The Application of System Dynamics to the Analysis of GERT Networks

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ABSTRACT

The relationship between system dynamics(SD) 。and other research areas is a subject of universal interest. Attention of the paper is to the possible links between SD and GERT (short for Graphical Evaluation and Review Technique).

A new simulating design for a class of GERT network is proposed and the equivalence of GERT networks to SD models established, thus a new solution to the network obtained.

Accorging to the approach, a GERT network is converted into a SD model in which levels are used to model the random variables associated with the network, such as the expected time to realize a node and the probability that it is realized. The resulting basic model can be used for calculations of any parameter of interest in the analysis of GERT networks.

Its advantages and implications are discussed.

I. INTRODUCTION

System dynamics (SD) is designed primarily for the analyzing and modelling of managerial, organizational, and socioeconomical problems. But now we focus our attention to the relationship, if any, between SD and GERT (abbreviated from Graphical Evaluation and Review Technique) and its applications.

GERT is a network technique evolved out of CPT/PERT, whereas SD is based on feedback control systems principles. However, it is not surprising that GERT is chosen as the partner of SD. First, the basic element of GERT network shown in Fig.1 is similar in composition to that of SD flowgraphs illustrated in Fig.2. Either of them consists of a line with arrow (arc or flow line) and two end points(nodes or levels). Its appeal, most importantly, stems from following considerations: the GERT network with Exclusive-or nodes on their receiving sides(GERT-E network) can be modelled by Semi-Markov processes (SMPs); on the other hand, SD models are Markovian(Sahin,1979) and Markov processes(MPs) are equivalent to a class of SD models(Sahin,1979), therefore, Markov processes may be the medium of establishing the link between GERT and SD. Of course, the variables considered by us are not limited to "transit probability" as in the case of MPs.

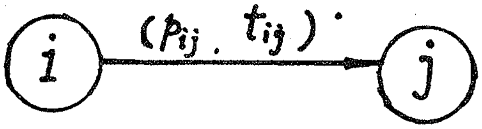

Figure.1 GERT element

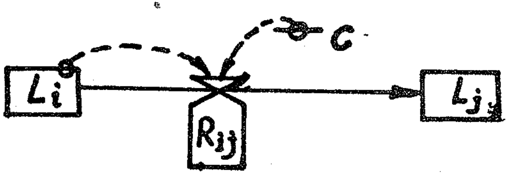

Figure.2 SD element

Implicit motivation of the research is from the wonder that any relationship between SD models and signal flow graphs (SFGs) exist there. It is well known that the type of GERT network mentioned above ,after a transformation of variables, can be viewed as a kind of SFGs. While Professor Jay W. Forrester had borrowed the concepts of SFGs to portray visually underlying cause and effect connections in the systems (Roberts, 1978, PP.5). Obviously, it is desirable to travel the full circle: from SFG graphical representation to the SD flowgraph to the GERT graphical represention and again to SFG.

As a result of the research, an equivalence of GERT to SD is established and a new solution to GERT networks is developed, the name of which is GERT-SD.

The presentation here assumes that the reader has a basic knowledge of GERT. Therefore, the concepts and conclusions concerned with GERT will be cited directly.

II.A BRIEF DESCRIPTION OF GERT

GERT network is a stochastic system characterized by states(nodes) and transitions (occurances of activities) from one state to another. Therefore, which state will be realized after nth transition is a random event. The main objective of analyzing a GERT network is to find the statistic outcomes of key nodes (often terminal nodes): the probabilities that they are realized and the distributions of the time to realize them and so on.

Consider mth route (path or chain) from an origin to a terminal (it may be,in fact, any node of interest) in a GERT-E network and index the nodes on the route 1, 2, ... Nm, S in such a way that 1 is origin, S terminal. According to the theoretical analysis (Sun, 1985) that the probability that S is realized is

$$Ps = \sum_{m} \left(\prod_{i=1}^{Nm} Pij \right) \tag{1}$$

in which, Pij is transition probability of activity i-j; and the expected time to realized the node s obeys normal distribution:

$$Ts \sim N \; (\; Es, \; Ds \;) \tag{2}$$

in which,

$$Es = \sum_{m} \left[\left(\prod_{i=1}^{Nm} Pij \right) \left(\sum_{i=1}^{Nm} Tij \right) \right] \tag{3}$$

and
$$Ds = \sum_{m} \left[\left(\prod_{i=1}^{Nm} Pij \right) \left(\sum_{i=1}^{Nm} \sigma_{ij}^{2} \right) \right] \tag{4}$$

in which Tij and σ_{ij}^{2} are respectively the expected time to complete activity i-j and its variance. It should be noted that equations (2)~(4) are suitable to other parameters such as cost as well as time.

