AUTONOMEOUS CHAOTIC BEHAVIOUR IN A GENERIC RESOURCE ALLOCATION PROBLEM

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Abstract. By analyzing the dynamics of resource allocation in a generic management system, this paper illustrates how chaotic behaviour can be internally generated in a typical System Dynamics model.

A company is considered to allocate resources to its production and marketing departments in accordance with shifts in inventory and/or backlog. When order backlogs are small, additional resources are provided to the marketing department in order to recruit new customers. At the same time, resources are removed from the production line to prevent a build-up of excessive inventories. In the face of larger order backlogs, on the other hand, the company redirects resources from sales to production. Delays in adjusting production and sales create the potential for oscillatory behaviour. If allocation of resources is strong enough, this behaviour is destabilized, and the system starts to perform self-sustained oscillations.

To complete the model, we have combined the above simple structure with a feedback which represents a loss of customers when delivery delays become unacceptably long. As costumer's reaction is increased, the simple one cycle oscillation becomes unstable and, through a cascade of period-doubling bifurcations, the system develops into a chaotic state. We present a relatively detailed analysis of this bifurcation series. Poincaré sections and return maps are constructed, and we discuss how these maps can be used to understand the observed qualitative shifts in system behaviour.

INTRODUCTION

It is a basic element of classical System Dynamics thinking that, due to their feedback structure and inherent adjustment delays, social systems tend to oscillate in response to external disturbances. It is usually assumed that these oscillations are damped, i.e. that social systems have stable equilibrium points. The rationale for this assumption is that growing oscillations predetermine a system for collapse and that, consequently, only such systems have survived which have stable equilibria.

This represents a characteristic control engineering point of view with notions carried over from dead and nearly linear systems. The assumption of stable equilibria is unnecessarily restrictive for living, social and biological systems and, more importantly, it prevents us from dealing with those processes through which new structures are created. As we know today (Nicolis et al. 1977, Allen 1980), evolutionary processes can only be understood in turns of transitions in which existing structures collapse and new structures are created.

It may be of interest to compare with recent developments, for instance, in physiology. Complementary to the concept of homeostasis which has dominated physiological thinking for such a long time, self-sustained oscillations are increasingly being recognized as playing a significant role for the function and regulation of normal physiological systems. Investigations performed during the last decade have thus revealed a variety of biological rhythms (Cosnard et al. 1983, Winfree 1980) with periods ranging from fractions of a second to several hours or even days.

Self-sustained oscillations belong to the most simple form of instabilities in nonlinear systems. From a physical point of view, the occurance of such oscillations in physiological systems is related to the fact that these are thermodynamically open systems which are maintained in a far-from-equilibrium condition through dissipation of energy (Nicolis and Prigogine 1977). Since physiological systems have evolved under selective pressure one can also speculate about the possible advantages of a rhythmic behaviour. In certain cases, periodic shifts allow the same biological structure to perform different functions. The self-sustained oscillations then act as a biological clock which serves to synchronize various processes. In other cases, overall efficiency and/or capacity may be increased by driving the system into an oscillatory behaviour.

Social systems are also thermodynamically open systems which evolve through unstable transitions and selective processes. Even for the "fully developed" social system, however, far from being detrimental, unstable phenomena can play an integrated role for the normal function. In analogy with physiological systems, rhythmic behaviour can serve to synchronize various processes or to improve the overall efficiency. It is also possible that oscillations can gard a system against long term drift.

The self-sustained oscillation which produce the economic long wave (Sterman 1985) may thus be regarded as a mechanism for synchronizing investments in new infrastructure. Likewise, the internally generated periodic behaviour in the classical anthropological study "Pigs for the Ancestors" (Rappaport 1968, Meadows and Meadows 1973) certainly played a significant role for the long-term population control of that society.

