EVOLUTION DYNAMICS OF SOME TWO-PHASE SYSTEMS

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ABSTRACT

We treat some problems of controlling the development of a two-phase system which is identified with the evolution of its inter-face. First we study the class of so-called decision change regimes and find out an estimate for the number of decision changes. Then we consider the possibility of optimal control in this class.

§1.PROBLEM STATEMENT

Behaviour of many socio-economics systems (population structure, industrial development and air pollution...) can be modelled by a system consisting of two-phase separated by a common inter-face (free-boundary). Moreover, the development of the system is, in a certain sense, indentified with the evolution of the free-boundary. So the problem of studying the dynamics and regulating the evolution of the free-boundary is of a great interest.

Denote by x and t the space- and the time-variables respectively, by u(x,t) the state of the system and by $x = s(t)$ the inter-face. The mathematical model of the system is following:

(1.1) $$\alpha_1 u_{xx}^- = u_t^-$$

$$\text{in } \Omega_T^-(s(.)) := \{(x,t) \in R^2 : a < x < s(t),\ t \in (0,T)\},$$

$$u^-(x,0) = \Phi(x),\ x \in [a,b],$$

$$\left.\begin{array}{l} u_x^-(a,t) = 0 \\ u^-(s(t),t) = 0 \end{array}\right\}\ t \in [0,T];$$

(1.2) $$\alpha_2 u_{xx}^+ = u_t^+$$

$$\text{in } \Omega_T^+(s(.)) := \{(x,t) \in R^2 : s(t) < x < c,\ t \in (0,T]\},$$

$$u^+(x,0) = \Phi(x),\ x \in [b,c],$$

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$$u_x^+(c,t) + \alpha u^+(c,t) = -f(t)$$

$$\Big\} \ t\in [0,T];$$

$$u^+(s(t),t)= 0$$

$$s(0) = b$$

(1.3) $\{$

$$\dot{s}(t)= -\delta_1 u_x^-(s(t),t)+ \delta_2 u_x^+(s(t),t), \ t\in [0,T].$$

Here $a< b< c$, $T> 0$, α_1, α_2, α, δ_1 and δ_2 are given positive constants.

The problem of finding a triple $\{u^-(.,.), u^+(.,.), s(.)\}$ which satisfies (1.1) - (1.3) is known in mathematical physics as the two-phase Stefan problem.

Let flux $f(t)$ be determined from the control equation

(1.4) $\beta\dot{f}(t)+ f(t)= p(t)$, $t> 0$, $f(0)= 0$,

where $p(t)$ is the decision of controller C to which the system reacts with a certain time lag $\theta \geq 0$ (we suppose that $s(t)= s_0(t)$, $t\in [-\theta,0]$, is observed previously). Moreover, there is a region desirable for $s(.)$:

$$s_1(t) \leq s(t) \leq s_2(t),$$

$s_1(.)$ and $s_2(.)$ being given such that

(A1) $a < s_1(t) < s_2(t) < c,$

$s_2(t) - s_1(t) \geq \mu > 0, \ t \geq 0.$

Finally, let S^0 (S^+) be the set of passive decisions (the set of active decisions) of the controller C:

(A2) $0 \in S^0 \subset [0,1]$, $1 \in S^+ \subset [0,1]$,

$p^0 \in S^0, \ p^+ \in S^+ \rightarrow p^0 < p^+.$

So, observing the evolution of $s(t)$, the controller C takes at time moment t

a passive decision $p^0 \in S^0$ if $s(t-\theta)= s_1(t-\theta)$ and the decision was active just before;

an active decision $p^+ \in S^+$ if $s(t-\theta)= s_2(t-\theta)$ and the decision was passive just before.

In the case $S^0 \equiv \{0\}$, $S^+ \equiv \{1\}$ and without any time lag this hysteresis behaviour has been considered by Hoffmann and Sprekels (1984). In the present paper we shall first develop the results of these authors to the case where S^0 and S^+ have only restriction (A2). Our special attention will be paid to estimating explicitly the time interval of realizalion of a decision change regime and the number of decision changes by a taken decision change regime and the parameters of the system. Then, in §3, we shall use the freedom in choosing decision change regimes to minimize the maximal deviation of inter-face s(.) from some given ideal one, or to minimize the total cost for going out from the desirable zone.

