variable Representation

Some reasons that make natural the use state-variable representation in System Dynamics are:

1. The exact match between the concepts of state variable $x(t)$ and rate of change $dx(t)/dt$ with the concepts of *Stock* and *Flow*, respectively.
2. It is a general representation that allows us to handle time varying and nonlinear systems.
3. Its realization and solution can be obtained using the concept of analog-computer simulation.
4. First-order differential equations of the state-space description are easily and accurately evaluated on a digital computer.

The state-variable representation is the structure of a dynamic system accomplished by means of a set of n first-order differential equations (Rohrs, Melsa, and Schultz, 1993, 28.) The state-variable description of a system is not unique. The system can be described by many different sets of state variables (Kailath, 1980, 53.) The general state-variable representation is given by:

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$

where $\mathbf{f}$ is a $n \times 1$ state nonlinear vector function, $\mathbf{x}$ is the $n \times 1$ state vector, n represents the number of states considered in the system, called the order of the system, $\mathbf{u}$ is an $m \times 1$ input vector, m is the number of inputs, and t denotes time dependency.

Linear systems are considered a special class of nonlinear systems. Linear systems take the form:

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) + b_1u(t) \\
x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t) + b_2u(t) \\
&\vdots \\
x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t) + b_nu(t)
\end{aligned}$$

where the n variables $x_i(t)$ are the *state-variables* and $u(t)$ is the exogenous input. Additionally, an *output expression* $y(t)$ as function of the states-variables is used to complete the representation:

$$y(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t)$$

The general linear state-space equations for an *n*-states, *m*-inputs, and *k*-outputs system has the form (Wiberg, 1971).

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\
\mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)
\end{aligned}$$

where $\mathbf{x}(t)$ is an *n*-vector; $\mathbf{u}(t)$ is an *m*-vector; $\mathbf{y}(t)$ is a *k*-vector; $\mathbf{A}(t)$ is an $n \times n$ matrix; $\mathbf{B}(t)$ is an $n \times m$ matrix; $\mathbf{C}(t)$ is an $k \times n$ matrix, and $\mathbf{D}(t)$ is a $k \times m$ matrix.

The state of a physical object is any property of the object, which relates input to output such that knowledge of the input time function for $t \geq t_0$ and state at time $t = t_0$ completely determines a unique output for $t \geq t_0$ (Wiberg, 1971). A state can be seen as the answer to the question: given

$\{u(t), t \geq t_0\}$ and the mathematical relationships of the abstract object, what additional information is needed to completely specify $\{y(t), t \geq t_0\}$?

The vector solution $\mathbf{x}(t)$ describes the trajectories on time of the internal variables. Elements of $\mathbf{x}(t)$ are the integrator outputs in any realization. It is clear that if the values of the integrator outputs are known for any given time $t = t_0$ and the inputs $\mathbf{u}(t)$ for $t \geq t_0$ also known then *all* present and future values of the outputs $\mathbf{y}(t)$, and integrator outputs (indeed, of any signal anywhere in the simulation) can be calculated.

The advantage of this state-space description is that there is no need to know the past of all the system to establish the present and future behavior. $\mathbf{x}(t_0)$ provides a sufficient statistical information to calculate the future $\{t \geq t_0\}$ response to a new input $\{\mathbf{u}(t), t \geq t_0\}$ without worrying about $\{\mathbf{u}(t), t \leq t_0\}$. In this sense, $\mathbf{x}(t_0)$ is a minimal sufficient statistics (Kailath, 1980.) Therefore, it is natural to call the integrator outputs at any time t the *state* of the system realization. This interpretation is not restricted to analog-computer realizations but also applies to any set of state-space equations, no matter how they are obtained -as realization of differential equations or as description of physical systems (Kailath, 1980, 63.)

Now the state variable concept is applied to provide a mathematical structure for the systems archetypes described by (Senge, 1990) in Appendix 2: *Systems Archetypes*.

III. Mathematical structures for Systems Archetypes

A. Eroding Goals

"A shifting the burden type of structure in which the short-term solution involves letting a long-term, fundamental goal decline" (Senge, 1990, 383.)

