GRAPH THEORY ANALITICS OF SD FLOW DIAGRAM

Associate Prof. Jia Ren'ang
(Jiangxi University, China)

The article advances and discusses a theory that using digraph theory to analyse SD flow diagram. First, on the basis of the concept strongly connected digraph generation out-tree and extreme out-tree, we prove two existence generation out-tree theorem and three relative propositions of flow diagrams generation out-tree. The method to define the extreme out-tree and feedback loop set are obtained in complicated SD flow diagrams, also the general laws are advaneed that each variable produces corresponding increment that in the feedback loop and extreme out-tree and at the different simulation moment, one increment Δx is given to certain variable x, All these conclusions are useful to the debugging and the result-analysis of SD model and are also useful to analyse the effects of controll variable in system.

I Basic Concepts

Before expounding the problems, we have to cite the following basic definitions.

Definition 1.1 A digraph is defined to be a pair (V,X), where (1) V is a non-empty finite set V=V(D), the elements of V is called vertices, and (2) X is a set X=X(D) of ordered pair of different elements, the elements of X is called arcs.

In accordance with the definition of digraph, the flow diagram in SD is a digraph D=(V,X), its sets of vertices are

$$V=\{Li \mid Li \text{ is level variable}, i=1,2,\dots,m\}\cup \{Ri \mid Ri \text{ is rate variable}, i=1,2,\dots,n\}\cup \{Ai \mid Ai \text{ is auxiliary variable}, i=1,2,\dots,k\}\cup \{Si \mid Si \text{ is supplementary variable}, i=1,2,\dots,g\}\cup \{ai \mid ai \text{ is constant}, i=1,2,\dots,q\}$$

and its set of arcs are

$$X=\{xi \mid xi \text{ is substance channel between two variables}, i=1,2,\dots,e\}\cup\{yi \mid yi \text{ is information channel between two variables}, i=1,2,\dots,h\}$$

Let's take SD flow diagram as digraph, the variable symbol in SD flow diagram is called variable point by a joint name, and substance and information channel between two variables are called arc by a joint name.

Definitions about in-degree and out-degree of variable point.

Definition 1.2 For the flow diagram D=(V,X), V is a set of variable points and X is a set of arcs, $x \in X$, $u,v \in V$, $x=(u,v)$, in which u,v are initial endpoint and terminal endpoint of arc x. We may call them endpoint by a joint name. The number of arc which u is taken as its initial endpoint is called out-degree of u, and it is denoted odDu or Odu. Also, the number of arc which u is taken as its terminal endpoint is called in-degree of u, and it is denoted idDu or idu.

Definitions about directed walk, path, circuit(loop) etc.

Definition 1.3 For the flow diagram D=(V,X), let's suppose there are alternate sequence of variable point and arc $v_0 x_1 v_1 x_2 v_2$

..., $x_n v_n$, among which $v_0, v_1, \ldots, v_n$ is variable point and $x_1, x_2, \ldots, x_n$ is arc.

a. For any x_i in sequence, v_{i-1}, v_i are taken as the endpoints, then this sequence is called directed semiwalk with length n, and variable points of the sequence are different, it is semipath.

b. For any x_i in sequence, v_{i-1} is taken as the initial endpoint, also v_i is taken as the terminal endpoint, then this sequence is called directed walk with length n, and if variable points are different in this sequence, it is called directed path.

c. Directed closed-walk is called directed loop.

d. If sequence is directed path, then v_0 can reach v_n.

According to definition 1.3, the feedback loop in SD Flow Diagram is directed loop.

Definition about connectivity.

Definition 1.4 In flow diagram D, u,v are any two variable points. If there is a semipath to join them, D is weakly connected, if there is at least a variable point can be reached by another, then D is unilateral connected; if they can be reached with each other, D is strongly connected.

Definition about out-tree.

In the following we call any parts of flow diagram D subgraph and D is the subgraph of itself.

Definition 1.5 If there is a variable point v in a subgraph of flow diagram D, and for any variable point of D, there exists one and only one directed path which is from v to u (for u=v, we consider that there exists only one path that its length is zero which is from v to u), then this subgraph is a out-tree, v is the root of out-tree.

Definition about spanning concept.

Definition 1.6 The directed semiwalk that include all variable points of flow diagram D is called spanning directed semiwalk. By analogy, directed walk, directed semipath, directed path, feedback loop, out-tree are called respectively spanning directed walk, spanning directed semipath, spanning directed path, spanning feedback loop, spanning out-tree.

We dont consider reference vertex (constant vertex ai) of flow diagram at following discuss, because the property of constant a_i in the problem we discuss can be realized by its corresponding variable point of level, rate or auxiliary variable, But in research we can make the explanation more simple without considering constant vertex a_i.

