Entrainment in a Disaggregated Economic Long Wave Model

by

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Abstract

This paper investigates how mode-locking and other highly nonlinear dynamic phenomena arise through the interaction of two capital-producing sectors in a disaggregated economic long-wave model. One sector might represent the construction of buildings and infrastructure capital with long lifetimes while the other represents production of machinery, computers, etc., with much shorter lifetimes. Due to the positive feedback associated with capital self-ordering, each sector in isolation produces a self-sustained oscillation with a period and amplitude determined by the characteristics of that sector. However, the sectors interact through their mutual dependence on each other's output for their own production. When this coupling is accounted for, the two sectors tend to synchronize or lock together with a rational ratio between the periods. While keeping the aggregate equilibrium characteristics of the system constant, we study how this locking occurs as a function of the difference in capital lifetimes and as a function of the strength of the coupling between the sectors. Besides mode-locking and quasi-periodic behavior, the observed phenomena includes cascades of period-doubling bifurcations, chaos, and intermittency. When the difference in capital lifetimes is very large, the system behaves like a one-sector model with a reduced capital content of production: Only one oscillatory mode remains, and it is much less pronounced than in the original one-sector model.

1. Introduction

John Sterman's simple long wave model (1985) has provided a theory of long-term economic fluctuations, based on the notion of the non-linear, disequilibrium investment accelerator. However, the simplicity of Sterman's model raises many questions about how the long-wave theory holds up in a more realistic model. (For instance, the model has no price system for capital.) The model should clearly be extended in a number of directions so as to provide a more detailed and realistic description.

This paper is concerned with the simple model's aggregation of capital into a single type. The real economy consists of many sectors employing different kinds of capital in different amounts. Parameters such as the average productive life of the capital produced and the relative amounts of different capital components employed may vary from sector to sector. In isolation, the buildings- and infrastructure-capital industry may show a temporal variation significantly different from that of, for instance, the machinery industry. What circumstances lead more realistic multi-sector models to behave in a similar fashion as the aggregate one-sector model, and what circumstances produce more complicated dynamic behaviors? To address this question requires a detailed look at the mechanisms that couple different sectors in the economy together, and a study of the implications of this coupling for different parameter values.

An early study by Kampmann (1984) took a first step in this direction by disaggregating the simple long-wave model into a system of two or more capital-producing sectors with different characteristics. Kampmann showed how a multi-sector system could produce a range of different behaviors, at times quite different from the original one-sector model. Moreover, analysis of the one-sector model with external forcing in capital demand has shown how the model produces mode-locking, period-doubling bifurcations, intermittency, chaos, and other interesting nonlinear dynamic phenomena (Mosekilde, et al. 1992, Sterman and Mosekilde 1993). It is thus reasonable to expect a multi-sector model to show great richness in behavior.

The present paper focuses on a two-sector model. One sector might represent the construction of buildings and infrastructure capital components with very long lifetimes while the other could represent the production of machines, transportation equipment, computers, etc., with much shorter lifetimes. The two sectors are coupled together through their mutual dependence on each other's output for their own production.

We consider the influence of two factors, the difference in the average lifetime of capital produced by each sector, and the degree of linkage between the sectors. One would expect that a significant difference in capital lifetimes would, *ceteris paribus*, lead to more complex fluctuations which may well differ substantially from the original 50-year cycle. Conversely, a stronger coupling between the sectors should lead to more uniform behavior, akin to that of the original single-sector model.

The following section presents the equations of the disaggregated model. Section 3 presents the simulation results and Section 4 our conclusions.

2. The Model

The model consists of two capital-producing sectors which use capital from itself and from the other sector as the only factors of production. Each sector receives orders for capital, both from itself, from the other sector, and from the consumer goods sector. Orders are backlogged until capital is produced and delivered. Apart from the obvious modifications needed to extend the model to more than one sector, and except for a few alterations in parameter values and function specifications which we felt were appropriate for a more detailed study, the disaggregated model is equivalent to Sterman's original model (Sterman 1985).

