AGGREGATION IN SYSTEM DYNAMICS

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Abstract

Stochastic aspects of systems have generally been ignored in most system dynamics studies except for purposes of sensitivity testing. Yet any model that claims to be more than simply an empirical description of a system must treat the underlying stochasticity explicitly in terms of its contribution to the dynamics. Recent work in chemical, biological and hydrodynamic systems has shown that the aggregation of stochastic effects can lead to novel behavior (self-organization in dissipative systems).

In this paper, an analogy between models of these physical systems and system dynamics models is developed, in which system dynamics models are seen to be an approximation (to lowest order in an expansion in system size) to a stochastic model for the system. The implications of theoretical results derived for the physical system models are evaluated for their application to system dynamics models. A research strategy to elaborate this approach to analyzing systems is proposed.

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Introduction

System dynamics models are based in part on the principle of conservation of flow of materials. A useful technique for the development of a model is to imagine the various pathways that a unit of material can take on its passage through the system. It is then generally assumed that the rates-of-flow along these pathways provide an adequate description of the system's operation if they respect the principle of conservation. However, a unit of material has no well-defined rate-of-flow so that this assumption involves the aggregation of events at a micro-level to provide a description of the system at a macro-level. The transition from one level of description to another has rarely been dealt with explicitly in the system dynamics literature, and then only sketchily (1,2). What is involved is a specification of the source of variability at the micro-level (stochastic effects) and a consistent treatment of these effects to determine their dynamic significance. Other fields of study have undertaken this kind of analysis and have shown that new dynamic behavior may arise in certain critical situations.

Besides the intrinsic interest in clarifying the aggregation process and the possibility of enlarging the range of models that can be used to describe dynamic processes, the analysis of the principles

of aggregation is of more immediate concern in system dynamics since there are a whole range of models currently available which can be classified in a hierarchy of aggregation. One need only consider the sequence: World3, the National Model, the Susquehanna River Basin model, the Urban Dynamics model and the various Industrial Dynamics models-to see that at each level an aggregation of effects and results from lower levels has been accomplished at least implicitly. A long-term goal of the analysis which is undertaken here will be to establish, if possible, methods for performing in a consistent manner the aggregation of micro-level models into macro-level ones.

In this paper we discuss briefly the sources of stochastic effects and their representations in system dynamics models and describe how these effects are related to the aggregation of micro-systems into macro-systems. Methods and results of aggregation analysis from other fields of study will be presented along with an evaluation of their pertinence for system dynamics models. We conclude with a brief description of further work to be done in this area.

Stochastic Effects in System Dynamics Models

The analysis of the flow of individual elements (products, persons, dollars, orders, etc.) through a system localizes the average flows into and out of the relevant levels in a model. At the same time, the analysis indicates the two main sources of stochastic effects which can be represented in the formulation of the rate equations, namely

i) exogenous influences on a decision-stream, for example new or hard-to-classify information sources, or uncontrollable material inflows or outflows, which are added to or subtracted from the average flow rate.

ii) exogenous influences on a policy-structure, for example variations in the weighting of different components of a policy such as the reaction to inventory versus backlog discrepancies, or in the combining of production factors.

These sources of stochasticity can be illustrated in a typical rate formulation for Production Rate PR based on Desired Inventory DI, actual Inventory I and an Inventory Adjustment Time IAT as shown in equation (1):

$$PR = \frac{DI - I}{IAT} + e \qquad (1)$$

The first source is represented by the random variable e which is simply added to the rate indicating a source of variability that is beyond the decision-maker's control. The specification of the random variable as a Gaussian with given mean and standard deviation by use of NORMRN is often used as is a smoothed Gaussian in an attempt to include auto-correlated 'error' effects(3). We will return to this source in the discussion below.

The second source can be represented by considering IAT to be a random variable thereby modeling a case where the application of a policy for production rate is perturbed by the effects of environmental or other exogenous forces so that the gap between desired and actual inventory is closed more, or less rapidly. This source of stochasticity has not been studied in system dynamics in any detail except, after a fashion, in certain sensitivity analyses(4). However in these cases, the stochastic nature was represented by a sample from an ensemble of models. The relationship to the behaviour of a model with variable coefficients requires an ergodic theorem which does not at present exist.