Example 1 Fig. 1.1 shows the flow diagram of SD World Model II and the spanning out-tree of this flow diagram is shown by Fig 1.2.

The root v of out-tree in the flow diagram (Birth Rate shown by Fig 1.2) can controll each variable of out-tree, with strongly practical significance. When one increment Δv is given a variable v, each variable produce corresponding new increment at the different simulation moment. Thus variable in practical system is found out, the crucial control variable is out. For example, in the World Model II, birth rate is a root of one of spanning out-tree, birth rate may controll each variable, moreover, we may known how to controll other variables by out-tree.

But not every flow diagram exists spanning out-tree, for example, SD traffic flow diagram in Fig 1.3 doesn't exist spanning out-tree. This just like that in the social economic system, also not every variable can controll any other variables in the system. So we discuss the following problems and get to three theorems.

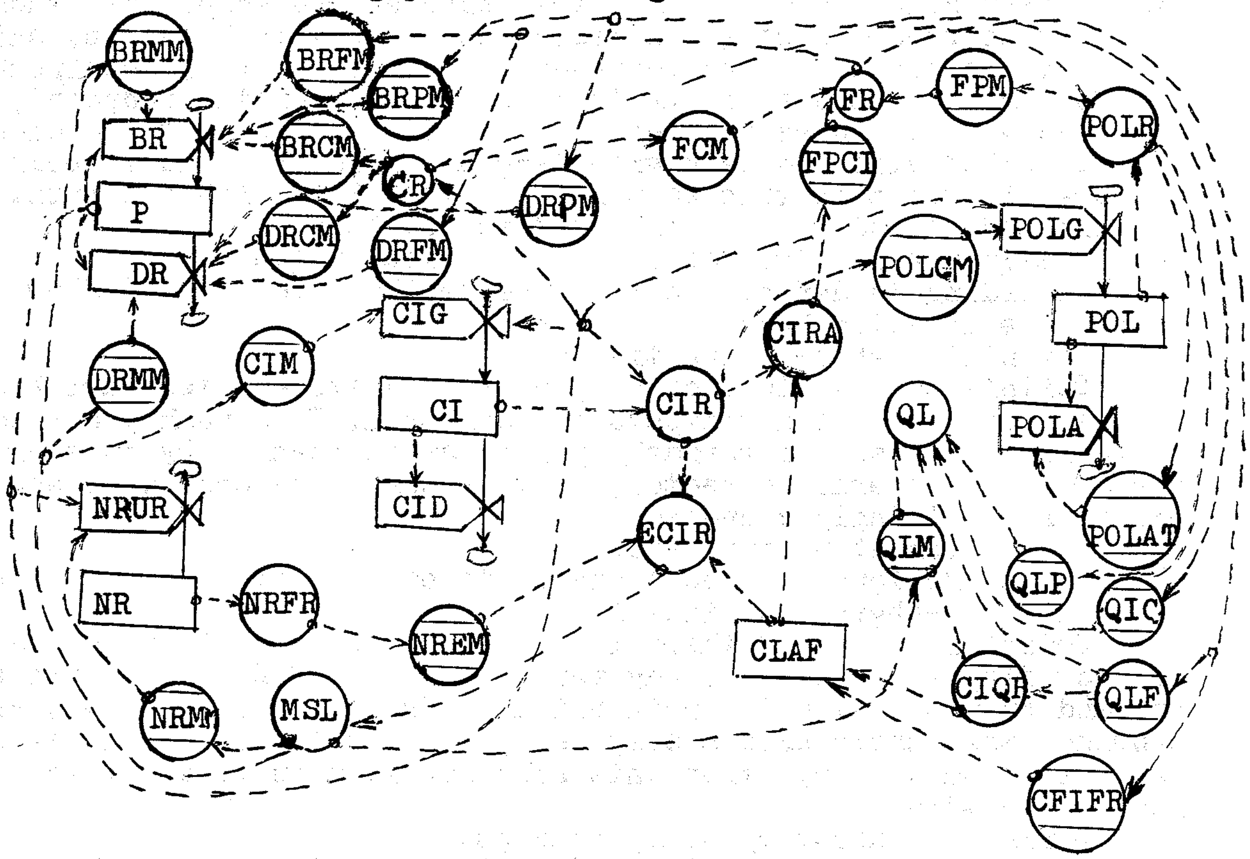

Fig 1,1 SD of the World Model II

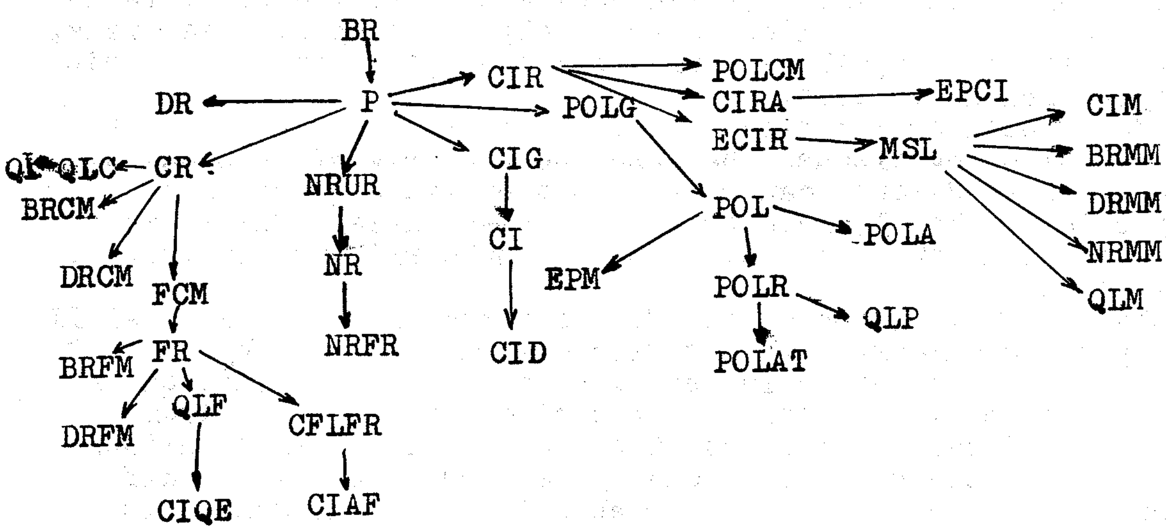

Fig 1.2 Diagram of spanning out-tree of SD World Model II

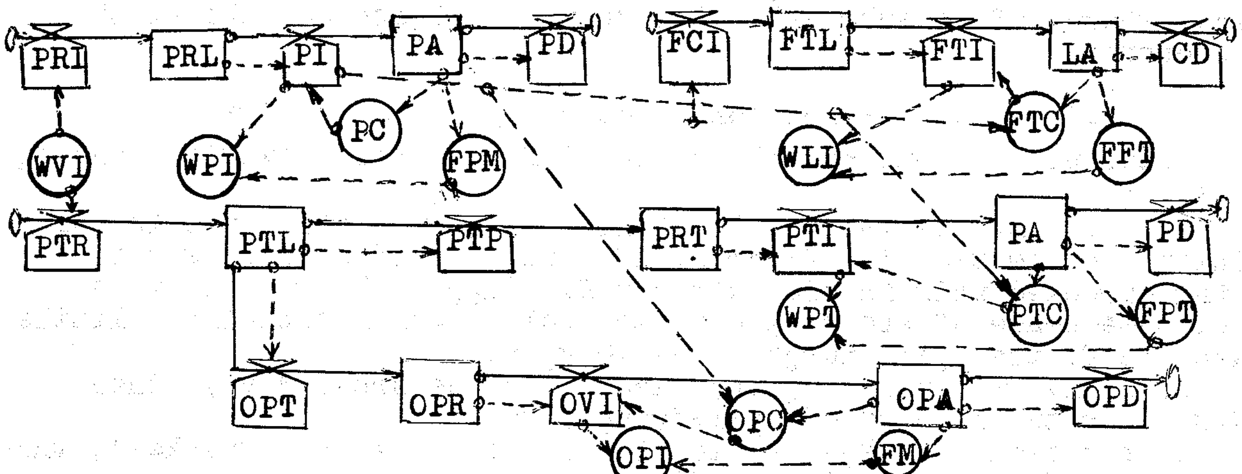

Fig 1.3 SD Flow Diagram of the traffic in Beijing

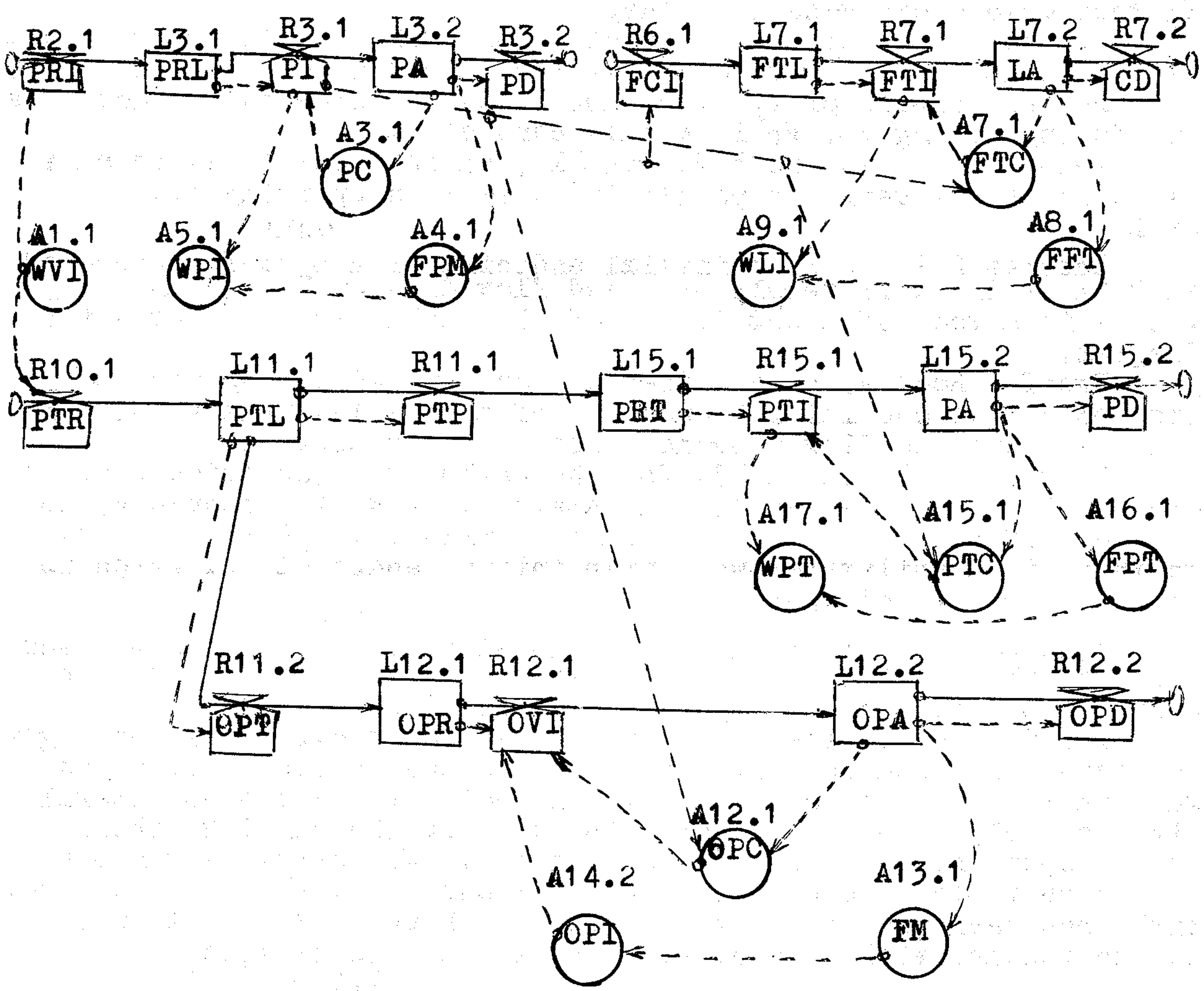

Fig 3.1

Note: SD Flow Diagram of the traffic in Beijing includes four subgraphes, they are path, freight transport, passenger traffic and other traffic subgraph. For details, please see the fifth reference.

II The existing condition of spanning out-tree

Theorem 2.1 Unilateral connected flow diagram exists spanning out-tree.

Proof By the theorem 5.6.2 in [I], a flow diagram D is unilateral connected flow diagram, if, and only if, there is a spanning directed walk in flow diagram D. Suppose $W = v_1 v_2 \dots v_n$ is point-sequence of spanning directed walk W_1 which contains a little arcs in D, then $j \neq 1$, $idv_j \geq 1$, $j \neq n$, $Odv_j \geq 1$.

a. If $idv_1 > 0$, for (v_j, v_1), leave out the arc (v_j, v_1), then W_1 becomes subgraph W_2 of satisfying $idv_1 = 0$.

b. For each (V_1, V_j), by leaving out each arc $(v_k, v_j)(k \neq 1)$, then W_2 becomes W_3 of $idv_j = 1$ in (v_1, V_j).

c. For each (v_j, v_k) satisfying each (v_1, v_j) in W_3, by the way as that in b that leave out the arc $(v_t, v_k)(k \neq j)$, then W_3 becomes W_4 that make every v_k satisfy $idv_k = 1$.

By analogy, subgraph W_n is obtained. The vertice sets of variable points in W_n is $\{v_0, v_1, \dots, v_n\}$, where $idv=0$, $idv=1$ $(j=2,3,\dots,n)$.

Because in-arc of v_j which hasn't been considered are not taken out during defing W_n, W_n is weakly connected.

According to theorem3.4.2 in [I], W_n is a out-tree, because it includes all variable points in D, it is a spanning out-tree of D.