Each sector $i = 1,2$ maintains a stock K_{ij} of each capital type $j = 1,2$. The capital stock is increased by deliveries of new capital and reduced by physical depreciation. Capital of type j depreciates exponentially with an average lifetime of τ_j. Capital output is distributed "fairly" between customers, i.e., the delivery of capital type j to sector i is the share of total output from sector j, x_j, distributed according to how much sector i has on order with sector j, S_{ij}, relative to sector j's total order backlog B_j. Each sector orders capital from both sectors, o_{ii} and o_{ij}, and receives orders for its product from both capital sectors, o_{ii} and o_{ji}, and from the consumer goods sector, g_i. Incoming orders accumulate in a backlog B_i which is then depleted by the sector's deliveries of capital x_i. Hence, the ten state variables in the model evolve according to,

$$\dot{K}_{ij} = \frac{x_j S_{ij}}{B_j} - \frac{K_{ij}}{\tau_j}, \forall i,j; \quad \dot{S}_{ij} = o_{ij} - \frac{x_j S_{ij}}{B_j}, \forall i,j; \quad \dot{B}_i = \left(o_{ii} + o_{ji} + g_i\right) - x_i, i \neq j \quad (1)$$

Each sector's output x_i is limited by its production capacity c_i. The capacity in each sector is assumed to be a constant-returns-to-scale Cobb-Douglas function of the individual stocks of the two capital types, with a factor share, $\alpha \in [0,1]$, of the other sector's capital type and a share $1-\alpha$ of the sector's own capital type, i.e.

$$c_i = \kappa_i^{-1} \cdot K_{ii}^{1-\alpha} \cdot K_{ij}^{\alpha}; j \neq i, \tag{2}$$

where κ_i is the constant "sector capital-output ratio" (see below). The parameter α thus determines the degree of coupling between the two sectors. (Note that the matrix of equilibrium capital stocks is assumed to be symmetric. The equilibrium flow input-output matrix, on the other hand, is not symmetrical unless the average lifetime of capital is the same for both types.)

The output from sector i, x_i, depends on the sector's capacity c_i, and the sector's desired output x_i^*. If desired output is much lower than capacity, production is cut back, ultimately to zero if no output is desired. Conversely, if desired output exceeds capacity, output can be increased beyond capacity, up to a certain limit. Specifically, the sector's output is formulated as

$$x_i = f\left(\frac{x_i^*}{c_i}\right) \cdot c_i, \quad f(r) = a\left(1 - \left(\frac{a-1}{a}\right)^r\right), \tag{3}$$

and a is a parameter which determines the maximum over-production possible. The function $f(.)$ differs slightly from Sterman's original piece-wise linear function. In particular, it allows production to exceed capacity for high values of its argument. The formulation was chosen to obtain an analytical, infinitely differentiable function.

Sector i's desired output x_i^* is assumed to be the value that would allow firms in that sector to deliver the capital on order B_i with the (constant) normal average delivery delay δ_i for that sector. Hence,

$$x_i^* = \frac{B_i}{\delta_i}. \tag{4}$$

Sector i's desired orders for new capital from sector j, o_{ij}^*, is assumed to consist of three components. First, all other things equal, firms will order to replace depreciation of their existing capital stock, K_{ij} / τ_j. Second, if their current capital stock is below (above) its desired level k_{ij}^* firms will order more (less) capital in order to remove the discrepancy over time. Third, firms consider the current supply line S_{ij} of capital and compare it to its desired level s_{ij}^*; if the supply line is below (above) desired, firms order more (less) in order to increase (decrease) the supply line over time. In total,

$$o_{ij}^* = \frac{K_{ij}}{\tau_j} + \frac{k_{ij}^* - K_{ij}}{\tau_i^K} + \frac{s_{ij}^* - S_{ij}}{\tau_i^S}, \tag{5}$$

where the parameters τ_i^K and τ_i^S are the desired adjustment times for capital stock and the supply line, respectively.