The main idea of GERT-SD is to view a GERT-E network as a Markov process and then model its steady state(according to Markov theory a Markov process will reach a steady state after enough times of transitions) by SD method, thus obtaining the statistic outcomes of the nodes being investigated. The resulting basic SD model includes two sub-ones, i.e. P-model and T-model, which, respectively, model the probabilities of nodes being realized and the expected times experienced when they are realized at those probabilities. The basic model can also be used for the calculations of other parameters and the discussion concerned can be seen in section V. In next section, P/T-models are developed.

III. THE BASIC MODEL OF GERT-SD

A. The mathematical model

Consider a GERT-E network with N nodes and let the probability that node j $(1<j<N)$ is realized at times n be $Pj(n)$ and he expected time experienced, correspondently, be $Tj(n)$. Here ,by the time n we mean the end of the nth transition (activity) of a network system. Thus we have stochastic arrays $(Pj(n))$ and $(Tj(n))$. A simulation method for them will be proposed.

Our basic assumption is that $(Pj(n))$ is the stationary distribution of the markov process $(Xj(n))$, in which $Xj(n)$ is the random variable determined by whether node j is realized at the time n, i.e.

$$Pj(n) = \sum_i Pi(n-1)\cdot Pij \qquad (1\leqslant i,j\leqslant N) \qquad (5)$$

in which Pij is the probability that activity i-j will be realized, given that node i is realized. we shall call it the "transition probability". The initial condition of (5) is

$$\sum_{i=1}^{N} Pj(0) =1 \qquad (6)$$

According to above assumption and the definition of $Tj(n)$, it can be derived that

$$Tj(n) = \sum_{i=1} [Tj(n-1)\cdot Pij + Pi(n-1)\cdot Tij\ Pij] \qquad (1\leqslant i,j\leqslant N) \qquad (7)$$

in which Tij is the expected time required to complete the activity i-j. (7)'s initial condition is

$$Tj(0) = 0 \qquad (1\leqslant j\leqslant N) \qquad (8)$$

(5) and (6) is the mathematical model regarding $(Pj(n))$ and $(Tj(n))$.

From Eq.(5)~(8), the following conclusion can be proved (Sun, 1985) if a terminal node S is chosen as an absorbing node, i.e. let Pss=1, when the network system, as a Markov process, reaches its steady state, we have

$$Ps(+\infty) = Ps \tag{9}$$

and

$$Ts(+\infty) = Ts \tag{10}$$

in which $Ps(+\infty)$ and $Ts(+\infty)$ are respectively the steady values of Ps(n) and Tj(n), and Ps and Ts respectively the probability and the expected time of the node S being realized.

B. The SD Description of the mathematical model: P/T-Models

Since the "transition probabilities" from node i ($1 \leqslant i \leqslant N$) to all its succeeding nodes must satisfy the condition

$$\sum_{j} Pij = 1 \qquad (1 \leqslant i, j \leqslant N) \tag{11}$$

as far as a GERT-E network concerned. In the light of the property, equations (5) and (7) can be changed respectively into the forms

$$Pj(n)=Pj(n-1)+\sum_{i(\neq j)} Pi(n-1)Pij-\sum_{k(\neq j)} Pj(n-1)Pjk \tag{12}$$

and

$$Tj(n)=Tj(n-1)+\sum_{i(\neq j)} Ti(n-1)Pij-\sum_{k(\neq j)} Tj(n-1)Pjk+\sum_{i} Pi(n-1)TijPij \tag{13}$$

In a SD model the general equation of level Lj is

$$Lj.K=Lj.J + DT*[\sum_{i(\neq j)} RINij.JK - \sum_{k(\neq j)} ROUTjk.JK] \tag{14}$$

Comparing (12) and (13) with (14), we set up the counterparts as indicated in Table 1.

TABLE 1

(14)	K	J	Lj.K	Lj.J	RINij.JK		ROUTjk.JK	DT
(12)	n	n-1	Pj(n)	Pj(n-1)	Pi(n-1)Pij		Pj(n-1)Pjk	1
(13)	n	n-1	Tj(n)	Tj(n-1)	Ti(n-1)Pij	Pi(n-1)TijPij	Tj(n-1)Pjk	1

The SD description of Eq.(12) is

$$\text{L} \quad Pj.k=Pj.J+DT*[\sum_{i(\neq j)} PRij.JK-\sum_{k(\neq j)} PRjk.JK] \qquad (15)$$

in which, R equations of PRij and PRjk are

$$\text{R} \quad PRij.KL = Pi.K * Pij \qquad (16)$$

$$\text{R} \quad PRjk.KL = Pj.K * Pjk \qquad (17)$$

Eq.(15) is the general form of level equations of P-model. Eq.(16) and (17) mean that the inflows and outflows of levels are linear functions of the levels from which they originate. DT=1 implys one transition in Markov processes is equivalent to one pace of simulation in SD models.