§2.REAL-TIME DECISION CHANGE REGIME

It was proved in Hoffmann and Sprekels (1984) that for every control p(t), $0 \le p(t) \le 1$, $t \in [0,T]$, the problem (1.1)-(1.4) has a unique solution $\{u^-(p(.);.,.),\ u^+(p(.);.,.),\ s(p(.);.)\}$ and moreover, $s(p(.);.)$ oscillates at most N^* times between $s_1(t)$ and $s_2(t)$ in [0,T,], N^* being finite an independent of p(.). In what follows it will be shown that N^* can be determined a priori.
From now on we shall write "regime" instead of "decision change regime", for short.

Definition 2.1. A sequence $p := \{p_i,\ i= 1,2,\dots,N^*\}$ is called a pre-regime provided that

$$p_0 \in \begin{cases} S^0 & \text{if } s(-\theta) \le s_1(-\theta), \\ S^+ & \text{if } s(-\theta) \ge s_2(-\theta), \\ S^0 \cup S^+ & \text{if } s_1(-\theta) < s(-\theta) < s_2(\theta), \end{cases}$$

and

$$p_{k+1} \in \begin{cases} S^0 & \text{if } p_k \in S^+, \\ S^+ & \text{if } p_k \in S^0, \end{cases}$$

$k= 0,1,\dots,N^*-1$.

Let p be a given pre-regime.

Definition 2.2. A sequence $r := \{p_k,\ t_k,\ k= 0,1,\dots,N \le N^*\}$ is said to be a regime (or a realization of pre-regime p) if the following relations hold:

$$t_0 := 0,$$

$$t_{k+1} \begin{cases} \inf M_{k+1} & \text{if } M_{k+1} \neq \varphi, \\ +\infty & \text{otherwise,} \end{cases}$$

$$M_{k+1} := \left\{ t \in (t_k, T) : s(p;t-\theta) = \begin{cases} s_1(t-\theta) & \text{if } p_k \in S^+ \\ s_2(t-\theta) & \text{if } p_k \in S^0 \end{cases} \right\}$$

$$s(p;t) := s(p(.);t),$$

$$p(t) = p(r;t) := p_k, \ t_k \le t < t_{k+1},$$

$$k = 0,1,2,\dots,N.$$

Consider a regime $r = \{r_i, t_i, \ i = 0,1,\dots,N\}$. Our goal is to establish some a priori estimates.

Denote by $f(r;.)$ the unique solution of (1.4) with $p(.) = p(r;.)$. One can see that $f(r;.)$ is monotone in every interval (t_k, t_{k+1}) and the following inequalities hold:

(2.1) $$\min\{f_k(r), f_{k+1}(r)\} \le f(r;t) \le \max\{f_k(r), f_{k+1}(r)\} = \begin{cases} f_k(r) & \text{if } f_k(r) \ge p_k, \\ f_{k+1}(r) & \text{if } f_k(r) < p_k, \ t_k \le t < t_{k+1}, \end{cases}$$

where

$$f_k(r) := f(r;k) = \exp(-t_k/\beta) \sum_{i=0}^{k-1} [\exp(t_{i+1}/\beta) - \exp(t_i/\beta)];$$

(2.2) $$0 \le F_k(r) \le f(r;t) \le F^k(r) \le 1,$$

where $$F_k(r) := \min\{f_i(r), \ i = 0,1,\dots,k+1\} \le f(r;t) \le \max\{f_i(r), \ i = 0,1,\dots,k+1\} =: F^k(r),$$ $$t \in [0, t_{k+1}];$$