Actual orders, however, are limited to be non-zero (no cancellation of orders) and the fractional rate of expansion of the capital stock is also assumed to be limited. These restraints are accounted for through the expression

$$o_{ij} = \left(K_{ij} / \tau_j\right) \cdot g\left(\frac{o_{ij}^*}{K_{ij} / \tau_j}\right), \text{ where } g(r) = \frac{b}{1 - b_1 e^{-c_1(r-1)} - b_2 e^{-c_2(r-1)}}, \tag{6}$$

and b, b_1, b_2, c_1, c_2 are parameters with values

$$b = 6;\ b_1 = \frac{27}{7};\ b_2 = \frac{8}{7};\ c_1 = \frac{2}{3};\ c_2 = 3. \tag{7}$$

The paramaters in (7) were chosen to approximate Sterman's original piece-wise linear function as closely as possible and yet obtain an analytic, infinitely differentiable function. The choice of parameters further assures that

$$g(1) = 1;\ g'(1) = 1;\ g''(1) = 0, \tag{8}$$

i.e., so that $g(r)$ has a "neutral" area around the steady state, r=1, where actual orders equal desired orders, which in turn equal current capital depreciation.

The desired capital stock k_{ij}^* is assumed to be proportional to the desired production level x_i^*. Thus,

$$k_{ij}^* = \kappa_{ij} \cdot x_i^*, \tag{9}$$

where κ_{ij} is a constant "capital-output ratio" of capital type j in sector i (see below).

In calculating their desired supply line s_{ij}^*, firms are assumed to take account of the current delivery delay for each type of capital. Their target supply line is the level at which the deliveries of capital, given the current delivery delay, would equal the current depreciation of the capital stock. The current delivery delay of capital from a sector is the sector's backlog divided by its output. Thus,

$$s_{ij}^* = \frac{B_j}{x_j} \cdot \frac{K_{ij}}{\tau_j}. \tag{10}$$

Finally, the orders from consumers to each sector g_i are assumed to be exogenous, constant, and the same for both sectors. The absolute values of the g_i's are unimportant; they are simply scaling factors. However, their values relative to each other are important, since they change the dynamics of the model considerably (see Kampmann 1984). This would be an obvious area of further investigation.

The capital-output ratios and average capital lifetimes are constructed in such a way that the aggregate equilibrium capital-output ratio and capital lifetime for the system as a whole remains constant, equal to their values in Sterman's original model. Specifically, the average capital lifetimes in the two sectors are

$$\tau_1 = \tau + \frac{\Delta\tau}{2};\ \tau_2 = \tau - \frac{\Delta\tau}{2};\ \tau = 20 \text{ years}, \tag{11}$$

where the bifurcation parameter $\Delta\tau$ is the difference in capital lifetimes (in years) between the two sectors. The capital-output ratios are

$$\kappa_{ii} = (1-\alpha)\kappa\frac{\tau_i}{\tau};\ \kappa_{ij} = \alpha\kappa\frac{\tau_j}{\tau};\ \kappa_i = \kappa_{ii}^{1-\alpha}\kappa_{ij}^{\alpha},\ i \neq j;\ \kappa = 3 \text{ years}. \tag{12}$$

This formulation assures that capacity equals desired output when both capital stocks equal their desired levels. It also assures that the equilibrium aggregate lifetime of capital and the equilibrium aggregate capital-output ratio equal the original parameters in the one-sector model. Moreover, is can be shown that the desired capital mix defined by (9) is the one that minimizes the steady state cost, given the desired output. (Implicitly, the units of capital are defined so that their (constant) prices are equal.)