Non-equilibrium conditions can also give rise to more complex behaviour. Chaotic phenomena can arise, for instance, as a limit cycle becomes unstable and develops through a cascade of period-doubling bifurcations (Feigenbaum 1979). We have already investigated this phenomenon in a simple model of urban migration (Mosekilde et al. 1985) as well as in a model of nephron pressure regulation (Jensen et al. 1986). We presume that similar phenomena occur in a number of classic managerial applications of System Dynamics (Roberts 1978). The purpose of this paper is therefore to show how chaos can develop in a generic resource allocation system. At the same time, by applying some of the mathematical tools available for describing chaotic phenomena, we develop a more complete understanding of the dynamical behaviour of the considered management system. It should be noted that there are several other forms of chaos which, so far, have received little attention in the System Dynamics literature.

THE MANAGERIAL SYSTEM

The problem to be studied relates to the management of manpower and other resources in the CRAM Computer Company which is situated just north of Copenhagen.*

In spite of a very significant growth potential, the owners of CRAM have decided to maintain the company more or less at its present size. This implies that very little net hiring takes place, and that manpower and other resources as much as possible are shifted between the various functions in the company to satisfy immediate needs. It may be hard for outside observers to understand the background for the above decision. Several good reasons can be given, however:

As described below, the company already experiences quite significant difficulties in managing present operations. The owners of CRAM all have an engineering background with little formal training in management, and they fear that if the company grows larger they will have to give up much of their direct control, and to call in a professional manager.

In this connection is has made a significant impression that two leading Danish computer firms have had to close down during recent years, apparently because they have been unable to cope with the experienced very high growth rates.

Finally, it is generally considered that the company has found a profitable niche which can be exploited without interfering with the interests of potential competitors. If CRAM were to adopt a more aggressive policy and expand to new markets, other, financially much stronger firms, could be tempted to initiate damaging countermeasures.

CRAM has specialized in certain forms of simulation hardware and software, simulators for various industrial processes, and application of simulation methods to solve problems for a variety of costumers ranging from hospitals and other public institutions to major Danish industrial firms.

It is characteristic for this type of products that very intensive sales work is required. Unfortunably, it also appears to be characteristic that orders come in at a strongly fluctuating and virtually unpredictable rate. After a quiet period in most of 1985 where much work was done to find new customers, a good deal of the country's highschools decided to order a copy of CRAM's school simulator. Almost at the same time, three major companies ordered training simulators, and interest in the company's simulation analyses increased significantly.

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* To secure anonymity, the company's name and other crusial data have been changed. It can be disclosed, however, that the company measures resources in man-hours/day. Inventory and backlog are measured in units of 10,000 DKK.

As a result, CRAM is presently fully occupied with completing these orders and little time is left for developing new products or recruiting new customers. Much of the problem derives from a feeling that CRAM's customers are very sensitive to delivery delays. If a desired product can not be delivered within very strict deadlines, the customers often loose interest. This is particularly bad because CRAM depends on its ability to maintain contacts with the same customers for a longer period. Once a customer becomes unsatisfied, he can almost be thought of as lost forever.

Under these conditions, the owners of CRAM consider it very fortunate that the company's manpower resources are as flexible as they are. To a large extent, the same persons produce and sell in the sense that each of CRAM's engineers are expected to bring home projects. This is particularly true for those involved in simulation analyses and software developments. Even the so-called hardware production, however, is almost entirely custom-design and requires a very close contact with the costumers throughout the production process.

Nontheless, as already noted, this flexibility has not been sufficient to secure a steady flow of orders. In the autumn of 1985, CRAM therefore decided to develop a System Dynamics model of its own operations. The basic flow-diagram resulting from this project is sketched in figure 1.*

The flow-diagram has four level variables: resources in production, resources in sales, inventory of finished products, and number of customers. In addition, to represent the time required to adjust production, a third order delay has been introduced between production rate and inventory. In total, the system has six independent state variables.