In most system dynamics models these sources of stochasticity are treated by replacing the stochastic variable by its average and ignoring the higher moments in the distributions of system variables. Typically one takes

$$\bar{e} = 0, \quad \overline{IAT} = IATN$$

where the overbar indicates an average over the relevant unconditional distribution. Other models may be based on combinations of these two sources. When modeling the behavior of individual decision-makers taken as a group, each decision-maker makes his decisions based on the information available to him, and both the information and the evaluation of it may vary over the population of decision-makers. A consistent aggregation over such a population would account for the auto-correlation of both sources of stochasticity[1].

Without exaggerating the importance of a proper treatment of stochastic effects and their aggregation, it must be recognized that the choice of level of aggregation and hence the specification of the kind and importance of stochastic effects touches two issues of some interest to model builders and users. The first concerns the realism of the relationships in the model and the generality or extent of the results of the model. More specific treatment of discrete, random events in a model increases the conformity of the model to the system at the expense of making each run of the model less significant, more peculiar. Hence the recent interest in the 'classical', discrete-event simulation literature for means to analyze in global terms the masses of data generated by that approach, an aggregation a posteriori(5). The second issue concerns the possibility of generating new dynamic behavior modes. A number of studies in chemical kinetics(6) and cytology(7) as well as

the well-known results in hydrodynamics and plasma physics(8) show clearly that motions of aggregates may differ in novel ways from the motions of components, even in a deterministic framework. When stochastic elements are joined to a deterministic model, as Allen does in a study of mutation and extinction of populations(9), more new behavior modes, otherwise inaccessible to analysis in a purely deterministic approach, are found. Evidently new behaviour raises the question of new policy, i.e. in what sense or to what extent are policy prescriptions changed by a proper treatment of stochastic effects, either aggregated or disaggregated.

In what follows we will deal with the first source of stochasticity which, in the form shown in equation (1) has come to be known in the physical sciences as the Langevin approach for including stochastic effects. This approach is somewhat ad hoc since no attempt is made to explain the stochasticity in terms of more fundamental mechanisms and hence the precise form of the distribution and the integration of the resulting stochastic differential equation in a rigorous way are open to a certain amount of arbitrariness. Van Kampen has shown that the arbitrariness in the method of integration is relatively unimportant but that the proper inclusion of stochastic effects is crucial(10,11). It is to a summary of a more adequate approach and of the main results from it that we now turn.

The Master Equation

Underlying the system dynamics approach is the idea of state-space models, i.e. that a sufficiently complete description of a system

can be made in terms of the values of certain variables (levels) at any point in time, joined to the relations describing the rates of change of these variables. In a generalization of equation (1) we can write

$$\dot{\underline{x}} = \underline{A}(\underline{x}) + \underline{\underline{B}}(\underline{x})\underline{e}(t) \qquad (2)$$

where $\underline{A}$ is a nonlinear vector function of the state vector $\underline{x}$, $\underline{\underline{B}}$ is a (possibly nonlinear) matrix function of $\underline{x}$, and $\underline{e}$ is a stochastic variable, usually taken to be Gaussian white noise[2]. As a representation of stochastic effects we interpret the term in $\underline{e}$ as giving little pulses to the value of $\underline{x}$. An alternative representation is in terms of differentials

$$d\underline{x} = \underline{A}(\underline{x}) + \underline{\underline{B}}(\underline{x})d\underline{w} \qquad (3)$$

If $\underline{\underline{B}}$ does not depend on $\underline{x}$, $\underline{\underline{B}}(\underline{x})=\underline{\underline{B}}_o$, equation (2) with a fixed initial value determines a unique stochastic process $\underline{x}(t)$. This is a Markov process whose transition probability, $P(\underline{x},t/\underline{x}_o,t_o)d\underline{x}$, from value $\underline{x}_o$ at t_o to the interval $\underline{x},\underline{x}+d\underline{x}$ at t obeys the Fokker-planck equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \underline{x}}\underline{A}(\underline{x})P + \frac{\underline{\underline{B}}}{2}\frac{\partial^2 P}{\partial \underline{x}\,\partial \underline{x}} \qquad (4)$$

The solution of this equation is a Gaussian distribution whose peak moves in the space $\underline{x}$ according to the solution for the deterministic (so-called 'macroscopic'[3]) equations and whose variance is determined by $\underline{\underline{B}}$. In this case, and this case only, the usual treatment of type (i) stochasticity in system dynamics models (whether linear or nonlinear) is justified. It is evident that the assumption that $\underline{\underline{B}}$ is independent of $\underline{x}$ is highly restrictive and unlikely to be true in real systems since it means that stochastic effects are of constant magnitude no matter the magnitude of $\underline{x}$[4].