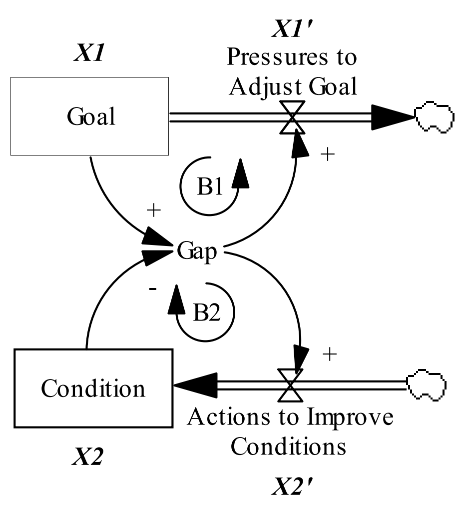


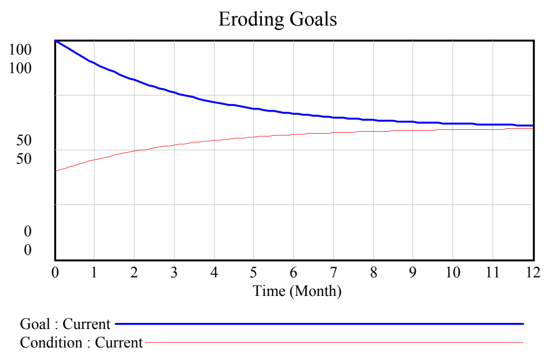
Fig. 1 Block diagram for Eroding Goal Archetype

$$x_1'(t) = -\frac{1}{T_1} x_1(t) + \frac{1}{T_1} x_2(t)$$

$$x_2'(t) = \frac{1}{T_2} x_1(t) - \frac{1}{T_2} x_2(t)$$

$$x_1(0) = x_{10} \ \ and \ \ x_2(0) = x_{20}$$

For this example, $T_1 = 5$ and $T_2 = 10$ are the corresponding time constants of the balanced loops. $T_2 > T_1$ to represent the delay included as definition in B_2 by the original archetype in (Senge, 1990, 383.) $x_{10} = 100$ and $x_{20} = 40$.



(01) Actions to Improve Conditions = Gap/10
(02) Condition= INTEG (Actions to Improve Conditions, 40)
(03) FINAL TIME = 12
The final time for the simulation.
(04) Gap = Goal-Condition
(05) Goal= INTEG (-Pressures to Adjust Goal, 100)
(06) INITIAL TIME = 0
The initial time for the simulation.
(07) Pressures to Adjust Goal = Gap/5
(08) SAVEPER = TIME STEP
The frequency with which output is stored.
(09) TIME STEP = 0.125
The time step for the simulation.

B. Escalation

"Two people or organizations each see their welfare as depending on a relative advantage over the other. Whenever one side gets ahead, the other is more threatened, leading it to act more aggressively to reestablish its advantage, which threatens the first, increasing its aggressiveness, and so on. Often each side sees its own aggressive behavior as a defensive response to the other's aggression; but each side acting "in defense" results in a buildup that goes far beyond either side's desires" (Senge, 1990, 384.)

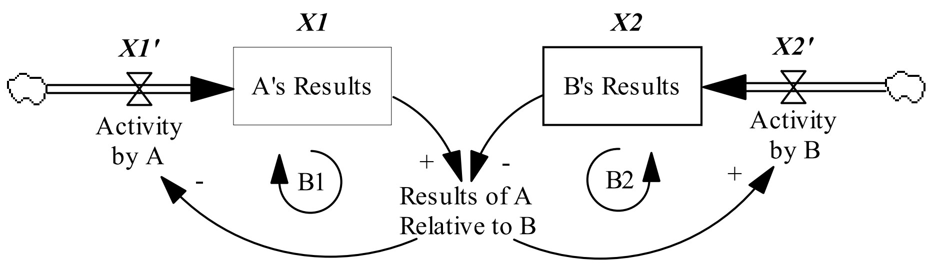


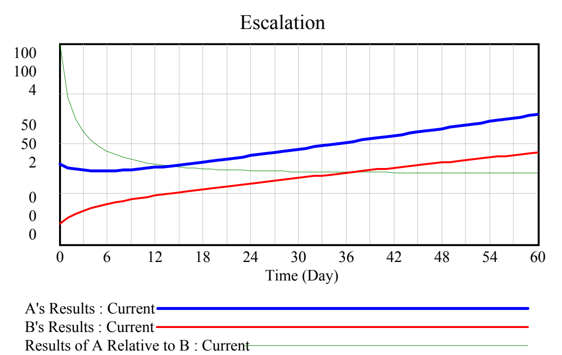
Fig. 2 Block diagram for Escalation archetype

$$x_1'(t) = a\left(R_a - \frac{x_1(t)}{x_2(t)}\right)$$

$$x_2'(t) = -b\left(R_b - \frac{x_1(t)}{x_2(t)}\right)$$

$$x_1(0) = x_{10} \;\; and \;\; x_2(0) = x_{20}$$

Where R_a and R_b refer to the desired relations $x1/x2$ by A and B, respectively. Parameters a and b are the x_i's rates of the description. For the example, $R_a = 2/1$, and $R_b = 1/1$, $a = 1$, $b = 1$, $x_{10} = 40$, and $x_{20} = 10$.