The rate of production is determined from resources in production through a nonlinear function which expresses a decreasing productivity of additional resources as the company approaches maximum capacity. The sales rate, on the other hand, is determined by the number of customers and by the average sales per customer-year. Customers are mainly recruited through visits of the company's engineers. The rate of recruitments depends upon the resources allocated to marketing and sales, and again it is assumed that there is a diminishing return to increasing sales activity. Once recruited, the customers are assumed to remain with CRAM for an average period AT, the association time.

A difference between production and sales causes the inventory to change. As previously discussed, CRAM will respond to such changes by adjusting its resource allocation. At times when the backlog of orders is small (inventories large) relative to a desired level, the company shifts resources into marketing and sales while, at the same time, cutting back on production.

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* To produce a somewhat more generic model, the present authors have replaced order backlog by inventory, order receiving rate by sales, and two table functions by their mirror images.

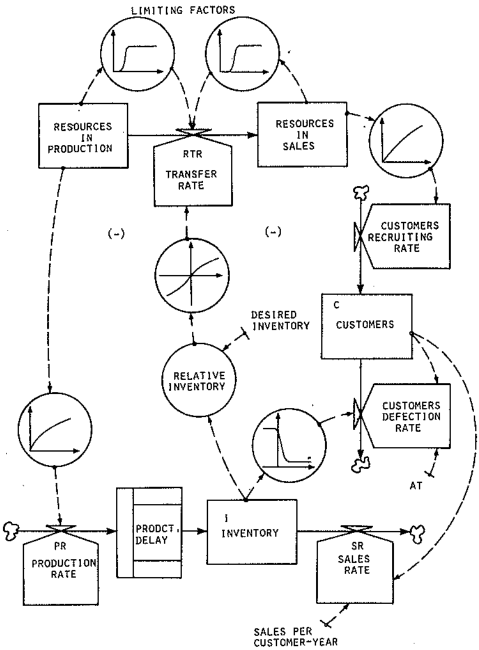

Figure 1. System Dynamics flow-diagram of resource allocation in a generic management system.

When the backlog of orders is higher than desired, on the other hand, resources are redirected from sales to production. A certain minimum of resources is always maintained, though, both in production and in sales. In the model, this is secured by means of the two limiting factors which stop transfer of resources when the manpower floor is approached.

As a final element, the model assumes that there is a feedback from inventory to costumers defection rate. If the inventory of finished products becomes very low, the delivery time becomes unacceptable to many customers, and as a consequence the defection rate is enhanced by a factor H. The following is a discussion of the development in system behaviour as H is increased from 10 to 36.

A DYNAMO-program for the investigated model is given on the next page. The simulations to be presented were performed in PASCAL with analytical representations of DYNAMO's table-functions. Certain of the parameter values in the DYNAMO-program may therefore deviate slightly from those used in the PASCAL-program. It has been found, however, that the DYNAMO-program is capable of producing a completely similar set of simulation results. The DYNAMO-program is not capable, of course, of producing the Poincaré sections and return maps discussed below.

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*  GENERIC RESOURCE ALLOCATION MODEL

NOTE  **  RESOURCE TRANSFER  **
L  RP.K=RP.J+(DT)(-RTR.JK)
N  RP=RPN
N  RPN=TOTALR/2
C  TOTALR=1000  man*hours/day
A  RS.K=TOTALR-RP.K
A  IRT.K=IRTN*TABHL(IRTT,(I.K-DI)/DI,-.3,.3,.1)
T  IRTT=-1/-.8/-.5/0/.5/.8/1
C  IRTN=1.0  man*hours/day/day
R  RTR.KL=IRT.K*CLIP(LFRP.K,LFRS.K,IRT.K,0)
A  LFRP.K=TABHL(LFTAB,RP.K/RPN,.2,.8,.1)
A  LFRS.K=TABHL(LFTAB,RS.K/RPN,.2,.8,.1)
T  LFTAB=0/.1/.2/.5/.8/.9/1