(2.3) $$0 \ge u_+(r;x,t) \ge -\tau_k \ge -\tau, \ (x,t) \in \mathrm{Cl}\Omega_k^{\ +}(s(r;.))$$

where

$$\tau_k := \max\{\alpha_3 F^k(r); \ |\Phi(x)|, \ b \le x \le c\},$$

$$\tau := \max\{\alpha_3; \ |\Phi(x)|, \ b \le x \le c\},$$

$$\alpha_3 := \min\{1, 1/\alpha\},$$

$$\Omega_k^+(s(r;.)) := \{(x,t)\in R^2:\ s(r;t)<x <c,\ 0<t<t_{k+1}\},$$

$k= 0,1,\ldots, N;$

(2.4) $$|\dot{s}(r;t)| \le \max\ \{\delta_i\Phi_i{}',\ i= 1,2;\ \delta_2F^k(r);\ \delta_2\alpha\tau_k\} =: E_k(r),\ t\epsilon\ [0,t_{k+1}],\ k= 0,1,\ldots,N;$$

(2.5) $$|\dot{s}(r;t)| \le \max\ \{\delta_i\Phi_i{}',\ i= 1,2;\ \delta_2;\delta_2\alpha\tau\} =: E,\ p(.),\ t\epsilon\ [0,T],$$

where

$$\Phi_1{}' := \max\ \{|\Phi'(x)|,\ a\le x\le b\},$$

$$\Phi_2{}' := \max\ \{|\Phi'(x)|,\ b\le x\le c\}.$$

It is worth noting that relations (2.4)-(2.5) enable us to estimate $\dot{s}(t)$ a priori and explicitly by a given regime and by the parameters of the system. To continue we need the function $\Phi(.)$ to have the following property:

(A3) $a-b < \Gamma < c-b$ where

$$\Gamma := (\delta_1/\alpha_1) \int_a^b \Phi(x)dx + (\delta_2/\alpha_2) \int_b^c \Phi(x)dx.$$

We are going now to formulate the main results of the section.

Result 1(concerning the time interval of relization of decision change pre-regimes).Every decision change pre-regime p is realized uniquely by a decision change regime

$r:= \{r_i,t_i,\ i= 0,1,\ldots,N\},\ t_N < T(p) \le t_{N+1}$, where $N= N(p)$

is the minimal from non-negative integers k satisfaying the following two conditions (i) - (ii):

(i)

(2.6) $$\sigma_{1,k-1}(r) := \sum_{j=0}^{k-1} \omega_j(r) \max\{f_j(r),\ f_{j+1}(r)\} < (1/\delta_2)(\Gamma-a+b),$$

$$(2.7) \qquad \sigma_{2,k-1}(r) := \sum_{j=0}^{k-1} \omega_j(r) \, [\alpha\tau_j - \min\{f_j(r), f_{j+1}(r)\}] < (1-\delta_2)(c-b-\Gamma),$$

$\omega_j(r) := t_{j+1} - t_j$;

(ii) at least one of the following inequalities holds:

$$(2.8) \qquad \sigma_{1,k}(r) \geq (1/\delta_2)(\Gamma-a+b),$$

$$(2.9) \qquad \sigma_{2,k}(r) \geq (1-\delta_2)(c-b-\Gamma),$$

(by definition $\sigma_{1,-1} = \sigma_{2,-1} = 0$, so (i)-(ii) hold automatically for $k = 0$).

Moreover, the time interval $[0,T(p)]$ of relization of decision change pre-regime p is defined as follows:

(iii) $\quad T(p) = T_1(p)$

where $\quad T_1(p) := \inf\{t \in [t_k, t_{k+1}] : \sigma_{1,k-1}(r) + (t-t_k)\max\{f_k(r), f_{k+1}(r)\} = (1/\delta_2)(\Gamma-a+b),$

if only (2.8) takes place;

$\quad T(p) = T_2(p)$

where $\quad T_2(p) := \inf\{t \in [t_k, t_{k+1}] : \sigma_{2,k-1}(r) + (t-t_k)[\alpha\tau_k - \min\{f_k(r), f_{k+1}(r)\} = (1-\delta_2)(c-b-\Gamma),$

if only (2.9) takes place;

$\quad T(p) = \min\{T_1(p), T_2(p)\},$

if both (2.8) and (2.9) take place.