For each sector, the adjustment and normal delivery times are scaled according to the average lifetime of capital produced by that sector. This means that when there is no coupling between the sectors ($\alpha = 0$), one sector is simply a time-scaled version of the other. We felt that this approach was the cleanest way to investigate the coupling of two oscillators with different inherent frequencies. Thus,

$$\tau_i^K = \tau^K \frac{\tau_i}{\tau};\ \ \tau_i^S = \tau^S \frac{\tau_i}{\tau};\ \ \delta_i = \delta \frac{\tau_i}{\tau};\ \ \delta = 1.5;\ \ \tau^K = 1.5;\ \ \tau^S = 1.5 \text{ years.} \qquad (13)$$

The adjustment times $\tau^K = \tau^S = 1.5$ years are shorter than Sterman's (his were $\tau^K = \tau^S = 3$ years). However, we chose these values both because they seem more realistic and because, together with the other modifications we have made, they yield a period and amplitude closer to the original model.

The one-sector version of our model, though slightly different from Sterman's original model, produces behavior very similar to it. Thus, with the above parameters, the equilibrium point is unstable, and the system quickly settles into a limit cycle with a period of approximately 47 years. Each new cycle begins with a period of rapid growth, where desired output significantly exceeds capacity. The capital sector is thus induced to order more capital from itself which, by further swelling order books, fuels the upturn in a self-reinforcing process. Eventually, capacity catches up with demand, but at this point it far exceeds the equilibrium level. The self-ordering process is now reversed, as falling orders from the capital sector leads to falling demand, which further depresses the capital sector's orders. Consequently, output quickly collapses to the point where only the exogenous goods sector places new orders. A long period of depression follows, where the excess capital is gradually depleted, until capacity finally reaches demand. At this point, however, capacity is below the its equilibrium level, and the cycle is ready to start anew.

3. Simulation results

In the simulations below, we examine the consequences of varying the difference in capital lifetimes $\Delta\tau$ for different values of the coupling parameter α. As described in Section 2, we have scaled all other parameters with the capital-lifetime parameters τ_1 and τ_2 in the two sectors in such a way that, for no coupling between the sectors, they are simply time-scaled versions of the original one-sector model. In the results that follow, sector 1 is always the sector with the longest lifetime of its capital output.

Introducing a coupling between the sectors will, apart from linking the behavior together, also change the stability properties of each individual sector, taking the other sector as exogenous. A high coupling parameter α implies that the strength of the capital self-ordering loop <u>within</u> a sector is small. In the extreme case $\alpha = 1$, it disappears altogether. A linear stability analysis around the steady-state equilibrium of the individual sector, taking the delivery delay of capital from the other sector is taken as exogenous and constant, shows that this equilibrium becomes stable for sufficiently high values of α. At this point, the behavior of the individual sector changes to a highly damped oscillation. As will become evident below, this stability effect of the coupling parameter has significant effects on the mode-locking behavior of the coupled system.

As long as the two sectors have fairly similar parameters, we expect synchronization (or 1:1 frequency locking) to occur, i.e., we expect that the two sectors will adjust themselves to another to yield a single aggregate economic long wave with the same period for both sectors. The same may be true for sufficiently high coupling strengths, irrespective of differences in sector parameters. An example of such synchronization is observed for $\Delta\tau = 6$ years and $\alpha = 0.25$ in Figure 1.