NOTE  **  PRODUCTION AND SALES  **
L  I.K=I.J+(DT)(DELAY3(PR.JK,PD)-SR.JK)
N  I=IN
C  PD=30  days
R  PR.KL=NP*TABLE(PRT,RP.K/TOTALR,0,1,.1)
T  PRT=0/.2/.4/.6/.8/1.0/1.2/1.33/1.40/1.46/1.50
C  NP=150  units/day
R  SR.KL=C.K*SPC
C  SPC=1.0  units/customer/day

NOTE  **  CUSTOMER RECRUITMENT AND LOSS  **
L  C.K=C.J+(DT)(CRR.JK-CDR.JK)
N  C=120  customers
R  CRR.KL=NCRR*TABLE(CRT,RS.K/TOTALR,0,1,.1)
T  CRT=.2/.5/.65/.80/.90/1.00/1.10/1.20/1.30/1.35/1.40
C  NCRR=0.04  customers/day
R  CDR.KL=(C.K/AT)(1+H*TABHL(DRT,I.K/DI,0,.1,.02))
T  DRT=1/.85/.6/.4/.15/0
C  AT=3000  days

NOTE **  BIFURCATION PARAMETER  **
C  H=10
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SIMULATION RESULTS

As indicated in figure 1, our managerial system is controlled by two interacting negative feedback loops. Combined with the delays involved in adjusting production and sales, these loops create the potential for oscillatory behaviour. If the transfer of resources becomes strong enough, this behaviour is destabilized and the system starts to perform self-sustained oscillations, limited in amplitude by the various nonlinear relations.

Figure 2 shows a typical example of such a limit cycle. We have here plotted the temporal variation of resources in sales (2a) together with a phase plot showing corresponding values of inventory and resources in sales (2b). For the bifurcation parameter we have used H=10. To accentuate the form of the stable attractor rather than of the initial transient, we have not started the plotting routine until, after about 24 oscillations, the transient has died out.

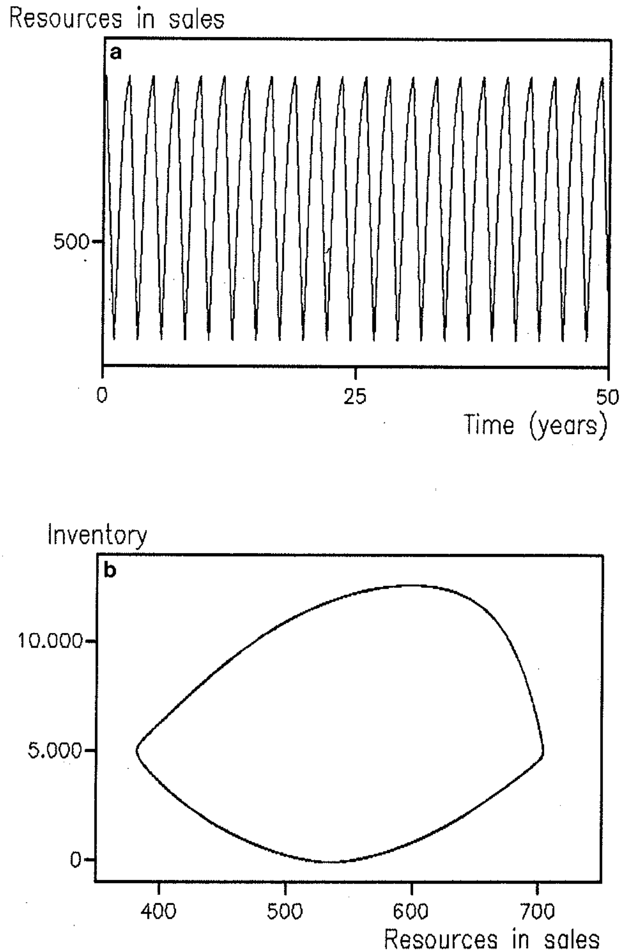

Figure 2. Time- and phase-plot of the one-cycle attractor obtained for H=10.