(01) A's Results = INTEG (Activity by A, 40)
(02) Activity by A = 1*((2/1)-Results of A Relative to B)
(03) Activity by B = 1*(Results of A Relative to B - (1/1))
(04) B's Results = INTEG (Activity by B, 10)
(05) FINAL TIME = 60
The final time for the simulation.
(06) INITIAL TIME = 0
The initial time for the simulation.
(07) Results of A Relative to B = A's Results/B's Results
(08) SAVEPER = TIME STEP
The frequency with which output is stored.
(09) TIME STEP = 1
The time step for the simulation.

C. Fixes that Fail

"A fix, effective in the short term, has unforeseen long-term consequences which may require even more use of the same fix" (Senge, 1990, 388.)

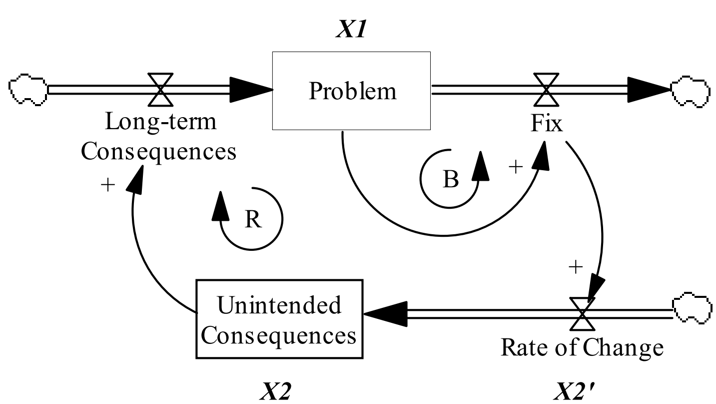
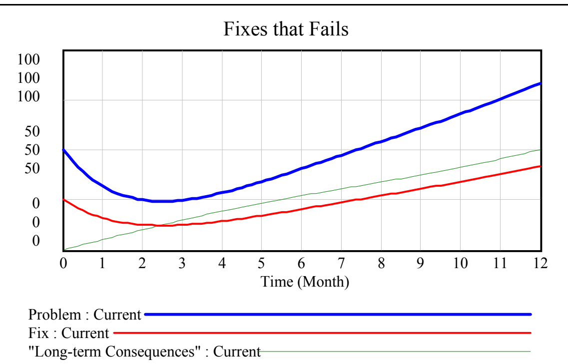


Fig. 3 Block diagram for Fixes that Fails archetype

$$x_1'(t) = ax_1(t) - bx_2(t)$$
$$x_2'(t) = cx_2(t-d)$$
$$x_1(0) = x_{10} \text{ and } x_2(0) = x_{20}$$

Where d is the delay in time units, and a, b, c are proportionality parameters. In this example, $a = 0.5$, $b = 0.5$, $c = 0.40$, $d = 5$, $x_{10} = 50$, and $x_{20} = 0$.



(01) Capacity= INTEG (Investment and Capacity, 80)
(02) Demand= INTEG (Net Rate of Change, 1)
(03) FINAL TIME = 12
The final time for the simulation.
(04) Growing Action= 1/100*Demand
(05) Growth and Underinvestment = Performance Standard-Performance
(06) INITIAL TIME = 0
The initial time for the simulation.
(07) Investment and Capacity = 0.01*Growth and Underinvestment
(08) Net Rate of Change = 0.75*Growing Action*DELAY3(Performance, 1)
(09) Performance = Capacity-Demand
(10) Performance Standard = 100
(11) SAVEPER = TIME STEP
The frequency with which output is stored.
(12) TIME STEP = 0.125
The time step for the simulation.

D. Growth and Underinvestment

"Growth approaches a limit which can be eliminated or pushed into the future if the firm, or individual, invest in additional "capacity." But the investment must be aggressive and sufficiently rapid to forestall reduced growth, or else it will never get made. Oftentimes, key goals or performance standards are lowered to justify underinvestment" (Senge, 1990, 389-390.)

