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Bulk Delays in System Dynamics Model

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Abstract

In a supply chain handling fast moving consumer goods one of the most commonly followed material ordering process involves releasing supply orders whenever inventory level falls below a predefined re-order level. There are two difficulties in modelling this process in system dynamics parlance. Firstly events in this process occur discretely whereas system dynamics emphasises continuous change. Secondly the time elapsed between occurrence of two successive events depend on inflow pattern. This violates an assumption that is implicit in the delays that are commonly used in system dynamics models. Delays in System Dynamics are modelled as k–th order Erlang functions which are characterised by two parameters i.e. the average time spent in the delay and the order of the delay. The order is actually the ratio between the square of average holding time and the variance of holding time. One of the important assumptions made here is that the average holding time remains independent of the inflow pattern. In many situations these provide a good representation of the reality. However in the material ordering process described the flow is delayed on number and the assumption does not hold.

In this paper we have tried to find a representation for the process that would not require making compromises either on the modelling accuracy or on the philosophical foundations of the methodology. Based on analysis of behaviour we have demonstrated that the re-order-level based ordering is equivalent to an Erlang process of order equal to the re-order-level. As a result it can also be modelled as an exponential delay. The average length of delay has to be kept as a variable to represent reality.

The work attains importance because of its utility in modelling supply chains where there has been a renewed interest in using system dynamics. In analysing the delays we have used a framework that is commonly used to analyse stochastic processes. Similar method of analysis could be applied to analyse other kind of processes. This is another important contribution of the paper.

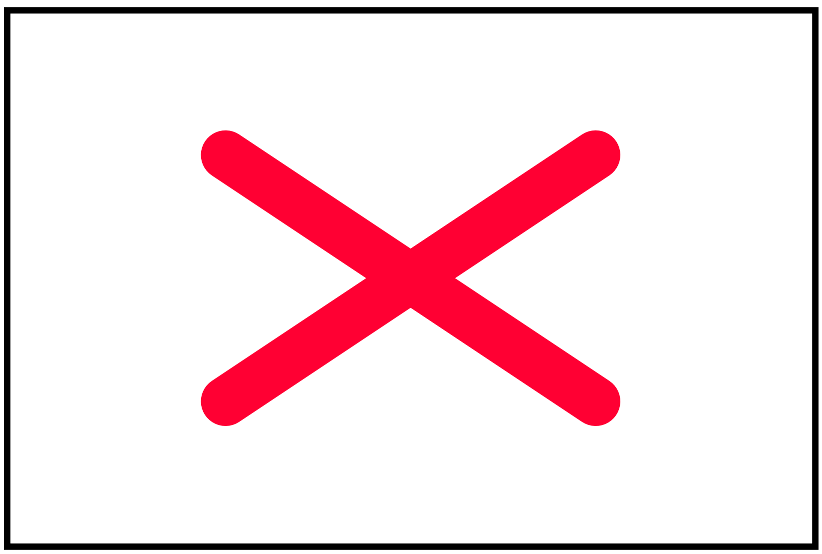
A. Introduction

Consider the supply chain of a fast moving consumer goods (FMCG) item in which sales are made from stock. It would be common to find an inventory management system where the orders for stock replenishment are released as soon as the inventory level falls below the Re-Order Level (ROL). Normally once the order has been placed, supply would be received after some delay. Since responsiveness to demand change is an important requirement of a supply chain handling FMCG items, ROL would be an important handle by which management would balance the rapidity with which of orders are placed vis-a-vis the cost of holding inventory. It may be assumed that ROL is held constant for a reasonable period of time.

In system dynamics, the inventory management system could be modelled in one of the two ways mentioned below.

In the first method, shown in figure 1, pending requisitions would be modelled as a stock (Pending_Requisitions). Each time a sale (Sales) is made, a requisition for supply would be created and the value of Pending_Requisitions increased. Supplier_Order, the outflow from Pending_Requisitions would be positive only when the value of Pending_Requisitions is equal to the threshold (Re_Order_Level). Suppler_Order would be defined as:

Supplier_Order = if(Pending_Requisitions >= Re_Order_Level, Pending_Requests/DT,0)



Modelling the ordering process in this manner would not be desirable because of the following reasons.