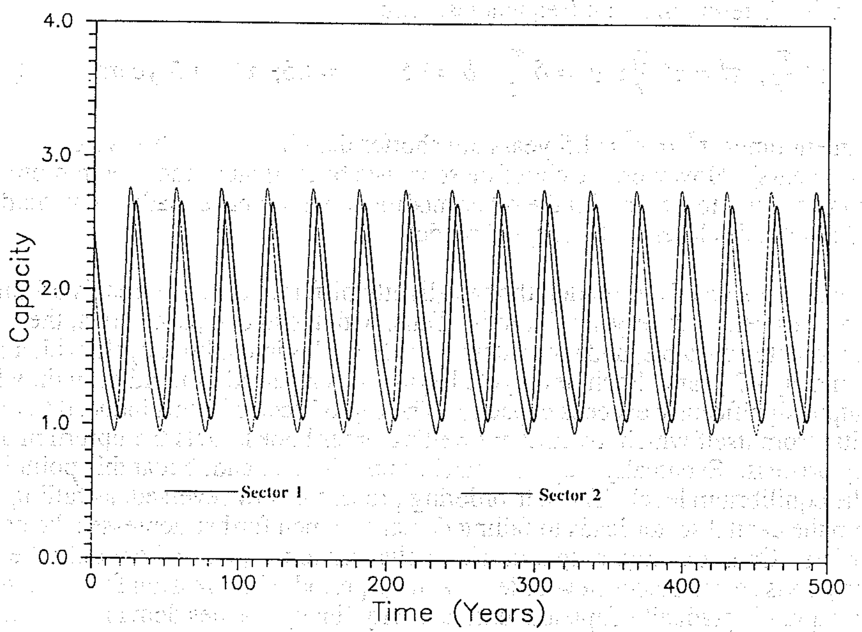

Figure 1: Synchronization (1:1 mode-locking) in the coupled two-sector model
The figure shows the capacity of the two sectors as a function of time in the steady state (after transients have died out). The difference in capital lifetimes $\Delta\tau$ is 6 years (i.e., the lifetime of capital type 1 and 2 is 23 and 17 years, respectively). The coupling parameter α is 0.25 in this and the following three figures. Due to the nonlinear coupling, the two sectors are locked into a single cycle.

If, with the same coupling parameter, the difference in capital lifetimes is increased to $\Delta\tau = 9$ years, we observe a doubling of the period. The two sectors now alternate between high and low maxima for their production capacities. This type of behavior is referred to as a 2:2 mode. It has developed out of the synchronous 1:1 mode through a period-doubling bifurcation (Feigenbaum 1978). As the difference in lifetimes is further increased, the model passes through a Feigenbaum cascade of period-doubling bifurcations (4:4, 8:8, etc.) to reach chaos at approximately $\Delta\tau = 10.4$ years. Figure 2 shows an irregular fluctuation existing for $\Delta\tau = 10.7$ years. Calculation of the largest Lyaponov exponent (Wolf 1986) confirms that the behavior is chaotic, yet it is interesting to note that the distance between cycles remains around 50 years even though the magnitude of each cycle varies a great deal.

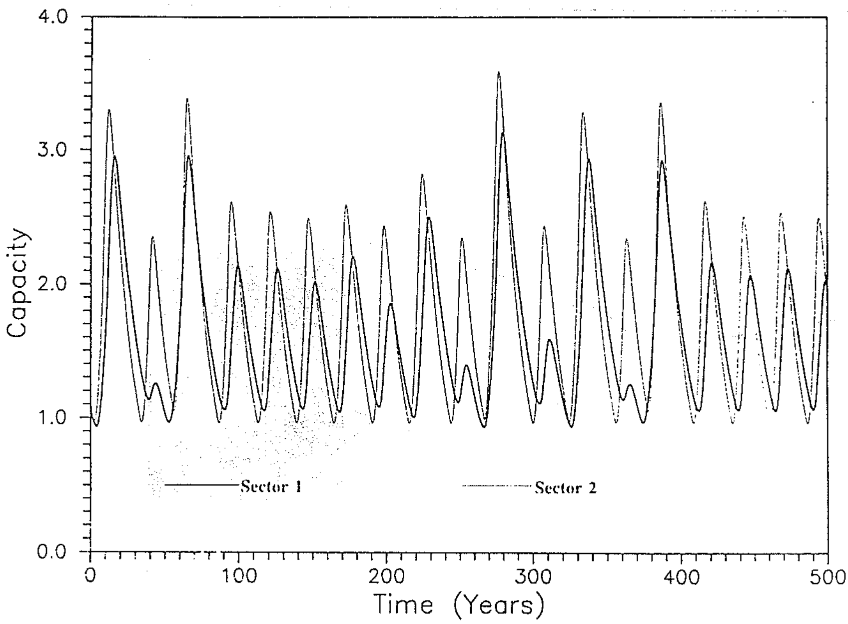

Figure 2: Chaotic behavior

For a difference in capital lifetimes $\Delta\tau$ of 10.7 years, the behavior is chaotic. (The coupling parameter α is still 0.25.) The model shows no regular periodic behavior, and initial conditions close to each other quickly diverge so that, in practice, the behavior is unpredictable. Note that the behavior nonetheless remains similar to the original uniform cycle, except that individual cycles vary greatly in size.