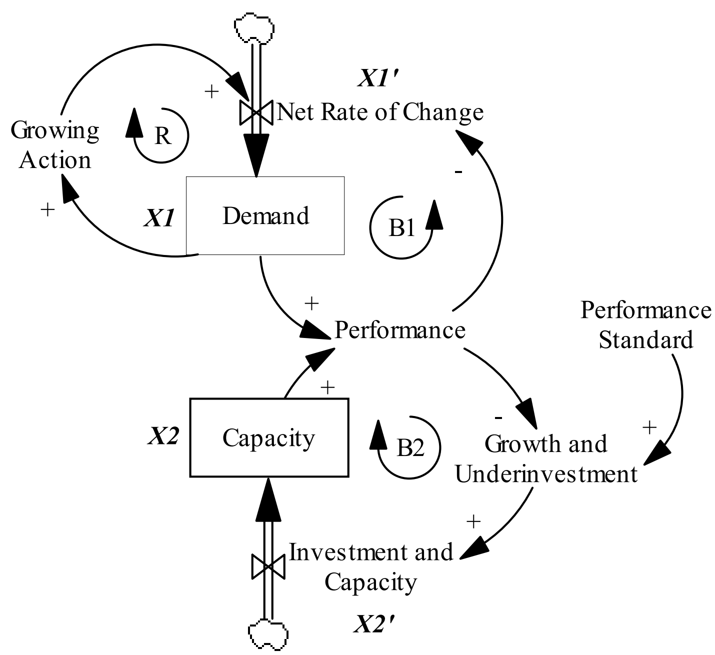


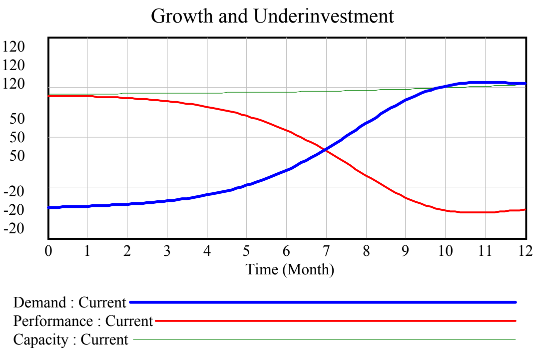
Fig. 4 Block diagram for Growth and Underinvestment archetype

$$x_1'(t) = \frac{a}{b} x_1(t)\left[x_2(t-d) - x_1(t-d)\right]$$

$$x_2'(t) = c\left[Ps + x_1(t) - x_2(t)\right]$$

$$x_1(0) = x_{10} \ \ and \ \ x_2(0) = x_{20}$$

Where Ps is the performance standard, and a, b and, c are model parameters. For this example,
$Ps = 100, a = 0.75, b = 100, c = 0.01, x_{10} = 1$, and $x_{20} = 80$.



(01) Capacity = INTEG (Investment and Capacity, 80)
(02) Demand= INTEG (Net Rate of Change, 1)
(03) FINAL TIME = 12
The final time for the simulation.
(04) Growing Action = 1/100*Demand
(05) Growth and Underinvestment = Performance Standard-Performance
(06) INITIAL TIME = 0
The initial time for the simulation.
(07) Investment and Capacity = 0.01*Growth and Underinvestment
(08) Net Rate of Change = 0.75*Growing Action*DELAY3(Performance, 1)
(09) Performance = Capacity-Demand
(10) Performance Standard = 100
(11) SAVEPER = TIME STEP
The frequency with which output is stored.
(12) TIME STEP = 0.125
The time step for the simulation.

__E. Limits to Growth - Model 1__

"A process feeds on itself to produce a period of accelerating growth or expansion. Then the growth begins to slow (often inexplicably to the participants in the system) and eventually comes to a halt, and may even reverse itself and begin an accelerating collapse.

"The growth phase is caused by a reinforcing feedback process (or by several reinforcing feedback processes.) The slowing arises due to a balancing process brought into play as a "limit" is approached. The limit can be a resource constraint, or an external or internal response to growth. The accelerating collapse (when it occur) arises from the reinforcing process operating in reverse, to generate more and more contraction" (Senge, 1990, 379.)