(a) This representation would render a part of the system operating in discrete fashion whereas the methodology of system dynamics, viewing the system from a holistic perspective, would like to visualise it as one where changes take place continuously (Forrester 1961).

(b) The time constant for the level-rate pair Pending_Requests * Supplier_Order, can not be defined easily. This would make selection of a DT value difficult. Improper selection of DT would leave the possibility that Supplier_Order instead of getting generated exactly when Pending_Requests equals Re_order_Level, would get generated later. The results would therefore be inaccurate.

Alternatively the ordering process could be modelled as a delay between Sales and Supplier_Order since buffering of purchase requisitions would introduce a delay in the system, which would be given by the following relation:

$$Delay(t) = ROL/Sales(t)$$

By the above relation T, which is the average of Delay(t), would be dependent on Sales. But no specific comments can be made about the order of the delay. Although most of the system dynamics model are not very sensitive to the order of the delay and in most real life situations it is sufficient to assume a third order delay, an inappropriate selection of order could result in faulty dynamic behaviour.

Either way, modelling this process in system dynamics using the available elements would require compromising on either the methodology's philosophical foundations or on the accuracy of representation. We felt that there is a need to find a better representation in system dynamics parlance because this ordering mechanism is quite common in FMCG supply chains and there has been a renewed interest in modelling of supply chains using system dynamics.

In this paper we have tried to find a representation of the process that overcomes the problems mentioned above. We realised that this process has all the characteristics of a delay with the difference that discharge from it takes place in bulk. We have therefore termed this as bulk delay and have tried to explore whether these could be represented as an exponential delay. Our exploration has been based mainly on a comparative study of the transient behaviour of different types of delays. Initially we have provided a review of exponential delays. Next we have defined a bulk-delay and have derived its transient behaviour. The transient behaviour has led to suggesting a representation for bulk delays. In conclusion we have provided a discussion on the analysis framework that we found to be quite powerful and could be used for analysing other processes.

B. A Review of Delays in System Dynamics

Delays are an important part of any system. They represent the time that elapses between a change occurring in some part of the system and the same getting reflected elsewhere. Delays play detrimental role in deciding the dynamic behaviour of systems. Modelling of systems therefore necessitates that delays are properly represented in order that real life behaviour can be replicated.

In System Dynamics, delays are modelled as an accumulation that divides a flow into two parts. A delay causes accumulation of the inflow. Flow coming into the accumulation is held for some time before being discharged (Forrester 1961, Coyle 1971, Mohapatra 1994). A delay is characterised by two parameters namely T, the time constant and k, the order. In case we assume that the flow consists of well identifiable discrete objects then T implies the average time an element is held inside the delay. Since system dynamics models consider only continuous flows of material and information, T is interpreted as the average time elapsed between some change happening in the inflow and the same getting reflected in the outflow. Order on the other hand implies the manner in which the outflow adjusts to the change in inflow. Between two delays of identical k, the one with higher T, takes more time to absorb the change. Again, between two delays having identical T the one with higher k starts responding later and most of the changes occur around T. In steady state delays do not have any effect at all. However whenever some change is inflicted on the system, delays decide the manner in which the system responds to the change.

Normally it is assumed that delay characteristics i.e. T and k, are independent of the inflow pattern or the quantity accumulated in the delay. This implies that in case change occurs in the input, the average time taken for the output to reflect that change does not depend on the rapidity with which change has taken place. In an inventory management situation where shipments are made based on orders placed and no capacity constraint exists, a system dynamics model would assume that in case there is a change in order booking rate, the shipment rate would follow suit after the delay. But the average length of time taken to convert orders into shipment would not depend on the manner in which change has occurred in order booking rate.