The SD description of (13) is

$$\text{L} \quad Tj.k=Tj.J + DT*[DRj.JK+\sum_{i(\neq j)} TRij.JK-\sum_{k(\neq j)} TRjk.JK] \qquad (18)$$

in which,R equations of DRj, TRij and TRjk are

$$\text{R} \quad DRj.KL = \sum_{i} Pi(n-1)*Tij*Pij \qquad (19)$$

$$\text{R} \quad TRij.KL= Ti(n-1)*Pij \qquad (20)$$

$$\text{R} \quad TRjk.KL= Tj(n-1)*Pjk \qquad (21)$$

(18)~(21) are the basic equations of T-model. The meaning of DT is the same as in the case of P-model and inflows and outflows are also the linear functions of the levels from which they are released. What is different from (15) is that Eq.(19) has an inflow DRj while there is not such a counterpart in Eq.(15), which is resulting from the fact that it takes time to transit from one node to another. From Eq.(19), DRj is the linear combinations of probability levels of the nodes preceding node j; it is diagramed as an inflow from a source.

C. Conversion Steps

Figure 3~4 gives an example of conversions from GERT to SD. Now let's state the basic rules of convrersions from GERT to SD.

Fig.3 Original

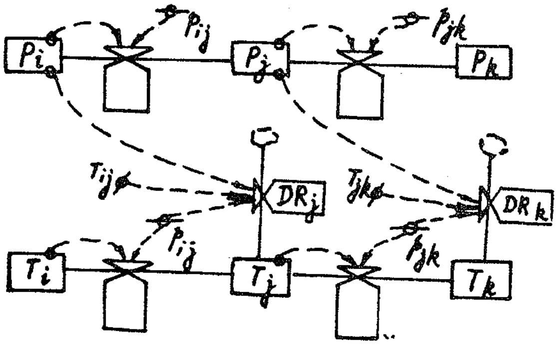

Fig.4 Equivalent

(i) Each node j in a GERT network is correspondent to two levels, Pj and Tj;

(ii) Each receiving branch of node j forms an inflow and each releasing branch forms a outflow; the inflows or outflows are all the linear functions of the levels from which they originate and the proportional constants are the transition probabilities of the correspondent activities; the general forms of them are (16)~(17) and (20)~(21).

(iii) Each receiving branch produces an additional inflow DRj, which is the linear combination of the probability levels of the nodes preceding node j, and the general form of which is (19).

(iv) DT=1. The simulation LENGTH is dependent on the structure of the network under consideration; its determination method will be presented in next section.

IV.THE DETERMINATION OF SIMULATION LENGTH

When feedback loops or self loops exist in a network, the simulation error is inevitable. However, so long as simulation LENGTH is long enough, error can be controlled in the permitted range. The method of determining LENGTH is given as follows.

A. The case of no feedback or self loop

In this case there are finite paths from an origin to any terminal, so it is possible to obtain the exact answer. Suppose that there are r paths and they include N1, N2, ..., Nr activities respectively, then

$$\text{LENGTH} = \text{Nmax} = \text{MAX}\ (\text{N1}, \text{N2}, \ldots, \text{Nr}) \qquad (13)$$

B. The case of feedback or self loop

It is clear that the probability levels of all terminal nodes must be 1 and any probability level of other nodes be 0 in the steady state of the network system being modelled. Thus the sum of probability levels of all non-terminal nodes when stopping simulation is a measure of the system deviating from its steady state. By controlling the deviation, we can obtain the relative accuracy of the outcomes associated with the system. It is owing to the existance of feedback/self loop that the probability levels of some nodes are not zero. Therefore, LENGTH in the case should be Nmax determined by Eq.(13) as if all feedback/self loops did not exist, plus the number of transitions, Nf, required to ensure that the total error resulting from all feedback/self loops is not beyond the given range, i.e.

$$\text{LENGTH} = Nmax + Nf \qquad (14)$$

Suppose there are m feedback nodes(a feedback node is the node in which a feedback or self loop is formed) and the permitted error of probability is Δp, then the average permitted error of each feedback node is

$$\Delta Po = \Delta p/m \qquad (15)$$

Consider a feedback node* i and let the sum of probabilities of its feedback/self branches is Pi, then the condition **to be satisfied is the number of transitions**

$$Ni \geqslant fi\,[\mathrm{Log}_{pi}(\Delta Po) + 1\,] \qquad (16)$$

in which, fi is the maxist of the numbers of branches of feedback loops originated from i, and in the case of self loops, fi = 1. Thus

$$Nf = \max_i(\ Ni\ ) = \max_i(fi\ [\mathrm{Log}_{pi}(\Delta Po)+1] \qquad (17)$$

