The Common Sense of the S-shaped Growth: Regarding the Exponential Increase

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We will propose that the expression, 'exponential function increase' causes confusion and that such phrases used to describe a state do not serve in modeling the structure of a phenomenon. We will propose that it is necessary to clearly state in details the rule of the increase when discussing modeling. Our presentation focuses especially on logistic curves and shows that the first increase in this curve is not exponential-like by the fluctuation index which was defined with the purpose of analyzing the instability degree of time series data (Appendix1).

What images do users usually envision when expressing 'exponential increase' or 'exponential function increase'? There is a tendency for users to borrow such expressions when there is a wish to project images of fast change or sudden increase of data. Whether the subject under discussion is an 'exponential function increase' or a 'high order function increase', they tend to think there is no impact to the latter part of the discussion. As long as the discussion proceeds and remains limited to a description of an adjectival state, then there is no issue.

However when the discussion shifts mid-ways to that indicative of increasing volumes or structure, it becomes necessary to distinguish whether the subject is in its truest sense an exponential function increase or a high order function increase.

Discussions on distribution of durable goods or spread of epidemics are proceeding without the above distinction being made.

In this presentation, we will propose that the expression, 'exponential function increase', readily causes confusion and that such phrases used to describe a state do not serve in modeling the structure of a phenomenon. We will propose that it is necessary to clearly state in details the rule of the increase when discussing modeling.

Our presentation focuses especially on logistic curves. A reconfirmation of the mathematical principle of the growth model is as follows.

Fixed model based on fixed growth rate:

$$dN/dt=RN \qquad \text{R: Growth rate (Constant)} \qquad (1)$$

As widely known, N becomes the exponential function of time t.

Generally, however, the growth rate decreases when the quantity reaches a certain level. In other words, R becomes the function of R (N) based on the population N. The simplest R (N) is a straight line. When R=a-bN, the equation of the model becomes the following.

$$dN/dt=N\,(a-bN) \qquad (2)$$

Then the curve of N materializes as a S-shaped growth curve or a logistic curve. The expression of ‘an exponential function like growth’ is often used about N which indicates the population. This appears to be for the following reasons.

1. Solution of the differential equation (2) has combination of exponential functions.
2. When t is very small, equation (2) is approximated by equation (1).

However, our research indicates that whichever part of this curve in the very early stage is examined, it does not increase exponentially. Then what increases exponentially? This expression is not used for the population but for the next equation. The relation of equation (3) appears in the midst of solving the differential equation (2).

$$N/(S-N) \quad \Rightarrow \quad \exp(at) \qquad S=a/b \qquad (3)$$

The ratio of the member N at each moment and the part (S-N) which is to be member after that does increase like exponential function. Today's confusion arises by the growing part of the S-shaped curve having been related to ‘an exponential function like increase’ while leaving this equation behind.

As for equation (2), it is possible to solve analytically. The solution is

$$N = S\exp(at) / (1+\exp(at)) \qquad S=a/b \qquad (4)$$

The graph of the variable N becomes a S-shaped curve. Since equation (4) contains an exponential function in both the numerator and the denominator, it is likely to be one of the reasons that we tie hastily the exponential like growth to the increasing part of this curve.

It is possible to show theoretically and statistically that the first part of the S-shaped curve is approximated with 3rd order polynomial better than with exponential function. (Appendix 2)

As long as only the increasing tendency is being analyzed qualitatively, the confusing use of S-shaped curve and exponential increase is considered not to cause such an unclear problem. The dangers that are to be caused by using such adjectival expressions are examined as follows.

There is an exercise explaining the differential equation model in the textbook (Reference [5]).

1. The number of accumulation of the AIDS patient in the United States is given. The model which is in proportion to the times of contact is built. The equation (1) that the number of patients becomes the well known exponential function is presented with the model.
2. The exercise ask you to look for the most suitable model through application of optimal curves, such as exponential function, 2nd order, or 3rd order polynomial expressions to the actual data. Then it points out that the 3rd order polynomial fits the actual data better than the exponential or other models.
3. The exercise also suggests that it is wonder the exponential increase does no result while the first simple model is suitable in understanding of the AIDS.

The data given by the beginning are simple increasing data, and it is difficult to expect a S-shaped curve. It is appropriate to add the upper limit of the number of infected persons to the equation (2) as the simple model which meets these data. It is better to use the early part (until the start of the function) which is calculated with the equation (2) from our calculation result. However, since the observed data is not exponential, there are possible dangers for those unfortunate students connecting the exponential like increase to the logistic curve to come up with other models. For example, the next equation gives the 3rd order polynomial solution.

