Fuzzy System Dynamics

Bjørnar Tessem & Pål Davidsen
Department of Information Science
University of Bergen
N-5020 Bergen, Norway

Abstract

Fuzzy numbers is presented as an alternative to probabilistic methods for the management of uncertainty in system dynamic models. Fuzzy numbers are particularly suitable to represent vagueness and qualitative values. Fuzzy numbers are used during simulation, but due to interactiveness among variables there is a need for global optimization methods. Some examples that illustrates the use of fuzzy numbers, both directly and as a means to represent qualitative values, are shown.

Keywords: System dynamics, fuzzy numbers, qualitative simulation, vagueness

1 Introduction.

This paper describes the need for and presents an implementation of a qualitative approach to the description, simulation, and analysis of complex, dynamic systems. The approach is based upon the fuzzy set theory and fuzzy numbers, which constitute an alternative to probability distributions, commonly used to represent uncertainties associated with dynamic systems.

Our implementation serves as a foundation for fuzzy extensions of system dynamics simulation languages and software. Such extensions can be implemented by overloading variables, parameters, operators, tables and other special functions and also the associated graphical symbols (Fishwick et al. 1991). In this paper, we do not consider explicit use of fuzzy relations, as discussed by Wenstøp (Wenstøp, 1976, 1979), but limit our discussion to expressions characterized by fuzzy parameters, operators and functions.

Having documented the need to utilize fuzzy numbers in system dynamics simulations, we discuss the issues of implementation - in particular issues associated with the interaction of variables caused by nonlinearities. We provide a few simple examples. An outline of further research concludes our discussion.

2 Fuzzy numbers in simulations. Why?

We need qualitative techniques because we are often unable or unwilling to describe system structure, state, and behavior with exact numerical precision. Our need to circumvent numerical representations originates from two sources: Our conceptions are often uncertain and commonly vague. If, for example, we have partitioned all states that the system can take, into disjunct sets, but not determined the set membership of the current state, then we are uncertain. If the system assumes the laws of probability, then we may represent our uncertainties in the form of probability distributions. If, on the other hand, we can or will not identify a set of disjoint sets of system states, then we are left to express ourselves vaguely - typically linguistically - such as *about 5, somewhere between 10 and 15*, or *in the middle*.

Moreover, numerical details often obscure significant **behavior patterns** and obstruct our identification and understanding of the dominating structural subcomponents underlying these patterns. In particular, this is true for the common cases of shifting equilibria and loop polarities in nonlinear systems. Consequently, we need qualitative methods to abstract from insignificant details and help us focus on the structural components underlying the significant dynamic characteristics, showing up in the time- and state-space of these systems.

In system dynamics, qualitative analyses, based upon stock-and flow, feedback loop, time- and state-space diagrams are quite common. The dynamic qualities of various nonlinear systems have been documented more rigorously in the literature. Based upon various combinations of graphical analysis (Aracil et al. 1992), (Davidsen 1992), (Toro et.al. 1992), empirical experiments (Sterman 1988), (Mosekilde et al. 1988), deterministic analysis (Aracil 1981 a,b, 1984, 1986; et al. 1984, 1988, 1989, 1991), (Toro et al. 1988 a,b), (Richardson 1984, 1986), simulations experiments where initial conditions are being perturbed (Mosekilde et al. 1983, 1985, 1988), (Rasmussen et al. 1985), (Sturis et al. 1988), statistical search in the parameter space (Toro et al. 1992b), and piecewise linear analysis (Toro et al. 1992a), the dynamics of equilibria, i.e. stability criteria, dominant feedback loop polarities, bifurcations, and aspects of chaos and self-organization have been studied extensively. An overview over quantitative methods for system dynamics models is given by Aracil and Toro (Aracil et al. 1991). The theories of fuzzy sets and fuzzy numbers have not been utilized in any of these studies.

Fuzzy sets theory and fuzzy numbers enable us to represent vague expressions symbolically. As noted by Fishwick (Fishwick 1991), those theories encompass probability distributions as a special case and unify features with respect to uncertainty in simulation. In fact, by using fuzzy numbers, we avoid the normalization required when applying probabilities. Moreover, it enables us to represent various levels of uncertainty or confidence by the use of alpha-cuts, to which we will return.

Fuzzy set and number theory allows us to consider a linguistic expression as an aggregate (lumped) representation of a reality that can be portrayed in greater detail using exact numerical representations (Zeigler, 1976). To ensure model integrity, the aggregation must constitute a homomorphic mapping so as to preserve the relationship between the input and the output trajectory of the model (Kuipers, 1986).