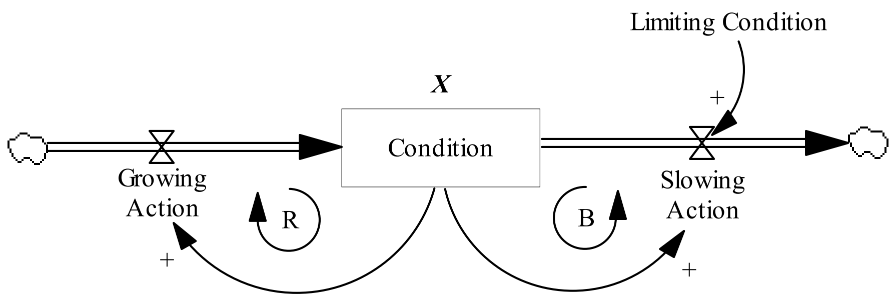


Fig. 5 Block diagram for Limits to Growth archetype

$$x'(t) = ax(t) - ax(t)\left[1 - \frac{L - x(t)}{L}\right]$$

reordering

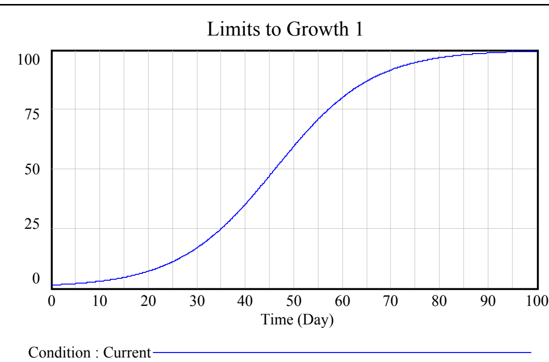
$$x'(t) = ax(t) - \frac{a}{L}x(t)^2$$

thus

$$x'(t) = a\left(1 - \frac{x(t)}{L}\right)x(t)$$

$$x(0) = x_0$$

Where L is the limit of growth and a is the maximum fractional growth. For this example,
$L = 100$, $a = 0.1$, and $x_0 = 1$.



(1) Condition = INTEG (Growing Action-Slowing Action, 1)
(2) FINAL TIME = 100
The final time for the simulation.
(3) Growing Action = 0.1*Condition
(4) INITIAL TIME = 0
The initial time for the simulation.
(5) Limiting Condition = 100
(6) SAVEPER = TIME STEP
The frequency with which output is stored.
(7) Slowing Action = 0.1*Condition*(1-((Limiting Condition-Condition)/Limiting Condition))
(8) TIME STEP = 0.125
The time step for the simulation.

Limits to Growth - Model 2

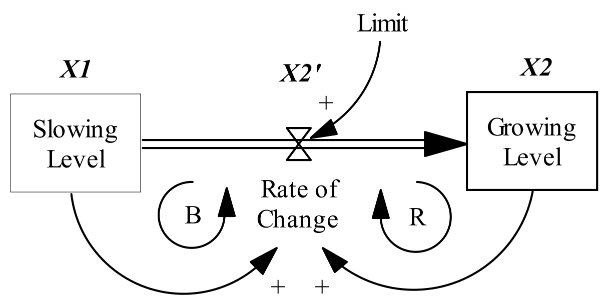


Fig. 6 Block diagram for Limits to Growth archetype - model 2

$$x_2'(t) = \frac{a}{L} x_1(t) x_2(t)$$

since

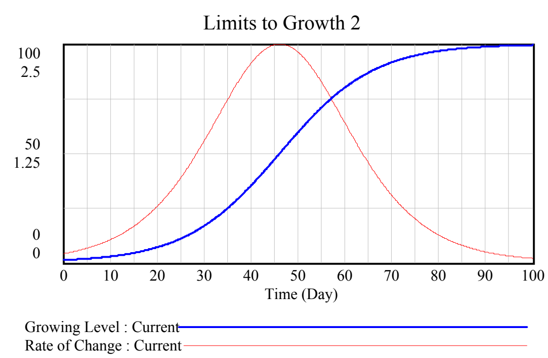
$$x_1(t) = L - x_2(t)$$

then by reordering

$$x_2'(t) = a\left(1 - \frac{x_2'}{L}\right) x_2(t)$$

$$x_1(0) = x_{10} \;\; and \;\; x_2(0) = x_{20}$$

Where L is the limit of growth and a is the maximum fractional growth. For this example,
$L = 100$, $a = 0.1$, $x_{10} = L\text{-}1$, and $x_{20} = 1$.



(1) FINAL TIME = 100
The final time for the simulation.
(2) Growing Level = INTEG (Rate of Change, 1)
(3) INITIAL TIME = 0
The initial time for the simulation.
(4) Limit = 100
(5) Rate of Change = (0.1/Limit)*Growing Level*Slowing Level
(6) SAVEPER = TIME STEP
The frequency with which output is stored.
(7) Slowing Level = INTEG (-Rate of Change, 100-1)
(8) TIME STEP = 0.125
The time step for the simulation.