Q.E.D

Theorem 2.2 v_1 is a initial endpoint of a spanning directed walk W_n of the unilateral connected flow diagram D, if, and only if, v_1 is a root of a out-tree T of the unilateral connected flow diagram.

Proof Root v_1 of out-tree W_n constructed during proving theorem 2.1 is one initial endpoint of a directed walk W of unilateral connected flow diagram. Necessity is tenable.

By theorem 5.6.2 in[I], for the unilateral connected flow diagram D, $v_2 v_3 \dots v_n$ exists unilateral walk W_{n-1}, because v_1 is the root of out-tree T, v_1 may connect with any $v_j (j=2,3,\dots,n)$. So there is a unilateral walk with initial endpoint v_1, which is consisted by v_1 and W_{n-1}.

Q.E.D

Theorem 2.3 A flow diagram D is strongly connected, if, and only if, there exists spanning out-tree in it and each variable point is the root of certain spanning out-tree T.

Proof By theorem 5.6.2 in [I], a flow diagram D is strongly connected, if, and only if, there is a spanning closed directed walk in it. Therefore, each variable point v exists unilateral directed walk with initial endpoint v_j. By theorem 1.2, there exists spanning out-tree with a root v_j . Necessity is tenable.

Vice versa, because the root of spanning out-tree can connect with each variable point of out-tree, each variable point in D can be reached with each other. Sufficiency is tenable.

Q.E.D

Strongly connected flow diagram posses very good property.

Proposition 2.1 In the strongly connected flow diagram, when one increment is given to any variable point, other variable points will produce corresponding increment.

For many flow diagrams, when their exogenous variable points and supplementary variable points are taken out, they can posses strong connectivity. For example, the World Model II is just like this. But it is very troublesome to judge the strong connectivity of a complicated flow diagram with the definition of strongly connected. The following two proposition can help us do it more simple.

Proposition 2.2 A feedback loop is a branch of strongly connected.

Proposition 2.3 Suppose $D_i=(V(D_i),X(D_i))(i=1,2)$ are two branches of strongly connected of flow diagram $D=(V(D),X(D))$, when one of the following conditions is satisfied, the derived subgraph $D_3=(V(D_3), X(D_3))$ of $V(D_1)\cup V(D_2)$ is the branch of strongly connected.

1. $V(D\ )\cap V(D\ )\neq\Phi$.
2. There exists arc x_1, whose initial endpoint is in $V(D_1)$ and terminal endpoint is in $V(D_2)$, at the same time, there also exists arc x_2, whose initial endpoint is in $V(D_2)$, and terminal endpoint is in $V(D_1)$.

So-called derived subgraph D_3 of $V(D_1)\cup V(D_2)$, D_3 should satisfy $V(D_3)=V(D_1)\cup V(D_2)$ and $u, v\in V(D_3)$, if (u,v) $X(D)$, then $(u,v)\in X(D_3)$.

We can use definition of strongly connected to prove the above proposition correct directly. In accordance with proposition 2.3, it is rather easy to judge that the World Model II is strongly connected.

III. The method to define the extreme out-tree of flow diagram

In this section, we discuss about the extreme out-tree of flow diagram.

Definition 3.1 In flow diagram $D=(V(D),X(D))$, if for any out-tree $T_i(V(T_i), X(T_i))$ of D, its out-tree $T=(V(T),X(T))$ unsatisfies $V(T)\subset V(T_i)$, then out-tree T is the extreme out-tree of D.

Not every flow diagram has spanning out-tree, but any one of flow diagram has extreme out-tree.

Next we first give the method to draw a diagram of extreme out-tree of branch of strongly connected. Here is a definition.

Definition 3.2 $D_1=(V(D_1), X(D_1))$ is the subgraph of $D=(V(D), X(D))$. If D is strongly connected, and for any variable point $V\in(V(D)-V(D_1))$, derived subgraph of $V(D\ )\cup\{V\}$ doesn't posses strong connectivity, then D_1 is the branch of extreme strongly connected of D.

A. The method to define spanning out-tree of branch of extreme strongly connected.

a. Choose any point V in branch of extreme strongly connected and leave out all in-arcs of point V.

b. Find out all U_i satisfied (V,U_i) and leave out all in-arcs not satified (V,U_i) from each U_i respectively, then define the second hierarchy variable point of out-tree (V is the first hierarchy variable point.)

c. Handle each variable point at the second hierarchy respectively according to "b", then we will get the third hierarchy variable point. By analogy, we can get the spanning out-tree T.

The exactitude of this defining method arises from the proof of theorem 2.1.

According to this method, let's consider the flow diagram of the World Model II shown by Fig 1.1. We can take Birth Rate as the roor of out-tree, then we may get the spanning out-tree shown by Fig 1.2.