$dN/dt=C*N^{(2/3)}$

And, even for those students coming up with the S-shaped curve, there is a possibility that they introduce unnecessary modification factors to modify the first part.

Our proposal is that, when expressing increasing data, it is better not to use the expression of 'exponential function like increase' except for the real exponential case. It is not so easy for beginners to appreciate whether the term 'exponential function like' is adjectival or not from the context.

We are interested in the analysis of the increasing tendency of time series data, and have been continuing the research. This time, we defined fluctuation index distinguishing the exponential growth and reports that the first increase in the S-shaped curve is not exponential-like by the indices.

References

[1] “Dynamic Modeling in the Health Science”, J. L. Hargrove, p134
[2] “Dynamic Modeling of Environmental System”, M. L. Deaton et al., p100
[3] “Business Dynamics”, J. D. Sterman, p118
[4] “Modeling the Environment”, A. Ford, p58
[5] “Should We Risk It”, D. M. Kammen and D. M. Hassenzahl, Princeton Univ. Press (1999)

Appendix 1 Data fluctuation index.

Consider general time series data.

$f(t_{n+1}), f(t_n), \cdots\cdots, f(t_1), f(t_0)$

Let the time step $h = t_i - t_{i-1}$ be fixed for simplicity. When there is less much misunderstanding, not ti but its suffix i indicates time.

Define the index based on the ratio of the changes of the time series data. The basic concept is expressed with the following equation.

(Present amount of change) =α* (Average of the past amount of change)

The value of α becomes large when the amount of change in the present system is equal to several times of the past. α is defined as the index which shows the degree of the instability of the system as follows.

$$\alpha(1) = \left(f(n+1) - f(n)\right)^2 / \left(f(n) - f(n-1)\right)^2$$
$$\alpha(2) = 2 \times \left(f(n+1) - f(n)\right)^2 / \left[\left(f(n) - f(n-1)\right)^2 + \left(f(n-1) - f(n-2)\right)^2\right]$$

Though it is possible to define using the average of three terms and more in the same way, the index based on two terms is discussed in this paper from the purpose of presenting as simple an index as possible.

Generally this index becomes the function of the independence variable (time) n and the time step h.

$$\alpha = G(n, h)$$

This index becomes fixed value for the well-known simple increase functions as follows.

1. Linear function: $at + b$ (a, b are coefficient)

The value of the index α becomes 1.

The index α of the linear functions is always 1 under the optional a, b, t.

2. Exponential function: ae^t (a is coefficient.)

The index becomes the function of the time step h only.

$$\alpha(1) = e^{2h} \quad (5)$$

$$\alpha(2) = \frac{2e^{4h}}{1+e^{2h}} \quad (6)$$

α (1) and α (2) become the function of h. In other words, the index value of exponential functions is fixed and is not affected by the value of the independence variable. Table 1 reports the calculation results of the relations of time step h and each α.

Table 1 Time step h and the α value

h	**0.25**	**0.5**	**0.75**	**1.0**
α (1)	1.6	2.7	4.5	7.4
α (2)	2.1	4.0	7.3	13.0

3. Power function: at^k (a, k are coefficient)

The index α becomes the function related to both time t and time step h. The index is found to approach 1 when time t increases upon calculating α for several power functions. However, large change is observed around the extreme value and the turning point. But, it is possible to show that the index becomes 1 in the region where the time is large enough. (See appendix) Therefore, when the independence variable is sufficiently large, it can be said that the power function is not increasing exponentially but is in the same as the linear function. Figure 1 shows the example of the third power. Although, the index α has larger value than that of the exponential function around the origin, it approaches 1 rapidly when leaves the origin. Using the numerical calculation, we have confirmed up to K=9 that the value of α for the k power function becomes smaller than that for the exponential function with t≧k .