A fuzzy number can represent qualitative or linguistic values, say the size of the national deficit, characterized as large, moderate, or small. We represent each fuzzy number in terms of a series of alpha-cuts that can be interpreted as confidence intervals. Pairs of real numbers represent the upper and lower bounds of such alpha-cuts, two numbers per alpha-cut per time-step. When we operate on fuzzy numbers, the equations of the model are interpreted in terms of fuzzy (interval) arithmetic operating on various combinations of these bounds. In fuzzy simulations, we must consider specifically interaction between variables that cause an over-extension of the confidence intervals when using standard fuzzy arithmetic. In this paper, we document techniques developed to ensure a proper simulation of fuzzy models.

Fuzzy dynamic models share some of the behavior characteristics of real models - in particular stochastic models subject to Monte-Carlo simulation. Their dynamic behavior is determined in much the same way by the underlying systems structure, i.e. by feedback, nonlinearities, and lags that contribute to the proliferation of uncertainty and vagueness throughout each model. The long range purpose of our research is to study, document and interpret the dynamic characteristics of fuzzy models in terms of real systems.

3 Fuzzy numbers in simulations. How?

In this section we describe how fuzzy numbers can be utilized in simulation. A fuzzy number is a concept based on the idea of *fuzzy sets* introduced by Zadeh (Zadeh, 1965). In order to understand the nature of fuzzy numbers we will need some definitions.

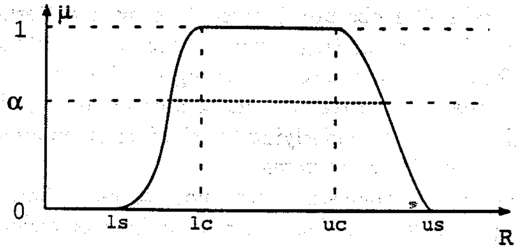

Figure 1: A fuzzy number.

A *fuzzy set* is a set with graded membership. A membership function $\mu_A : U \rightarrow [0,1]$ defines the membership grades of the elements of some universe U in the fuzzy set A. If $\mu_A(x) = 1$ then is x a full member of A, if $\mu_A(x) = 0$ then is x not a member of A.

An *α-cut* of a fuzzy set A is the classical set $A_\alpha = \{x|\mu_A(x) \geq \alpha\}$, α is a number in $[0,1]$.

A *convex fuzzy set* A is a fuzzy set on a continuous universe such that for all α, A_α is a convex classical set.

A *fuzzy number* is a convex fuzzy set with $U = R$.

The *support* of fuzzy number A is the interval [ls,us] where

$$\text{ls} = \inf\{x|\mu_A(x) > 0\} \quad \text{and} \quad \text{us} = \sup\{x|\mu_A(x) > 0\}$$

The *core* of a fuzzy number A is the interval [lc,uc] where

$$\text{lc} = \inf\{x|\mu_A(x) = 1\} \quad \text{and} \quad \text{uc} = \sup\{x|\mu_A(x) = 1\}$$

The definitions are illustarted in figure 1.

A fuzzy arithmetic has been developed for fuzzy numbers, see f.ex. Kaufmann and Gupta's book (Kaufmann et al. 1991). Fuzzy arithmetic can be viewed as a generalization of Moore's interval arithmetic (Moore, 1966). For each α the α-cut of a fuzzy number is an interval. If, furthermore, $C = f(A, B)$ is the fuzzy number resulting from performing a fuzzy arithmetic operation f on fuzzy numbers A and B, then for each $\alpha \in [0,1]$, the result C_α would be the result of the interval arithmetic operation f on A_α and B_α. This implies that for practical computation with fuzzy numbers, the numbers are represented by a finite set of α-cuts, and operations are performed by applying interval arithmetic to each α-level.

The interval arithmetic versions of the standard arithmetic operations are as follows:

$$\begin{aligned}
[a,b] + [c,d] &= [a+c, b+d] \\
[a,b] - [c,d] &= [a-d, b-c] \\
[a,b] * [c,d] &= [a*c, b*d],\ U = R^+ \\
[a,b]/[c,d] &= [a/d, b/c],\ U = R^+
\end{aligned}$$

Fuzzy numbers usually represent ignorance with respect to the exact value of some variable. If there is some kind of interactiveness between the values of variables, this may lead to too wide intervals after the application of interval arithmetic. A typical example is $A - A$ which one expects to have the value 0. However, the result of applying interval arithmetic is $[a_l - a_u, a_u - a_l]$, when $A = [a_l, a_u]$. There may also be more subtle interactiveness between variables that may lead to over-pessimistic intervals.