B. The method to define extreme out-tree of flow diagram.

a. To define the branch of extreme strongly connected of flow diagram D with the definition of strongly connected and proposition 2.2, 2.3, and to mark different branches of strongly connected by the method that give numbers to each variable point. The way is suppose $D_i=(V(D_i), X(D_i))$ is the code number i branch of extreme strongly connected, then level, rate, and auxiliary variable point in D are L_{ij}, R_{ik}, A_{it} respectively, j,k,t are a numbers of level, rate and auxiliary variable in D respectively.

b. Suppose $D_1, D_2, \ldots, D_n$ are n branches of extreme strongly connected, and draw connected arc of branch of each extreme strongly connected, only draw one for the same direction arc. At the same time, suppose $D_1, D_2, \ldots, D_n$ are variable points, then one no loop digraph G is got.

c. Find out the extreme unilateral digraph of digraph G. If digraph G has $G_1, G_2, \ldots, G_K$ extreme unilateral digraph, then flow diagram D exists K extreme out-tree.

d. Draw the corresponding spanning out-tree of each digraph G_i by the way that draw spanning out-tree of branch of extreme strongly connected. Each spanning out-tree of this is the extreme out-tree of D.

Example 2. Draw extreme out-tree of traffic flow diagram in Fig 1.3.

P.S a. To define the branch of extreme strongly connected by the definition and proposition 2.2, 2.3 of strongly connected. Then give code numbers to each variable points. (Shown by Fig 3.1)

b. Suppose branch D_i of each extreme connected is variable point, and draw no loop digraph G as Fig 3.2 .

c. To draw the extreme unilateral digraph G_1, G_2, G_3 of G as Fig 3.2 .

d. To draw the corresponding spanning out-tree of each G_i.

C. Property of extreme out-tree.

Proposition 3.1 Give one increment to any variable point u of extreme out-tree of flow diagram D, only the following two kinds of variable points will produce the corresponding increment in flow diagram.

a. the following variable points of u

b. the preceding variable point v of u, this v is torminal endpoint of arc whose initial endpoint is u or following variable point of u.

This property is useful to the debugging of SD model.

IV: The defining method of feedback loop of the finished flow diagram

The flow diagram is set up gradually. A complicated system model is always a complicated flow diagram. For example, the flow diagram of the World Model II is complicated, so it is rather troublesome to list all feedback loop of a complicated flow diagram, but for the quantitative analysis and qualitative analysis system, it's very important to find out the sets of feedback loop. So we advance the gradually-reduce method. First, here is a definition.