A more detailed illustration of the route to chaos is provided by the bifurcation diagram in Figure 3, which plotts the maximum production capacity attained in sector 1 as a function of the lifetime difference $\Delta\tau$. The coupling parameter α is kept constant and equal to 0.25. The diagram shows a rich fractal structure similar to the classic period-doubling and chaos seen in a wide range of simple non-linear models (see e.g. {Mosekilde, 1988 #25}). For instance, inspection of Figure 3 shows that, in the chaotic regime, we find periodic windows deriving from the 1:1 solution such as, for instance the 10:10 and the 7:7 solutions existing around $\Delta\tau = 11.6$ years and $\Delta\tau = 11.85$ years, respectively.

The parameter phase diagram in Figure 4 gives an overview of the dominant modes for different combinations of the lifetime difference $\Delta\tau$ and the coupling parameter α. The zones of mode-locked (i.e., periodic) solutions in this diagram are referred to as Arnol'd tongues (Arnol'd 1965). Besides the 1:1 tongue, the figure shows a series of 1:n tongues, i.e., regions in parameter space where sector 1 completes precisely 1 long-wave oscillation each time sector 2 completes n oscillations. Between these tongues, regions with other commensurate wave periods may be observed. An example is the 2:3 tongue found in the area around $\alpha = 0.15$ and $\Delta\tau = 12$ years. Similar to the 2:2 period-doubled solution on the right-hand side of the 1:1 tongue, there is a 2:4 period-doubled solution along part of the right-hand edge of the 1:2 tongue. Most likely, similar period-doubling structures can be found on ever finer scale along each of the other tongues, producing a fractal structure, but we have yet to explore this hypothesis in detail.