Limits to Growth - Model 3

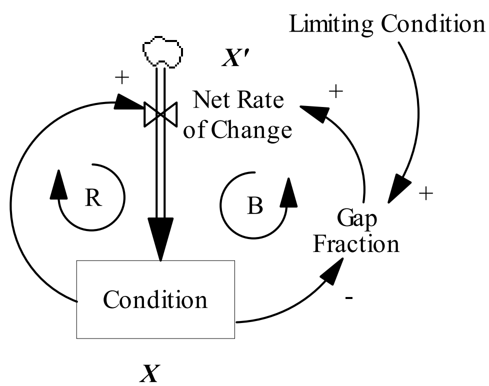


Fig. 7 Block diagram for Limits to Growth archetype - model 3

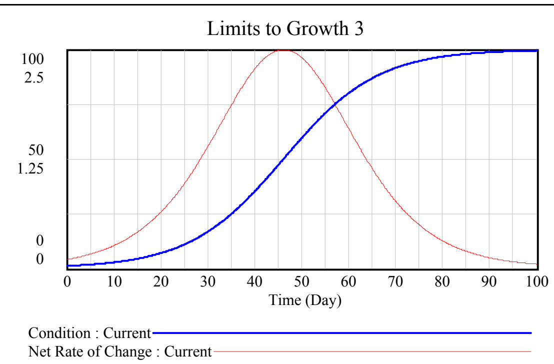
$$x'(t) = ax(t)\left(\frac{L - x(t)}{L}\right)$$

reordering

$$x'(t) = a\left(1 - \frac{x(t)}{L}\right)x(t)$$

$$x(0) = x_0$$

Where L is the limit of growth and a is the maximum fractional growth. For this example,
$L = 100$, $a = 0.1$, and $x_0 = 1$.



(1) Condition = INTEG (Net Rate of Change, 1)
(2) FINAL TIME = 100
The final time for the simulation.
(3) Gap Fraction = (Limiting Condition-Condition)/Limiting Condition
(4) INITIAL TIME = 0
The initial time for the simulation.
(5) Limiting Condition = 100
(6) Net Rate of Change = 0.1*Condition*Gap Fraction
(7) SAVEPER = TIME STEP
The frequency with which output is stored.
(8) TIME STEP = 0.125
The time step for the simulation.

F. Shifting the Burden

"A short term "solution" is used to correct a problem, with seemingly positive immediate results. As this correction is used more and more, more fundamental long-term corrective measures are used less and less. Over time, the capabilities for the fundamental solution may atrophy or become disabled, leading to even greater reliance on the symptomatic solution" (Senge, 1990, 381.)

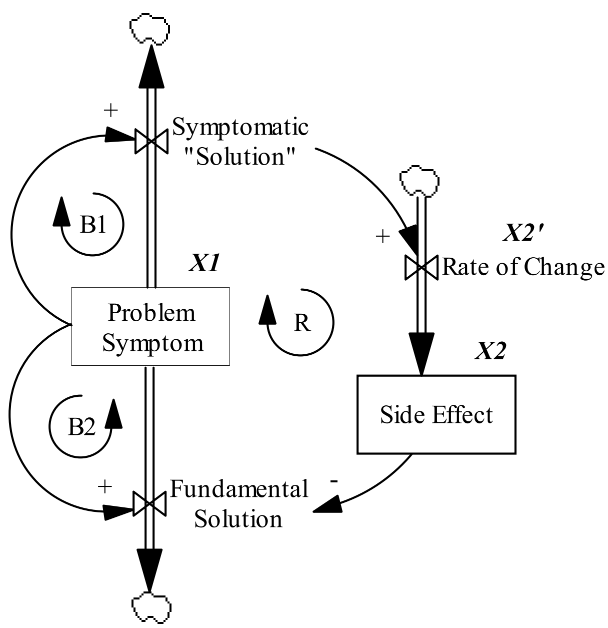


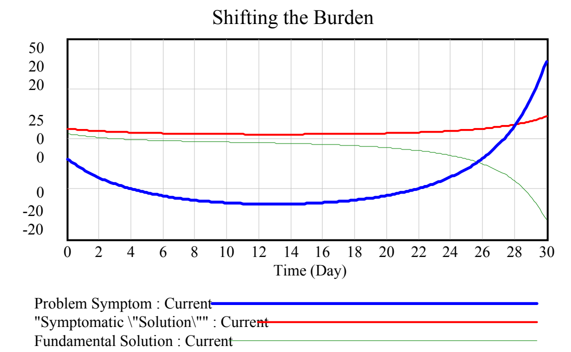
Fig. 8 Block diagram for Shifting The Burden archetype

$$x_1'(t) = -ax_1(t) - [b - cx_2(t)]x_1(t)$$

reordering, the system is:

$$x_1'(t) = -(a+b)x_1(t) + x_1(t)x_2(t)$$
$$x_2'(t) = adx_1(t)$$
$$x_1(0) = x_{10} \ and \ x_2(0) = x_{20}$$

For this example, $a = 0.10$, $b = 0.05$, $c = 1.0$, $d=0.01$, $x_{10} = 20$, and $x_{20} = 0$.