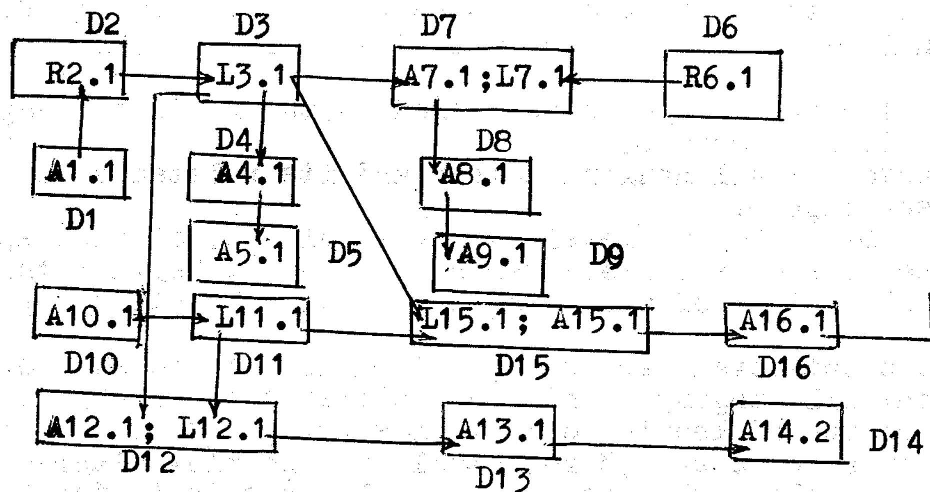

Fig3.2 Stronglt connected points

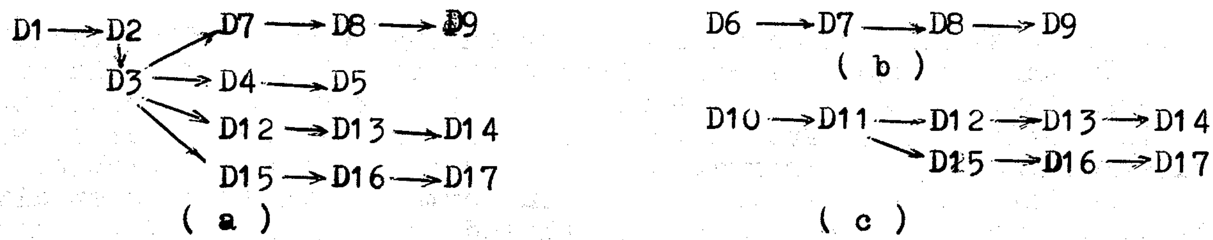

Fig 3.3 The branch of extreme unilateral connectivity

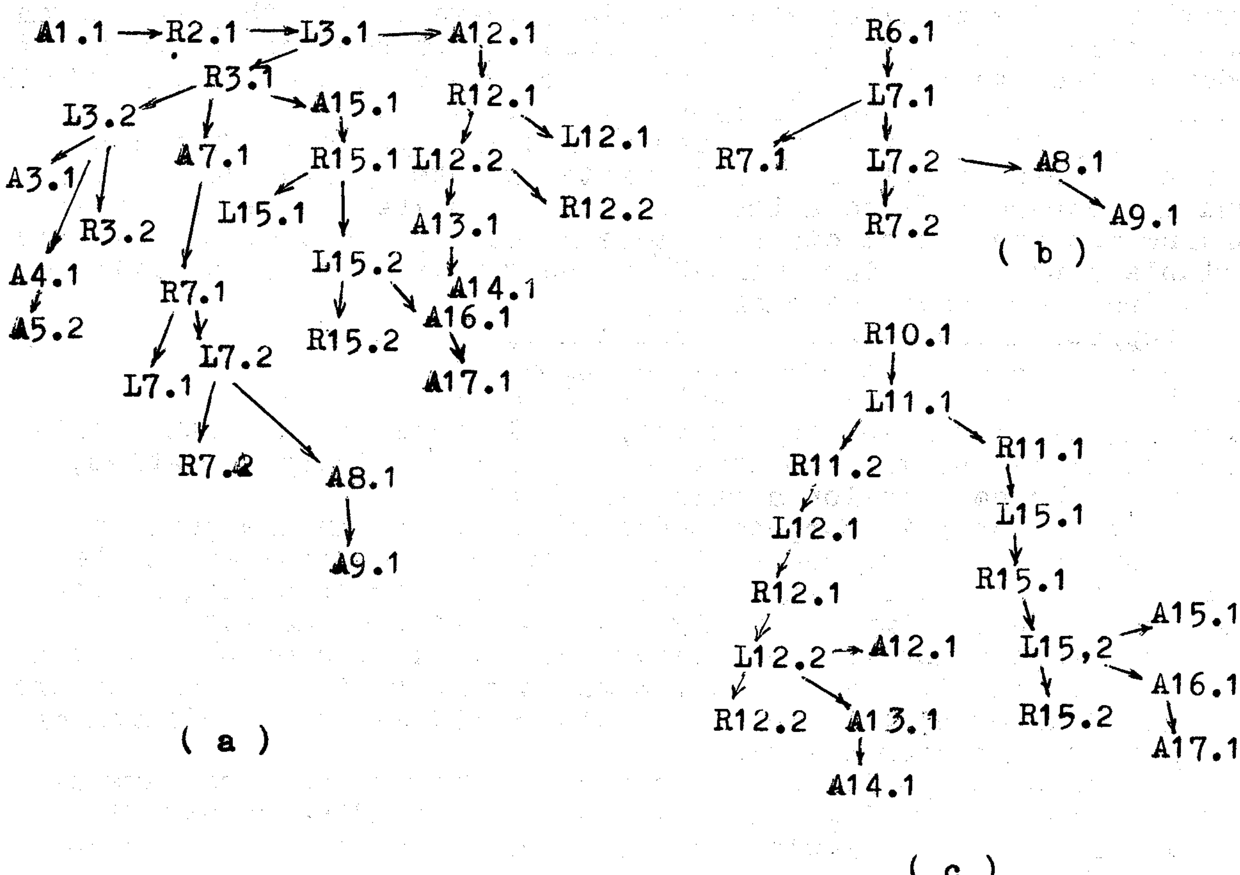

Fig3.4 All extreme out-tree of connected flow diagram shown by Fig 3.1

Definition 4.1 In the flow diagram D=(V(D),X(D)), if u∈V(D) and satisfy with idu=0 or Odu=0, then u is the hanging variable point of D.

The step of the gradually-reduce method in the feedback loop sets of finished flow diagram D.

Step 1. Leave out all hanging points and its related arcs of D to get the subgraph D1.

Step 2. For D1, take any level variable point L_i of subgraph D (it can also be rate or auxiliary variable point), begin with L_i find out all feedback loop of including L_i by the exhaustive way one by one.