Figure 3 shows a similar set of simulation results obtained for H=13. In the time-plot we now observe alternating high and low minima and maxima. The period of the stable attractor has thus doubled. In the phase-plot we observe how the attractor has folded itself and now closes only after two revolutions.

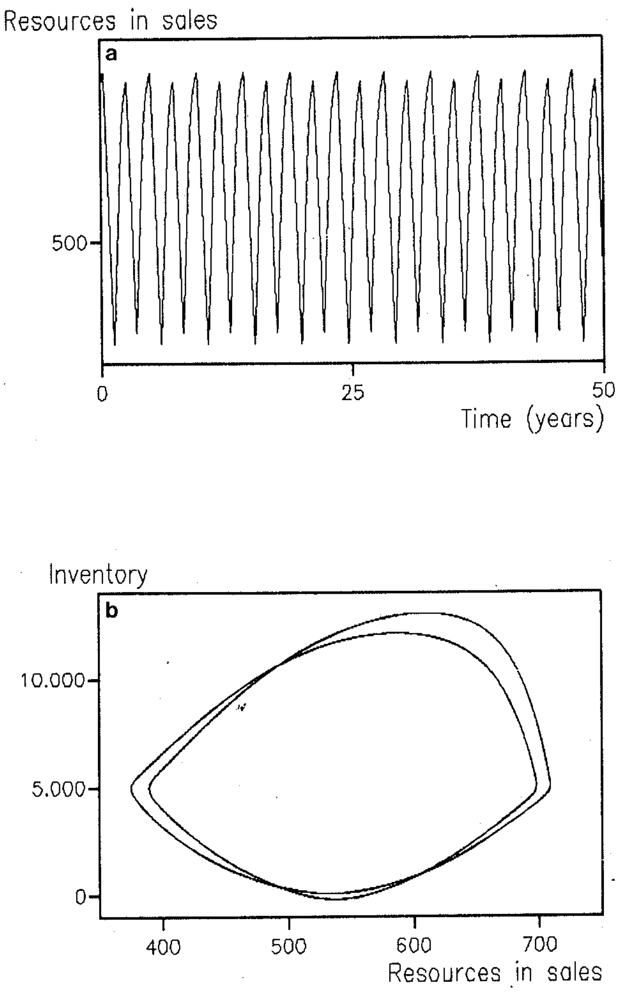

Figure 3. Time- and phase-plot of the two-cycle attractor obtained for H=13.

If the bifurcation parameter is increased to H=28, we obtain the results of figure 4. There are now 4 different maxima: two high maxima and two low maxima. In the phase-plot, the attractor appears to have folded twice, a 4-cycle.

As H is further increased, the bifurcation process continues until for H~30, the threshold to chaos is exceeded. Now, the stable attractor no longer closes to itself, the period has become infinite, and the time variation of resources in sales seems to be random. This is shown in figure 5.