(01) FINAL TIME = 20
The final time for the simulation.
(02) Fundamental Solution = (0.05-1*Side Effect)*Problem Symptom
(03) INITIAL TIME = 0
The initial time for the simulation.
(04) Problem Symptom = INTEG (-Fundamental Solution-"Symptomatic \"Solution\"", 20)
(05) Rate of Change = 0.01*"Symptomatic \"Solution\""
(06) SAVEPER = TIME STEP
The frequency with which output is stored.
(07) Side Effect = INTEG (Rate of Change, 0)
(08) "Symptomatic \"Solution\"" = 0.1*Problem Symptom
(09) TIME STEP = 0.125
The time step for the simulation.

G. Success to the Successful

"Two activities compete for limited support or resources. The more successful one becomes, the more support it gains, thereby starving the other" (Senge, 1990, 385.)

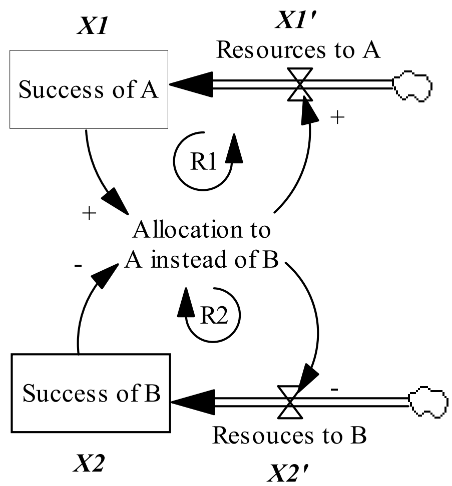
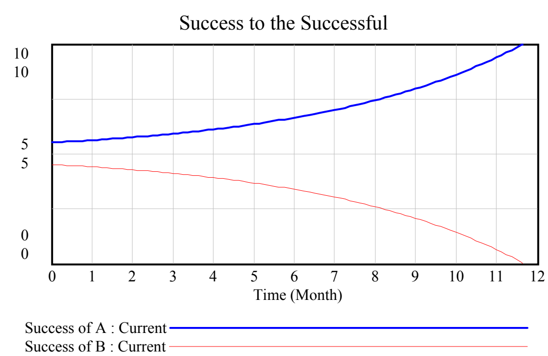


Fig. 9 Block diagram for Success To The Successful archetype

$$x_1'(t) = ax_1(t) - ax_2(t)$$
$$x_2'(t) = -bx_1(t) + bx_2(t)$$
$$x_1(0) = x_{10} \ \ and \ \ x_2(0) = x_{20}$$

For this example, $a = 0.1$, $b = 0.1$, x_{10} = 5.5, and $x_{20} = 4.5$.



(01) Allocation to A instead of B = Success of A-Success of B
(02) FINAL TIME = 12
The final time for the simulation.
(03) INITIAL TIME = 0
The initial time for the simulation.
(04) Resouces to B = -0.1*Allocation to A instead of B
(05) Resources to A = 0.1*Allocation to A instead of B
(06) SAVEPER = TIME STEP
The frequency with which output is stored.
(07) Success of A = INTEG (Resources to A, 5.5)
(08) Success of B = INTEG (Resouces to B, 4.5)
(09) TIME STEP = 0.125
The time step for the simulation.

H. Tragedy of the Commons

"Individuals use a commonly available but limited resource solely on the basis of individual need. At first they are rewarded for using it; eventually, they get diminishing returns, which causes them to intensify their efforts. Eventually, the resource is either significantly depleted, eroded, or entirely used up" (Senge, 1990, 387.)