Thanks to **Result 1** every decision change pre-regime p can be identified with its unique realization $r = \{r_i, t_i, i = 0,1,\ldots,N = N(p)\}$, as we shall do henceforth.

Set for $0 \leq t_* < t^* \leq T$

$$\mu(t_*,t^*) := \min\{s_2(t) - s_1(t), t_*-\theta \leq t \leq t^*-\theta\},$$

$$\mu(t^*) := \mu(0,t^*), \; \mu_k(p) := \mu(t_{k+1});$$

$$D_i(t_*,t^*) := \max\{|s_i(t), t_*-\theta \leq t \leq t^*-\theta\}, i=1,2,$$

$$D(t_*,t^*) := \min\{D_i(t_*,t^*), i=1,2\},$$

$$D^i(t^*) := D^i(0,t^*), \; i=1,2,$$

$$D(t^*) := \min\{D^i(t^*), i=1,2\},$$

$$D_k(r) := D(t_{k+1}),\quad D := D(T)$$

and let $N(t_*,t^*)$ denote the number of decision changes in the interval $[t_*,t^*]$.

Result 2 (concerning the number of decision changes and their frequency). The following estimates hold:

$$(2.10)\qquad N(t_*,t^*) \le 1 + \frac{t^*-t}{\mu(t_*,t^*)}[E + D(t_*,t^*)],$$

$$0 \le t_* < t^* \le T;$$

$$(2.11)\qquad \infty_k(r) \ge \frac{\mu_k\ (r)}{E_k(r)+D_k(r)},\quad k= 0,1,\dots,N(r).$$

§3.OPTIMAL DECISION CHANGE REGIME

It follows from the results obtained in §2 that the control to the system being considered is completely determined by the decision change pre-regime p and it remains to choose the latter. In the sequel we shall try to optimize this operation, i.e. to use the freedom in choosing p to minimize the maximal deviation of s(.) from a given ideal interface $\sigma(.)$:

$$(3.1)\qquad J_1(p) := \max\ \{|s(p;t)-\sigma(t)|,\ 0 \le t \le T(p)$$

or the total cost which must be paid for leaving the desirable zone:

$$(3.2)\qquad J_2(p) := \int_0^{T(p)} J_{p,s}(t)dt$$

where

$$J_{p,s}(t) := \begin{cases} \pi_1(s_1(t)-s(t)) & \text{if } s(t) < s_1(t), \\ 0 & \text{if } s_1(t) \le s(t) \le s_2(t), \\ \pi_2(s(t)-s_2(t)) & \text{if } s(t) > s_2(t), \end{cases}$$

π_1 and π_2 being given positive coefficients.

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Denote by PR the set of all possible decision change pre-regimes corresponding to a taken pair (S^0, S^+). Thus we are led to the following optimization problem:

(OP) Minimize $J_1(p)$ (or $J_2(p)$) subject to $p \in PR$.

Result 3. Every optimizing sequence of decision change pre-regimes $\{p^{(n)}, n = 1,2,\ldots\}$, i.e.

$$J_1(p^{(n)}) \to m := \min \{J_1(p), p \in PR\}$$

as $n \to \infty$, contains a subsequence $\{p_{(k)}, m = 1,2,\ldots\}$ which converges to an optimal decision change pre-regime p_0:

$$p_{(k)} \to p_0 \quad \text{as } k \to \infty \quad , J_1(p_0) = m.$$

In practice this result allows us to orient our decision to an optimal one and to find an approximate solution.
Let us omit the mathematical proof of **Results 1-3**, because it needs much more than one page.

CONCLUSION REMARK. The present paper summarizes a qualitative study of the problem. In practice, especially if we confine ourselves to some finite sets S_0 and S_1, the technique applied facilitates the analysis of simulation and gaming process to obtain, on the one hand, the complete information about possible decision change pre-regimes, and on the other hand, an optimal one.

REFERENCES

Hoffmann K.-H and Sprekels J. 1982. Real-time control of the free boundary in a two-phase Stefan problem. Numer. Funct. Anal. and Optimiz. 5(1): 47-76.