System dynamics simulation models typically involve a large number of functions and operations on variables that are not mutually independent. If we ignore these interdependencies, the resulting values of such functions and operations become too wide.

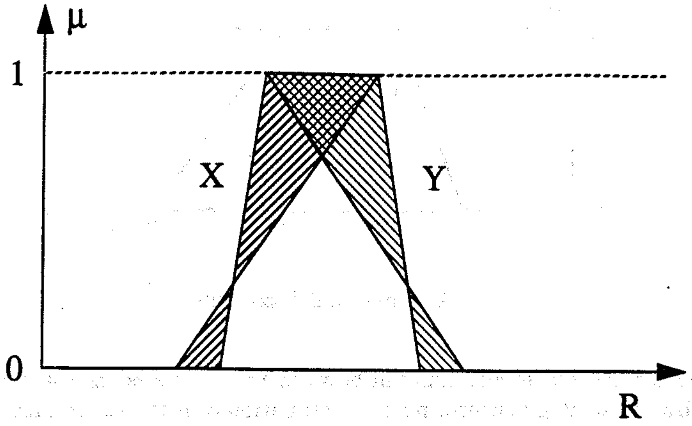

Figure 2: The distance between two fuzzy numbers.

Another feature of dynamic models is that functions in general are non-monotonic. This also prohibits the use of interval arithmetic, and search for other methods are needed.

The problem of computing α-cuts for a variable $X = f(X_1, X_2, \ldots, X_n)$ can be formulated as

$$X_\alpha = [\inf f(x_1, x_2, \ldots, x_n), \sup f(x_1, x_2, \ldots, x_n)], x_i \in X_{i,\alpha}$$

This means that we have two global optimization problems, which in general have no particular structure, and thus leads to no particular solution strategy. The strategy chosen by us in our first experiments is simple, but error sensitive when used on function with steep gradients. It consists of two steps, first compute f on all vertices of the n-dimensional box that our bounds describe, then select a large number of uniformly distributed random points inside the box, and compute f for each of these. Select the largest of the computed values as the upper bound of X_α, and the smallest as the lower bound of X_α. An overview of global optimization methods can be found in f. ex. (Gill et al. 1981).

One of the main purposes of this work is to facilitate qualitative simulation. The relationship between qualitative values and fuzzy numbers is based on the ideas on possibility theory by Zadeh (Zadeh, 1978, 1979). The idea is that variables are allowed to have a finite set of qualitative values, such as {VERY HIGH, HIGH, MEDIUM, LOW, VERY LOW}. To each of the qualitative values we have assigned a fuzzy number, which is then used for simulation.

The resulting fuzzy numbers may be transformed back to qualitative values by using the qualitative value closest to the computed. A suitable choice for distance between numbers is the distance given in (Kaufmann et al. 1991, pp 100-109). Let $X_l(\alpha)$ and $X_u(\alpha)$ be the lower and upper bounds of X_α respectively. Then, the distance between two fuzzy numbers X and Y is

$$d(X,Y) = \int_0^1 (|X_l(\alpha) - Y_l(\alpha)| + |X_u(\alpha) - Y_u(\alpha)|)d\alpha$$

The shaded area in figure 2 represents the distance between to fuzzy numbers X and Y. The doubly shaded area counts twice. Examples of qualitative simulation will be given in the examples section.

The work of Fishwick (Fishwick, 1991) is similar to what we have presented here. He distinguishes between 3 methodologies; Monte Carlo-simulation, uncorrelated simulation and correlated simulation. The first is based on the knowledge of probability distributions. Uncorrelated simulation is simulation where fuzzy arithmetic is used straight forward, which often leads over-pessimistic fuzzy numbers. In correlated simulation he assumes knowledge about which values of the initial parameters that correspond to each other. Sometimes this may be a too strong assumption.

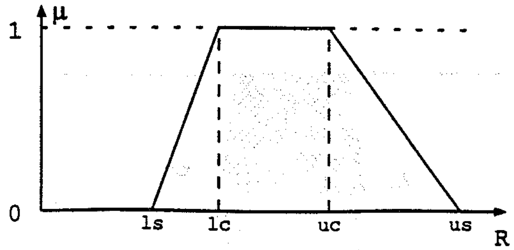

Figure 3: A trapezoidal fuzzy number.

The method we have presented is something in between the correlated and uncorrelated method. The global optimization strategy ensures no over-estimation of intervals at each times step. The method, however, does not assume correlation between the values of parameters between each time step. This assumption, however, would often be reasonable.