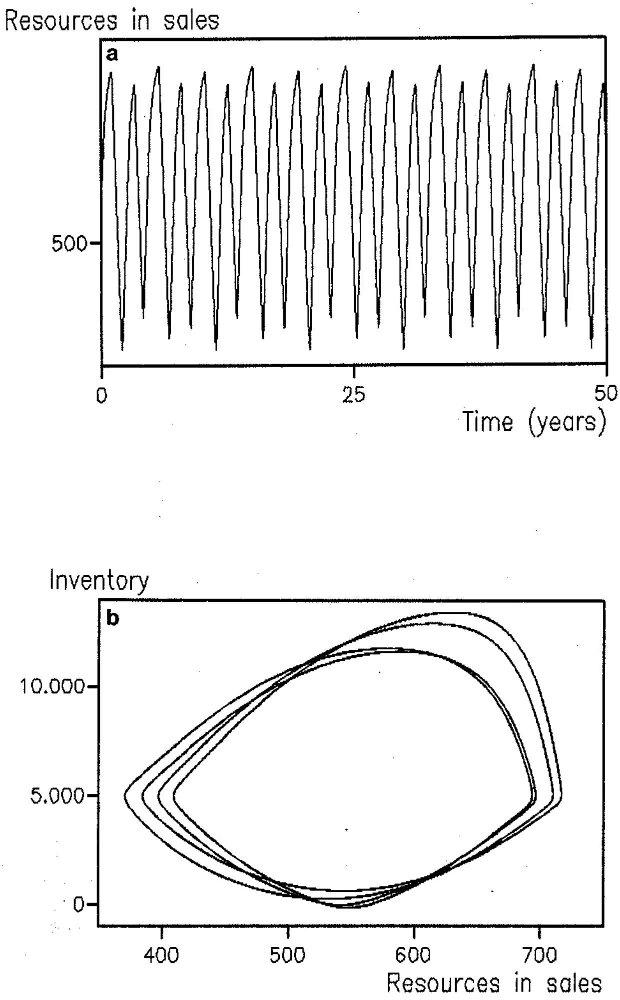

Figure 4. Time- and phase-plot of the 4-cycle attractor obtained for H=28.

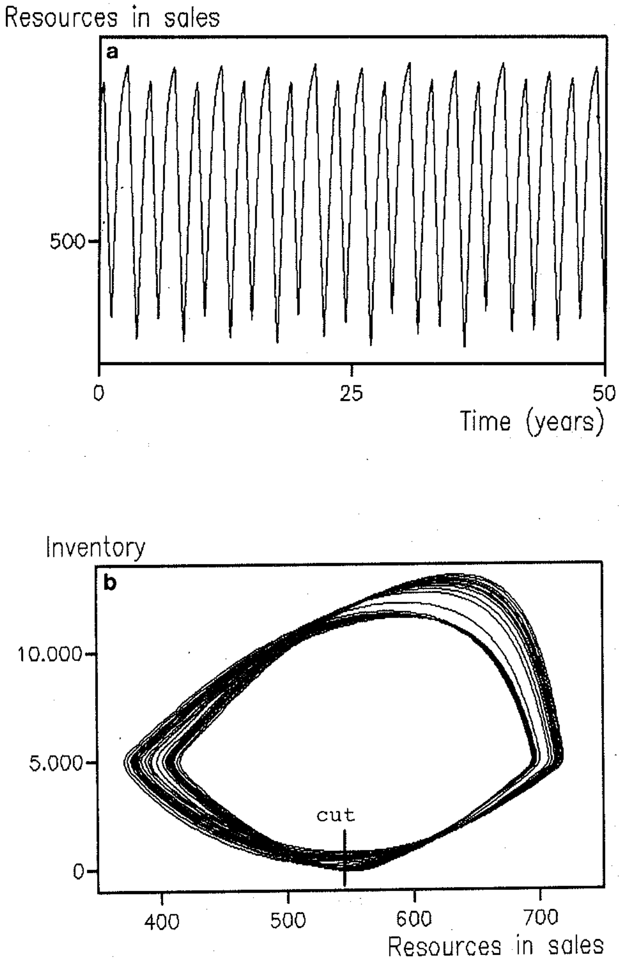

Figure 5. Time- and phase-plot for the chaotic attractor obtained for H=36.

To investigate the chaotic behaviour in more detail we have determined the points of intersection between the attractor and a 5-dimensional plane in phase-space approximately perpendicular to the attractor. Figure 6 shows the results of such a Poincaré section performed in the region where inventory is at minimum (the position of the cutting plane is sketched in figure 5).