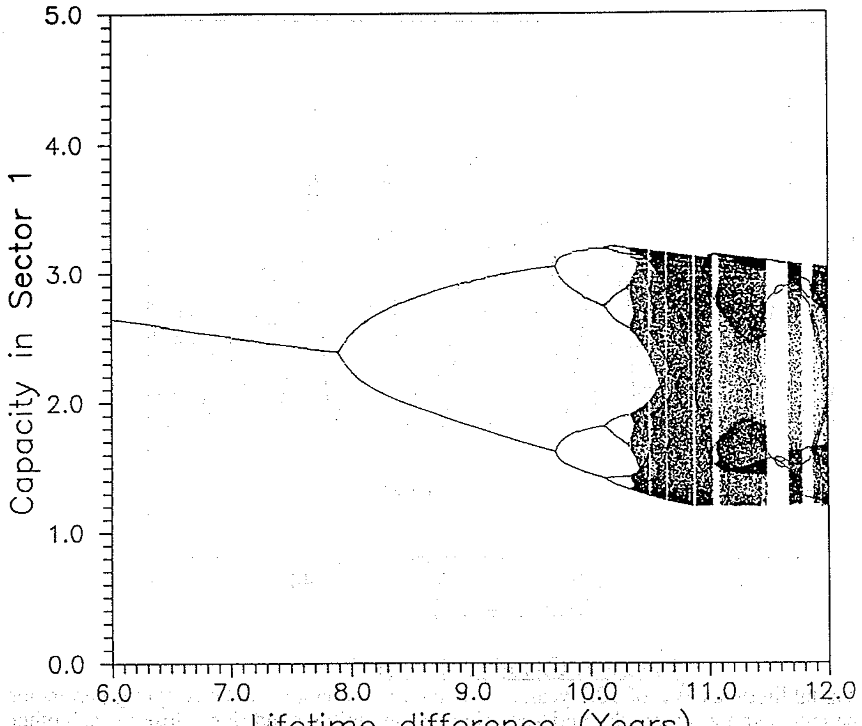

Figure 3: Bifurcation diagram for increasing $\Delta\tau$ and constant α

The figure shows the local maxima attained for the capacity of sector 1 (the longer-lived capital producer) in the steady-state behavior for varying values of the lifetime difference $\Delta\tau$. The coupling parameter α is held constant at 0.25. For a given $\Delta\tau$, a single value in the diagram indicates a uniform limit cycle; two-values indicate a period doubling with a smaller and larger cycle, etc. In chaotic regions, the number of local maxima is infinite since no individual cycles are identical.

Figure 4 reveals that the synchronous 1:1 solution extends to the full range of the lifetime difference $\Delta\tau$ for sufficiently high values of the coupling parameter α. We believe that the reason for this is the stabilizing effect of high values of α on the individual sectors. When α is sufficiently large, the equilibrium of the individual sectors, taking the other sector as exogenous, becomes stable. (For reference, two curves have been drawn in Figure 5, defining the regions in which one or both of these individual equilibria are stable: For a given value of the lifetime difference, values of α above the curve result in a stable individual sector equilibrium.)

As α increases, the overall behavior is more and more derived from the coupled capital self-ordering feedback, and less and less from the autonomous self-ordering mechanism in each individual sector. Thus, for high values of α, there is less competition between the two individual, autonomous oscillations, and therefore less prevalence of the normal mode-locking phenomena.

For large differences in capital lifetimes and low values of the coupling parameter α, the short-lived sector (sector 1) completes several cycles for each oscillation of the long-lived sector (sector 2). However, as α is increased, the short-term cycle is reduced in amplitude, and, for sufficiently high α's, it disappears altogether, resulting in a synchronous 1:1 solution. The locally stabilizing effect of high values of α creates a complicated distortion of the Arnol'd tongues. For instance, the figure reveals that both the 1:1 region and the 2:2 region are folded down above the other regions for high values of $\Delta\tau$.