4 Examples.

The examples presented here are all simple population models, illustrating typical results that we can expect from simulations with fuzzy numbers. The fuzzy numbers used initially are all so called ***trapezoidal*** fuzzy numbers, examplified in figure 3. These can be represented by the quadruple (ls,lc,uc,us) which are named the lower support, lower core, upper core, and upper support values respectively. The first example has the following equations:

```
INIT(Population) = (80000,90000,100000,100000),
Fertility = (0,0,0.02,0.04),
Mortality = (0.02,0.04,0.06,0.08),
Births = Population * Fertility,
Deaths = Population * Mortality,
Population = Population - Deaths + Births
```

The simulation in 100 steps leads to the result given in figure 4. The curves indicate how the upper support, upper core, lower core, and lower support values develop. We observe that the upper support grows exponentially. This happens when fertility takes its upper support value and mortality its lower support value. The other curves, however, approaches 0, and we may, for instance, conclude that the population most likely will die out.

The second example is an example with qualitative values. Assume both Fertility and Mortality is a number in the interval [0,0.1]. Assume further that this interval can be split into the following qualitative values:

```
VERY LOW = (0.00,0.00,0.01,0.02)
LOW = (0.00,0.01,0.02,0.03)
RATHER LOW = (0.01,0.02,0.03,0.04)
SOMEWHAT LOW = (0.02,0.03,0.04,0.05)
SLIGHTLY BELOW MIDDLE = (0.03,0.04,0.05,0.06)
SLIGHTLY ABOVE MIDDLE = (0.04,0.05,0.06,0.07)
SOMEWHAT HIGH = (0.05,0.06,0.07,0.08)
RATHER HIGH = (0.06,0.07,0.08,0.09)
HIGH = (0.07,0.08,0.09,0.10)
VERY HIGH = (0.08,0.09,0.10,0.10)
MIDDLE = (0.03,0.04,0.06,0.07)
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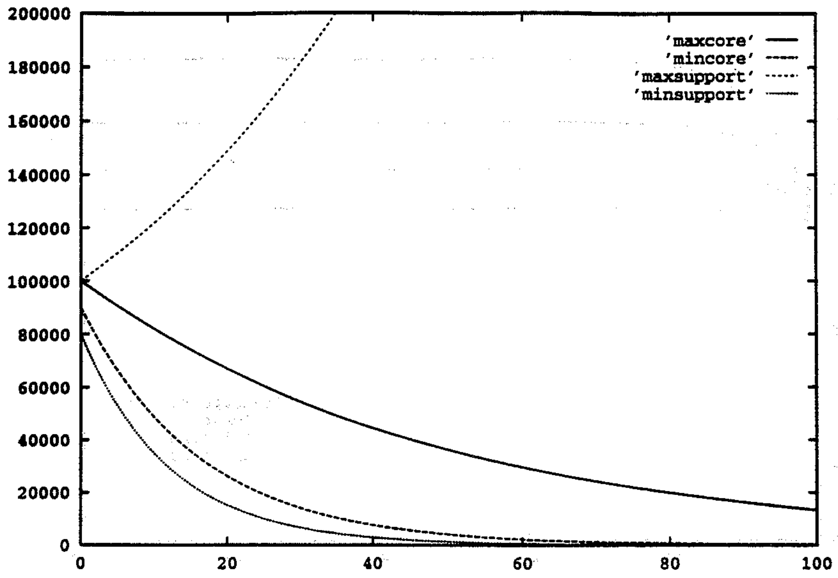

Figure 4: The development of a population.

```
BELOW MIDDLE = (0.00,0.00,0.04,0.05)
ABOVE MIDDLE = (0.05,0.06,0.10,0.10)
UNKNOWN = (0.00,0.00,0.10,0.10)
```

The equations used involved increasing stress with increasing population:

```
INIT(Population) = 90000,
BasicFertility = SLIGHTLY ABOVE MIDDLE,
BasicMortality = SOMEWHAT LOW,
NormalPopulation = 100000,
Stress = (Population / NormalPopulation)**2,
Fertility = BasicFertility / Stress,
Mortality = BasicMortality * Stress,
Births = Fertility * Population,
Deaths = Mortality * Population,
Population = Population + Births - Deaths
```

The simulation leads to the result given in figure 5.

If we now split the possible values of Population, say [0,200000], into a set of fuzzy numbers in the same way as for Fertility and Mortality. Then we get a very good match with the qualitative value SLIGHTLY ABOVE MIDDLE.

The third example is one where Stress is a non-monotonic function depending on Population. High and low populations lead to high stress. The equations are:

```
INIT(Population) = (20000,30000,30000,40000)
MinStressPopulation = 50000,
NormalPopulation = 100000,
LowestStress = 0.95,
BasicFertility = (0.04,0.05,0.06,0.07),
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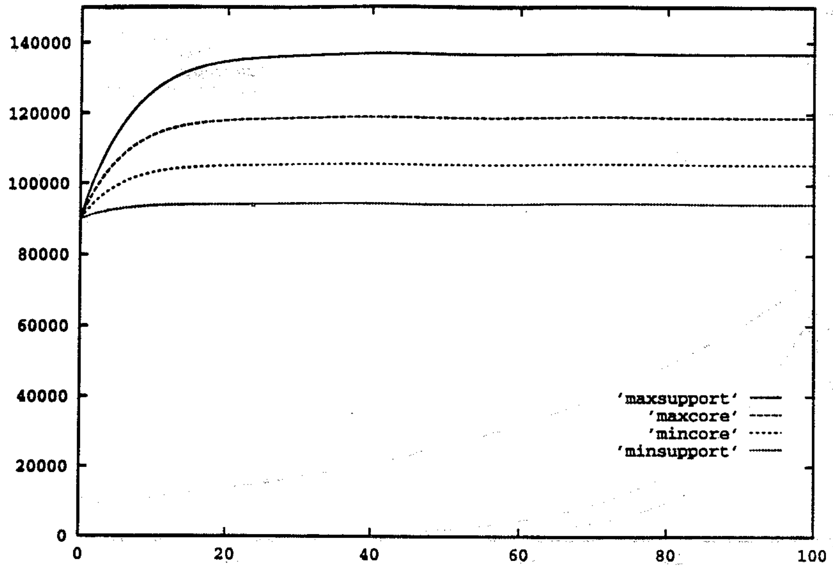

Figure 5: The development of a population.

```
BasicMortality = (0.02,0.03,0.04,0.05),
BasicStress = ABS(Population - MinStressPopulation) /
              NormalPopulation,
Stress = BasicStress ** 2 + LowestStress,
Fertility = BasicFertility / Stress,
Mortality = BasicMortality * Stress,
Births = Fertility * Population,
Deaths = Mortality * Population,
Population = Population + Births - Deaths
```

The result is given in figure 6. The population seems to stabilize somewhere around 100000, but may also die out.

5 Conclusion.

We have discussed the use of fuzzy numbers to represent uncertainty in system dynamics models. The method gives us a simple framework for simulation when the initial data is only vaguely defined, and it does also represent a good alternative to qualitative simulation with symbolic methods.

The weakness of the model as it is explained here, is the lack of correlation of parameters between time steps. This should be fixed. Furthermore, at another level, one should investigate the dynamic properties of fuzzy dynamic systems. Possibly, this kind of work can, together with artificial intelligence techniques, support the process of characterizing dynamic models.

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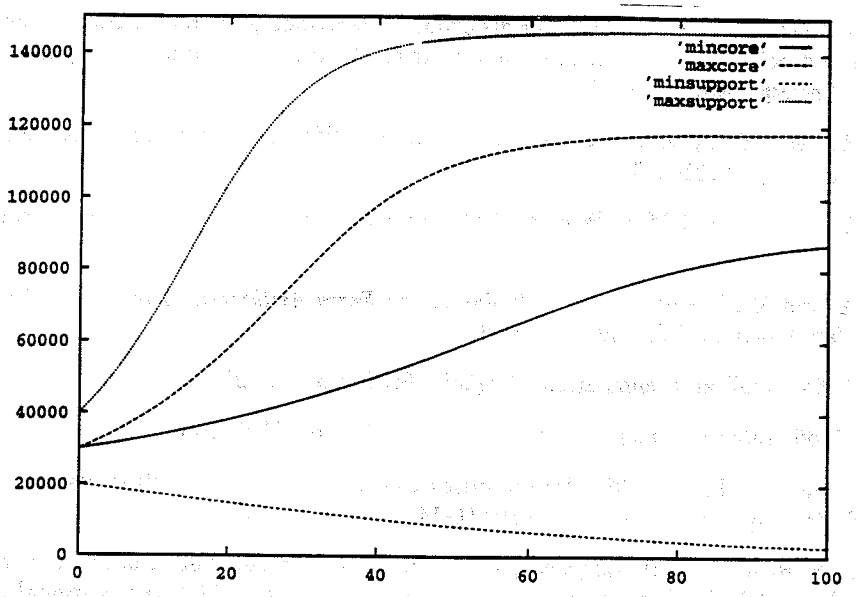

Figure 6: The development of a population.

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