In system dynamics a delay of order k is approximated as a cascade of k first order exponential delays (Meadows 1974, Mohapatra 1976, Hamilton 1980). For a k-th order delay, having a time constant T the transient response is given by the following (Mohapatra 1976):

$$OUT_k(t)/L(0) = [k/T]^k t^{(k-1)} e^{-kt/T}/(k-1)! \qquad [1]$$

Where $OUT_k(t)$ is the outflow from the delay at time t and L(0) is the value at time 0 of the first level in a chain of k levels. Defining t_k^* as the time at which maximum outflow occurs from a k-th order delay Mohapatra (1976) has derived the following expression for a k-th order exponential delay:

$$t_k^* = (1 - 1/k)\, T \qquad [2]$$

In equation [1], if we set L(0) equal to 1 then we derive the impulse response of a k-th order delay which is given by the following:

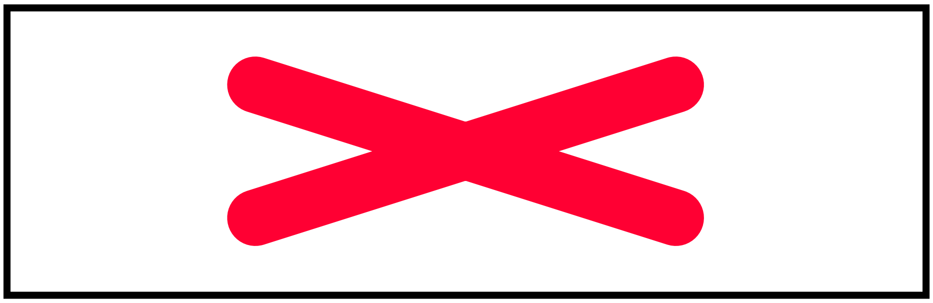
$$OUT_k(t) = [k/T]^k t^{(k-1)} e^{-kt/T}/(k-1)! \qquad [3]$$

Equation [3] is equivalent to the holding time p.d.f. of a k-th order Erlang distribution. The average holding time is equal to T and a variance is equal to T^2/k. Based on this equivalence we can make the generalised statement that the holding time p.d.f. for a delay also gives the transient response of the delay. With increasing value of k, variance of the elapsed time reduces. For a pipeline delay (k approaching $\propto$) the variance vanishes. The equivalence between a k-th order delay and a k-th order Erlang function enables estimation of k from observed values of T and variance σ_T. For k = 1, equation [3] reduces to

$$OUT_k(t) = (1/T)e^{-t/T}$$

This is the p.d.f. of an exponential distribution.

The above relations could be derived from another angle. Suppose the system handles discrete elements. Then one could equate a k-th order delay to a k + 1 state transient Markov process that has k transient states and only one recurrent state (Mohapatra & Roy 1991). In this each state represents one intermediate level inside the delay. In each transition k/T fraction of the content of i-th level gets transferred to the i+1-th level in the chain. From this equivalence we can draw the state transition diagram for a k-th order exponential delay and the same is given in Figure 2.



The path length from state i to state j gives the p.d.f. for time taken for travelling from state i to state j (Howard 1971). The path length from state 1 to state k+1 thus gives the holding time p.d.f for the delay. It can be easily shown that the same would be that of an Erlang distribution with parameters k and T. The average length of delay can also be worked out from the diagram. Since on any transition k/T fraction moves from one state to the next one, the average time to move from the state i to state i+1 is T/k. The average time to move from the first state to k+1-th state is thus given by the following relation:

$$\nu = \Sigma\, 1/p_i \qquad [4]$$

From visual inspection one can see that ν equals to T.

C. Modelling the ROL based Ordering

We have seen that the ROL based ordering actually introduces a delay in the system that has the following characteristics.