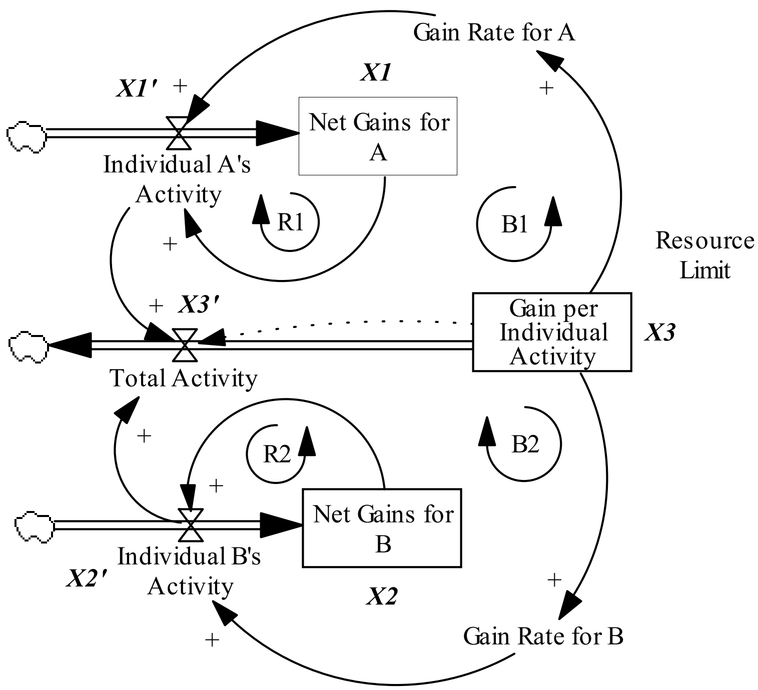


Fig. 10 Block diagram for Tragedy of the Commons archetype

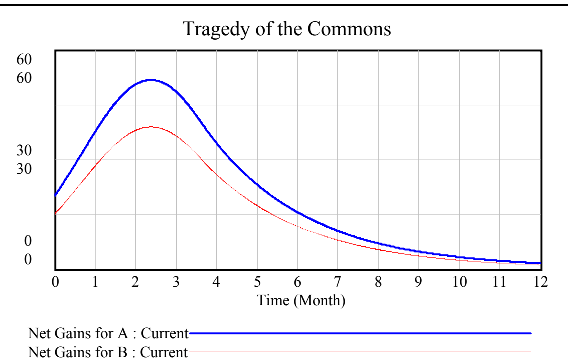
$$x_1'(t) = \left(\frac{x_3(t)}{a} - b\right)x_1(t)$$

$$x'_2(t) = \left(\frac{x_3}{c} - d\right)x_2(t)$$

$$x_3'(t) = x_1(t-e) + x_2(t-e)$$

$$x_1(0) = x_{10},\ \ x_2(0) = x_{20},\ and\ x_3(0) = x_{30}$$

For this example, $a = 1000*12$, $b = 0.4$, $c = 1000*12$, $d = 0.4$, $e = 5$, $x_{10} = 20$, $x_{20} = 15$, and $x_{30} = 100$.



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(01)    FINAL TIME  = 12
        The final time for the simulation.
(02)    Gain per Individual Activity = INTEG (-Total Activity, Resource Limit)
(03)    Gain Rate for A = Gain per Individual Activity/1000*12-0.4
(04)    Gain Rate for B = Gain per Individual Activity/1000*12-0.4
(05)    Individual A's Activity = Gain Rate for A*Net Gains for A
(06)    Individual B's Activity = Net Gains for B*Gain Rate for B
(07)    INITIAL TIME  = 0
        The initial time for the simulation.
(08)    Net Gains for A = INTEG (Individual A's Activity, 20)
(09)    Net Gains for B= INTEG (     Individual B's Activity, 15)
(10)    Resource Limit = 100
(11)    SAVEPER  = TIME STEP
        The frequency with which output is stored.
(12)    TIME STEP  = 0.0078125
        The time step for the simulation.
(13)    Total Activity = IF THEN ELSE( Gain per Individual Activity>0,
          DELAY3(Individual A's Activity   +Individual B's Activity, 5 ) , 0 )
```

IV. Conclusions

Proposed mathematical structures for systemic archetypes has been presented. The mathematical structures are based on systems of differential equations and the concept of state variable representation.

Formal models of dynamic systems represented by differential equations have proved to be effective elements to transmit knowledge, ideas, and experiences among different disciplines of science.

System Dynamics as a modeling method that enhances learning of complex systems by using feedback principles and computer simulation models have the great potential to become a conventional provider of formal models for the social sciences. The following and natural step seems to be to represent mathematically System Dynamics models.

Then a wider channel of communication may be established among other disciplines by sharing System Dynamics models in the form of differential equations, and taking advantages in the same way of those models developed and considered as deep knowledge by other areas.

So the goals of System Dynamics would not change at all. As it is known, learning and policy designs is what SD is about. However, the scope of the implementation stage would widen with the availability of a second representation of the results of the modeling process. The benefits of having this second standard communication mechanism might be substantial.

V. References

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Back to the Top