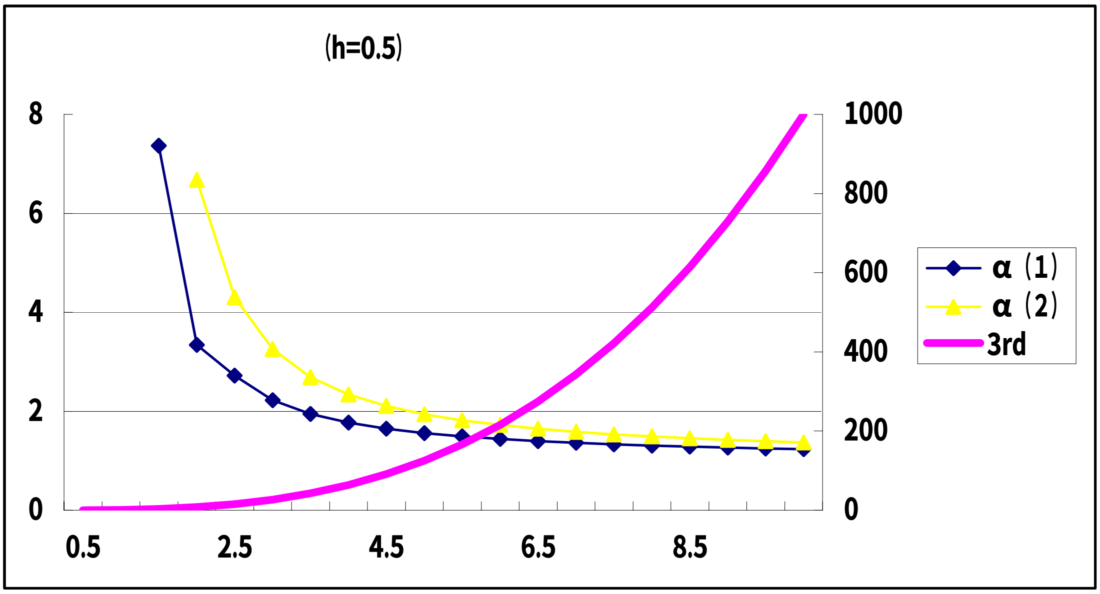

Figure 1　Third order function and index α

Appendix 2　Logistic (growth) curve

We applied the proposed index to the logistic curve. Figure 2 shows the index α (1) and α(2) with the time step h=0.25. Index α(1) is 1 in all the ranges. And it has the value smaller than 6. α(2) is 2 as well. Because it is less than 0, it cannot be said that the first portion of this graph is exponential function like increase.

To verify this, we performed the application of optimal curves in the range[0,2] by using the statistics analytic software (SPSS). Data is calculated using the equation in the Figure 2 with initial value (X=0.1 at h=0). Since index α(2) is larger than 1 and is smaller than 2, we tried the 2nd order and the 3rd order polynomial expressions which have larger increasing rate than 1st order linear function and have lower increasing rate compared to exponential functions. Both the 1st order linear function and the exponential function are also indicated for the comparison. The result by the decision coefficient became the order indicated in the following table. (Table 2)

Table 2 Approximate curve (range[0,2]) of logistic curve

	Decision coefficient	Adj. decision coefficient	Standard error
3^{rd} order	0.99999	0.99998	0.00056
2^{nd} order	0.99996	0.99995	0.00087
Exponential	0.99560	0.99498	0.03681
1^{st}order	0.98092	0.97820	0.01799

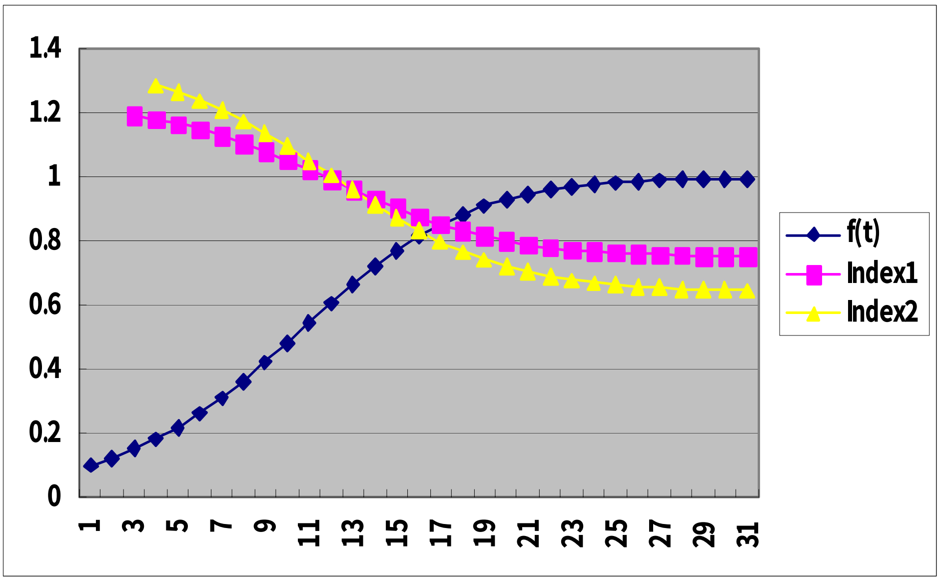

Figure2 Logistic curve and the index

The curve is plotted by the difference equation; $X = X(1-X)$.

(Initial value is 0.1 at h=0)

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