Now, LENGTH can be determined from Eq.(17), (13) and (14). For the convenience of application, Table 2 gives Ni evaluated at different ΔPo and Pi when fi=1(correspondent to the case of self loops). If the values of ΔPo or Pi faced are not in the list enumerated, Ni should be chosen from the table valves whose ΔPo and Pi are respectively less and greater than the faced ΔPo and Pi.

TABLE 2

(fi=1)

Ni Pi Po	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.01	3	4	4	5	5	6	6	7	7
0.001	4	5	6	6	7	8	9	10	11
0.0001	5	6	7	8	9	10	12	12	15
0.00001	6	8	9	10	11	12	14	16	18

V.OTHER FUNCTIONS OF THE P/T MODELS

The function of P/T-models is not limited to the calculation of Ps and Es. It can be for computions of other parameters.Now let's give a brief introduction.

A.The Variance of Time Distribution

Review (3) and (4) and it is clear that the expression of the variance of time distribution is similar to that of the expected time in structure. If we replace Pij with Pij in P-model and Tij with σ_{ij}^2 in T-model, the steady time level of the terminal node is the variance of the time distribution.

B.Other attibutes

The attibute parameters with activities of a GERT network, except time, may be cost, power and other resources. Since the computing method of these attibutes is the same as that of time, the P/T-models an be used to treat them. The operation concerned is to input the parameter of each activity under consideration to the correspondent variable Tij in T-model and rerun the basic model. The steady level of a terminal is the expected value of that parameter. The above-mentioned method of computing time variance is also suitable to other attibutes.

C. The Expected Number of executions

In the analysis of GERT networks, it is of interest to know the expected number of executions of a given node, branch(activity) or portion of the graph, because the expected number of executions of a given element represents the extent to which the element is critical in the network. Now we will say that the basic model of GERT-SD is also useful in this respect. The operation concerned is to assign 1 to the variable Tij associated with a given branch and 0 to all other Tij in the network if the branch is concerned; or 1 to Tij of the receiving branches of a given node and 0 to all other Tij if the node concerned; or 1 to Tij of branches of a given portion of the network and 0 to all other Tij if the portion of the network concerned, and then run the basic model. As a result, the steady levels of the terminal node output answers.

VI. EVALUATIONS OF GERT-SD AND DISCUSSION

Except the GERT-SD developed in the paper, two types of solution to GERT networks, i.e. the analytic solution and GERT simulation (GERTs) exist. The analytic solution is concerned with too much mathematics, such as Moment Generating Funtion and SFG theory, so that it is not easy for unsophisticated managers to accept. When GERT-SD is used , however, the SD description associated with each node or branch of a network is regular and the regulation is simple in mathematics. Importantly, DYNAMO used by the method is easy to grasp and, in fact, only a part of functions of DYNAMO are concerned. Moreover, sensitivity analysis of parameters becomes easier than the case of analytic method because of the function RERUN of DYNAMO.

Compared with GERTs, GERT-SD method saves much computer time in simulation. This is because GERTs is a random simulation approach based on the time-consuming Monte Carlo techniques, whereas GERT-SD is a deterministic modelling and can output statistic outcomes through a single time of simulation. An example(Sun, 1985) shows that the computer time consumed by GERT-SD is only 1/53 that by GERTs. This advantage is especially clear in sensitivity analysis of parameters.

There is one point to be argued. It seems that the SD flowgraphs of complex networks will become so complicated that the diagraming of the SD flowgraphs is almost impossible. Our argument is that it is not always necessary to a skilled user to illustrated the SD flowfraph because of the regulation of conversion rules of GERT to SD. Another approach to release the point is to compile a general program in line with the principles of GERT-SD and, fortunately,such a program had been written with FORTRAN(Sun,1985).

VII. SUMMARY

GERT and SD are two methodologies evolved out of different philosophies to treat seemingly different classes of problems.

The equivalence between them has inaugurated new and fertile horizons in both fields. GERT-SD will increase the potentialities of GERT due to its advantages over other existing solutions. Conversely, the applications of GERT-SD to practical problems can strengthen the position of SD as a management science tool. Moreover, since a GERT-E network can be transformed into a Signal Flowgraph as indicated in the introduction of the paper, the equivalence of GERT-E to SD makes a similar relationship between SFGs and SD expected.

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