(i) It appears in a physical flow

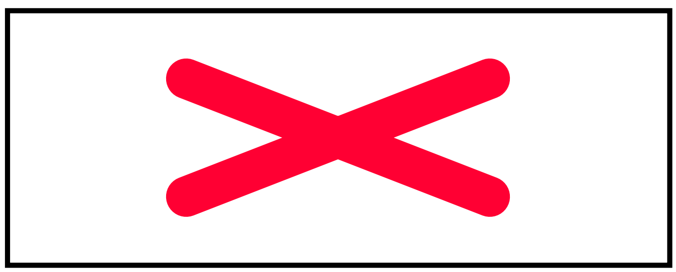
(ii) Accumulation takes place inside the delay

(iii) Transitions take place as and when the accumulation inside the delay reaches a threshold (N)

(iv) This delay is also characterised by the time constant (T). This represents the average time an unit spends in the delay

We can see that the assumption of T being independent of the inflow pattern does not hold. Examples of this abound in physical (automatic flushes that discharge water when the level exceeds a certain level) and industrial systems (material ordering based on reorder quantity, capacity ordering based on shortage crossing certain threshold). In all of these systems T, the time delay between inflow and outflow does depend on the inflow pattern. For the purpose of generalisation we would henceforth refer to the process as bulk delay.

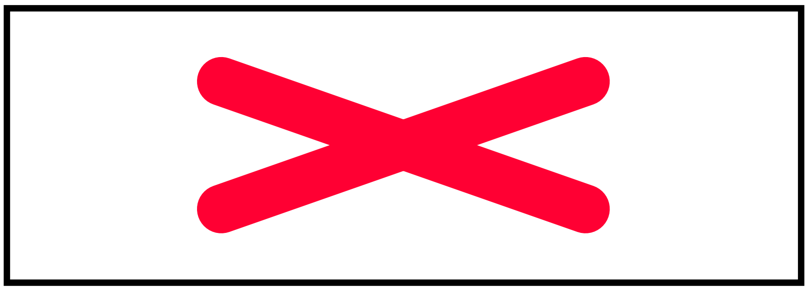
We would use the probabilistic framework mentioned in the previous section to understand the behaviour of the process. For the purpose of analysis we assume a system given in Figure 3., one that only has a bulk delay.



We assume that the flow consists of discrete elements and that the inflow follows a Poisson distribution. The assumption of the Poisson process is valid because Sahin (1979) has shown that the memoryless property is inherent in the stock and flow structure of system dynamics models. For the Poisson process we define λ as the average flow rate. Thus for the inflow all characteristics of a Poisson process can be assumed. In our analysis, we shall use one of those which says that for an average inflow rate of λ per unit time the probability that a transition takes place during a small time interval DT is equal to λ DT.

Consider this system to be in a state when there is nothing stored inside the delay. Suppose we observe the system after DT. Since the inflow is a Poisson process probability that one element has flown into the accumulation is λ DT. We denote this as the transition to state 1. The probability that no inflow has occurred and system continues to be in state 0 is $(1 - \lambda$ DT). We could also interpret this as if the level value has increased by the same amount. Extending this further we say that the level inside the delay transitions through values 0,1,2,…N. In terms of state change we say that the system transitions from state i (i = 0, 1, 2, ….N – 2, N – 1) to the next higher state i + 1 with a transition probability equal to λ. However the transition from state N – 1 to N requires different treatment. After state N –1, the system reverts to state 0 (the entire content is emptied) as soon as the value reaches N but this state 0 is a recurrent state. In order to differentiate this recurrent state from the transient state 0, we take another state and call this state 0'.

The states-transitions diagram explaining the above is shown in figure 4.



As before the path length from state i to state 0' could be used to derive the p.d.f. of the time taken to reach the recurrent state from state i. In this case though we can obtain the p.d.f. from inspection because we observe that the state-transition diagram in Figure 4 is structurally equivalent to the one for a k-th order exponential delay shown in Figure 2. This equivalence allows us to say that a bulk delay is equivalent to an Erlang process of order N that has a p.d.f. as given below:

$$f(t) = \lambda^N t^{N-1} e^{-\lambda t}/(N-1)! \qquad [5]$$

Since equation [5] is the holding time p.d.f. for the process given in Figure [4], we can now say that equation [5] completely describes the transient response of a bulk delay. It is thus an exponential delay of order N, the threshold value. The time constant T is equal to N/λ. One notes that the value of T matches the value of average delay obtained by inspecting figure 4 following the expression given in Equation [4]. The variance (σ) of holding time is given by N/λ^2.