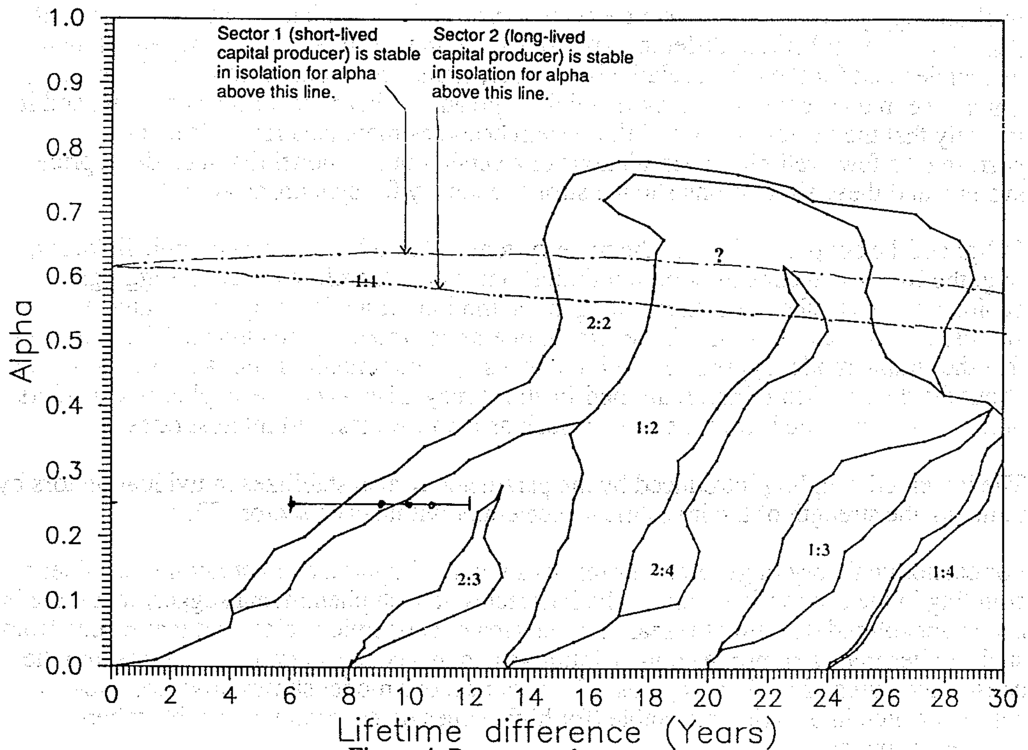

Figure 4: Parameter phase diagram

The figure summarizes the steady-state behavior of the two-sector model for different combinations of the coupling parameter α and the lifetime difference $\Delta\tau$. A region labeled "p:q" indicates the area in parameter space where the model shows periodic mode-locked behavior of p cycles for sector 1 and q cycles for sector 2. (However, other solutions may coexist at the same point in the diagram, depending on the initial conditions of the model.) The question mark indicates that the details of the diagram are still under exploration. In particular, regions of chaotic behavior have not yet been outlined in detail. The dashed curves across the diagram indicate the value of α above which each sector in isolation (with the other sector treated as exogenous) becomes stable and the cycles are created solely by the interaction of the two sectors. This effect implies that, for large α, synchronous behavior becomes more and more prevalent. Finally, the line at α=0.25 and $6 \leq \Delta\tau \leq 12$ years locates the region examined in the previous figures.

4. Conclusions

The present paper represents work in progress and the results are necessarily incomplete. In particular, the details of the phase diagram in Figure 4 remain to be explored. However, even at this preliminary stage, we can draw implications for both the validity of the simple long-wave model and for economic theory in general.

By employing only a single capital-producing sector, the simple long wave model represents a simplification of the structure of capital and production. In reality, capital is composed of diverse components with very different characteristics. We have focused on the difference in the average lifetime of capital, and it is clear from our analysis that a disaggregate system with diverse capital components exhibits a much wider variety of fluctuations. Strong and moderate degrees of coupling have the effect of merging individual cycles into a more uniform coherent cycle, although it is not clear whether realistic parameter assumptions would lead to complete synchronization.

However, our analysis has relied only on the coupling introduced by the input-output structure of capital production and has ignored the many other sources of linkage. The most obvious links are created by the price system. If, for instance, one type of capital is in short supply, one would expect the relative price of that capital to rise. To the extent that sectors can substitute different types of capital in their production, one would then expect demand for the other, relatively cheaper, capital components to rise. Thus, an imbalance in one sector would more quickly spread to other parts of the economy, and it is likely that the overall motion of the system becomes more coherent. (We have performed a few preliminary simulations of a version of the model that includes a price system, and these simulations show a strong tendency for synchronization.)

In light of the coupling effect of the price system and of other macroeconomic linkages, (e.g. the Keynesian consumption multiplier) one would therefore expect disaggregate capital systems to show a coherent long-wave motion for a wide range of parameter values, and the basic validity of the simple one-sector model seems intact. Thus, the fact that the simple model aggregates capital into a single commodity is not a cause for doubts about the theory. More important modifications may arise when one explicitly considers other factors excluded from the model, such as labor, wages, and interest rates.

The increased coupling introduced by the parameter α also stabilizes individual sectors by reducing the strength of the investment accelerator within each sector. This

For economic theory in general, our results show the importance of studying non-linear coupling in the economic system. The intricacies of such phenomena suggest that there is a vast unexplored domain of research in the area of economic cycles, and that results from such studies may well prove counter-intuitive and, hence, generate new insights into the causes and cures of business cycles. For instance, our model shows how two sub-systems which in isolation are stable (for high values of α) become unstable when allowed to interact.

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