Step 3. Leave out level variable point L_j and its related arcs, and then leave out all hanging points and related arcs which are produced in the course of leaving out, then subgraph D_2 of D is got.

Step 4. Execute the step 2,3 repeatedly to the flow diagram D_2, and execute in cycles, then we may get all elements in feedback loop sets of finished flow diagram. (The example is omitted.)

V. General laws on Δ x increment producing corresponding variable

In this section, we will discuss the calculation formula and general laws on each variable produces corresponding increment that in the feedback loop and extreme out-tree and at the different simulation moment, one increment Δx is given to certain variable.

Definition 5.1 Variable y is in certain simulation step divition, because there exists Δx of related variable x, and Δy=y(now available value)-y(original value)≠0, then the variable y produces a corresponding increment Δy of order 1. If from j_0 simulation step divition to j simulation step divition, there exists the phenomenon that k simulation step divition produced corresponding increment of order 1, then variable y produced k corresponding increment of order K j simulation and is denoted by $\Delta^{(K)}y$(k=1,2....).

When y is multifactor variable of $x_1, x_2, ... x_n$ and if from the simulation step divition j_0 to j, factor x_i makes y produce corresponding increment $\Delta^{(k)}y$ and k isn't smaller than the orders of corresponding increment that other factors x_j make y produce, then we say variable y produce increment $\Delta^{(k)}y$ to the simulation step divition j

A. Problems about feedback loop

Suppose there exists feedback loop (1) in SD F.D.

$$\begin{array}{l} L_n \leftarrow R_{n-1} \leftarrow A_{n-1} \leftarrow L_{n-1} \leftarrow \ldots\ldots \leftarrow R_3 \leftarrow A_3 \leftarrow L_3 \\ A_n \rightarrow R_n \rightarrow L_1 \rightarrow A_1 \rightarrow R_1 \rightarrow L_2 \rightarrow A_2 \rightarrow R_2 \end{array}$$

where L_i. A_i. R_i denote respectively level,auxiliary and rate variable and A_i may be the abbreviation of many auxiliary variables, R_i may be the combination n rate variables.

Proposition 5.1 Suppose there are Li,Ai,Ri in the path of F.D.. D and before them one variable x exists increment Δx at the j_0 simulation divition, then orders of their corresponding increments are same in simulation divition $j>j_0$ at each moment.

The proposition is true because of the provide of DYNAMO language in SD. That is, at the same simulation divitionj,Ai and Ri are calculated after calculating Li. This result is the foundation of that next two theorem are true.

We know by proposition 5.1 that only after we get the laws of Li's changing, we will get the laws of corresponding Ai and Ri.

Theorem 5.1 In simulation, with the condition without considering feedback effect and in Rn of feedback loop (1), we add increment with thus laws (shown by table 1) from j_0 simulation divition.

j	j_o	j_o+DT	j_o+2DT	j_o+3DT	...	LENGTH
Rn	Rn	Rn	Rn	Rn	...	Rn

where j_o is integer multiple of DT, then the orders of corresponding increment that each variable produce in the feedback loop (1) should satisfy the following formula,

For $\Delta^{(k)}Li(i=1,2,...,n)$,

$$k=\begin{cases}0 & j_o \le j \le j_o+(i-1)DT \\ [j-j_o-(i-1)DT]/DT, & j_o+iDT<j\le LENGTH\end{cases}$$

For $\Delta^{(k)}Rn$, considering feedback effect,

$$k=\begin{cases}0 & j_o \le j \le j_o+(n-1)DT \\ [j-j_o-(n-2)DT]/DT & j_o+nDT<j\le LENGTH\end{cases}$$

other Ai and Ri produce corresponding increments with the same orders of Li.

B. The problems about out-tree

Definition 5.2 Suppose T is the out-tree including n variables of SD flow diagram, then out-tree T is out-tree of n orders.

Corollory 5.1 Suppose v is the root of n out-tree T of SD flow diagram. If one increment Δv is given to v just at the simulation step divition j_o(j_o is integer multiple of DT), then

a. When v is non-level variable point, at most to the step divition (j_o+nDT), each variable point of out-tree will produce corresponding increment; When v is level variable point, at most to the step divition j +(n-1)DT, each variable of out-tree will produce corresponding increment.

b. If there are k level variables from v to v_l, when v is non-level variable, v_l will produces a corresponding increment at most to the simulation divition (j_o+kDT). When v is level variable point, it can be reduced a step; v_l will produce at least corresponding increment of (n-k) orders when it simulates to the divition (j_o+nDT).

We have acheived very good results in applying above-metioned theory in debugging the model of System Dynamics in Jianxi provicial scientific program.

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