Since f(t) tends to zero as t tends to infinity we say that in the steady state the delay does not have any effect on the overall system behaviour.

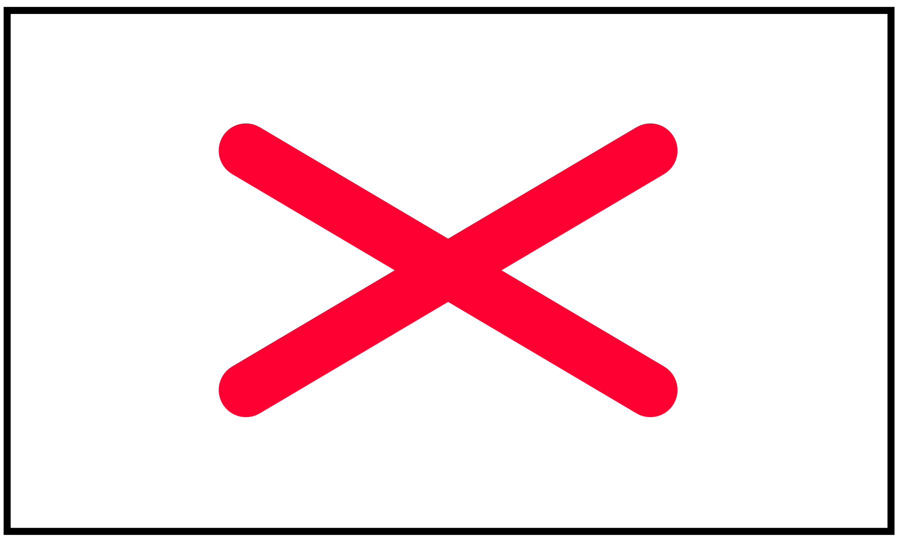
We define t_N* as the time at which maximum output would occur for a bulk delay of threshold N. The t_N* could be obtained by taking the first derivative f(t) and setting that to zero.

$$t_N^* = (N - 1) / \lambda$$
$$= (1 - 1/N)\, T, \qquad \text{where } T = \lambda / N \qquad [6]$$

With these generalised results of bulk delay made available we now proceed to model this in system dynamics. Such a delay, we have seen, can be modelled as an exponential delay with order set equal to the threshold value and a variable time constant set equal to the ratio of the threshold value and the average arrival rate.

We apply this in modelling the ROL based system described in the beginning of this paper. We could now model this in the manner given in Figure 5. In the modified model Pending_Requistion is shown as a delay of order equal to Re_Order_Level. Time Constant T is the ratio of Re_Order_Level and Sales. Supplier_order is given by

Supplier_Order = DelayMtr(Sales,T,Re_OrderLevel)



Representing the ordering process in this manner has the following advantages:

(1) The ordering is represented as a continuous process.

(2) The known values of average length of delay (ratio of Re_order_Level and expected value of Sales) and delay order (Re_order_Level) enables more accurate estimation of DT and consequently more stable simulation results.

D. Conclusions

While handling real life situations one comes across processes which can not be straightaway modelled using existing elements of system dynamics method. But deeper analysis into the behaviour of the system enables us to find equivalence. In this paper we demonstrated how this could be done.

We considered a process, which at first appeared to be discrete and difficult to model using existing elements. However investigation into the behaviour led us to an equivalence of the process and exponential delay that has been very widely in system dynamics models. The process could thus be modelled quite accurately.

In analysing the process we used a framework that is commonly used in the case of stochastic processes. It was interesting to note that although system dynamics models appear deterministic in nature, they lend themselves to analysis in probabilistic framework. We feel that the method of analysis is quite general to be applied to whole variety of processes. The rich repertoire of analytical results available for stochastic processes can then be used to find similar results for system dynamics models which have so far depended mostly on simulation based analysis.

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