In the present case, the Poincaré section has two branches in accordance with the double-band nature of the attractor in figure 5. In a revolution along the attractor, a point on one

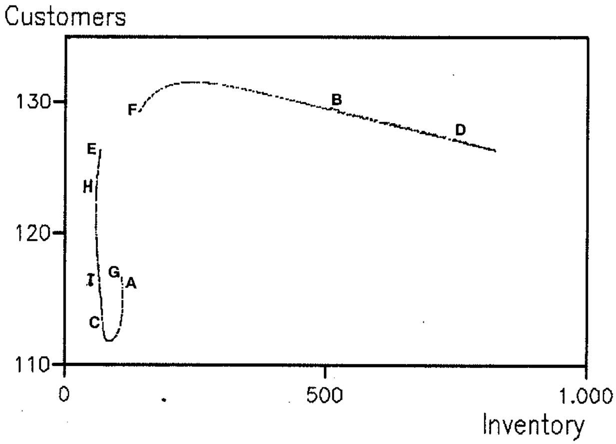

Figure 6. Poincaré section of the chaotic attractor near the point where inventory is at minimum. The Poincaré cut has been projected into the plane determined by inventory and customers.

branch is mapped into a point on the other branch such that, for instance, A → B → I, C → D → H, and E → F → G. While continuously becoming wider, the band thus partly folds back onto itself every second revolution. This expansion and folding process is characteristic of chaotic systems, and a closer look therefore reveals that each branch has a layered substructure resulting from a large number of subsequent foldings.

If this substructure is neglected it is possible to construct a one-dimensional return map for the system. This can be done by approximating the Poincare section by a continuous curve along which we can measure distances. More precisely we have introduced s_n as the distance from point E in figure 6 to the projection of point n in the Poincaré section onto the measuring curve. The measuring curve is parameterized to have a total length of 1.

Figure 7 shows the obtained return map, i.e. the distance s_{n+1} plotted as a function of s_n. The maximum of the return map represents the folding properties of the dynamical system. The division of the return map into two branches again reflects the double-band nature of the attractor.

By constructing a return map as in figure 7, the original six-dimensional nonlinear dynamical system has effectively been represented by a one-dimensional discrete iteration process. From the return map it is possible to derive a number of characteristic features of the dynamical system. In particular, in those cases where the return map has a second order parabolic maximum, the transition to chaos is asymptotically determined by two universal constants α = 2.5029 --- and δ = 4.6692 ---, independent of the nature of the original system (Feigenbaum 1979).

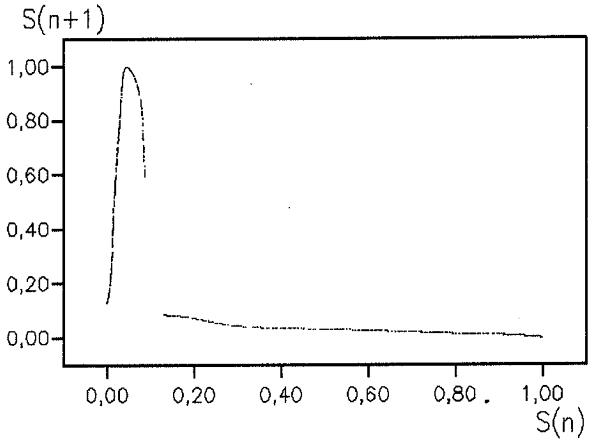

Figure 7. Return map of the considered managerial system for H=36. The maximum does not appear to have a simple, second order parabolic form. Although the universal route to chaos through subsequent period-doubling bifurcations is followed, it is not clear whether the asymptotic approach to chaos for this system will be characterized by the Feigenbaum constants (Mosekilde and Rasmussen, 1986).

CONCLUSION

During certain stages of development, social and biological systems may exhibit damped oscillations in response to external disturbances. In other stages, exponential growth may be dominant, and in yet other stages various forms of strongly nonlinear behaviour can occur. By analyzing internally generated chaotic behaviour in a simple management system, this paper has tried to contribute to a deeper understanding of the complicated forms of behaviour which can occur in strongly nonlinear managerial systems. This extends results obtained in some of our previous publications (Mosekilde et al. 1985, Jensen et al. 1986). We would like to stress, however, that formal modeling of the most important processes in all living systems, i.e. the evolutionary processes in which new structure unfolds through unstable transitions, still remains a practically unexplored area of research.

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