Van Kampen(10) remarks that, in the general case that the influence of stochastic effects depends on the value of the state variable,

'equation' (3) as it stands makes no sense as a definition of a stochastic process, $\underline{x}$, since the jumps or pulses due to the stochastic process change the value of the state which simultaneously determines the size and direction of the pulses. To resolve the ambiguity, an appeal to a micro-level description of the source of stochasticity is made and two cases are distinguished:

i) The stochasticity is external to the system in that it can in principle be removed or 'turned off';

ii) The stochasticity is internal or intrinsic to the system in that it is due to the interactions of elements in the system and cannot be removed even in principle.

In the case of external stochastic sources which are connected to the system by the matrix $\underline{\underline{B}}(\underline{x})$, it is possible to define a process for which the auto-correlation time t_c is non-zero (this makes the process non-Markovian) and "take the limit" $t_c \longrightarrow 0$ to determine the appropriate interpretation of equation (2)[5]. If we can assume that the motion of the system is described by

$$\dot{\underline{x}} = \underline{A}(\underline{x})$$

when the noise is turned off, the first term in the series corresponds to equation (2) with the value of $\underline{\underline{B}}(\underline{x})$ determined as one-half the sum of the values of $\underline{\underline{B}}$ before and after the 'jump' imposed by the stochastic impulse. This is the so-called Stratonovich interpretation, or approach to the integration of stochastic differential equations[6].

In the case of internal sources of stochasticity, no precise definition of a deterministic equation (5) can be determined and the stochastic process $\underline{x}$ must be analyzed as such. If we can assume that the process $\underline{x}$ is Markovian, then we can establish an equation for the

probability distribution of $\underline{x}$, $P(\underline{x},t)$ which has the general form

$$\frac{\partial P}{\partial t} = \int [W(\underline{x}/\underline{x}')P(\underline{x}',t) - W(\underline{x}'/\underline{x})P(\underline{x},t)]d\underline{x}' \qquad (6)$$

This is the so-called Master Equation describing the 'gain' and 'loss' of probabilty P in terms of the transition probability function $W(\underline{x}/\underline{x}')$ such that

$W(\underline{x}/\underline{x}')\Delta t$ = probability that the system moves from $\underline{x}'$ to $\underline{x}$ in the time interval $t, t+\Delta t$

All of the dynamics of the system is contained in the specification of W (which is just a'local'description of P). For many systems W and P are discrete distributions in $\underline{x}$ so that the integral in (6) reduces to a (generally finite) sum of terms.

For example, consider a stock, x_1 which receives units at an average rate of a units per time period (a is constant), and from which deliveries are made at an average rate $a_{11}x_1$ per t.p. In a stochastic model we identify elementary transition events between states x_1' and x_1 (addition of one unit, delivery of one unit, no change in the number of units). Then we determine the probability of occurrence of each event, here taken to be equal to the average rate of change (inflow, a, or outflow, $a_{11}x_1'$) times a time increment Δt which is short enough that <u>at most</u> one transition event can occur. Taking the events and their probabilities as shown in Table 1, we can establish an expression for $P(x_1,t)$ in terms of P at time $t - \Delta t$(the Markovian assumption) and then by standard arguments(12) we derive an equation for P equivalent to equation (6) namely

$$\frac{\partial P}{\partial t} = aP(x_1-1,t) - (a+a_{11}x_1)P(x_1,t) + a_{11}(x_1+1)P(x_1+1,t) \qquad (7)$$

State	Event	Probability
x_1-1	$x_1-1 \to x_1$	$a\,\Delta t(1-a_{11}x_1\Delta t)P(x_1-1,t-\Delta t)$
x_1	$x_1 \to x_1$	$(1-a\,\Delta t)(1-a_{11}x_1\Delta t)P(x_1,t-\Delta t)$
x_1+1	$x_1+1 \to x_1$	$(1-a\,\Delta t)a_{11}x_1\Delta t P(x_1+1,t-\Delta t)$

Table 1: Elementary transition events and probabilities for the one-level stock model

Introducing the generating function

$$f(s,t) = \sum_{x_1=0}^{\infty} s^{x_1} P(x_1,t) \quad , \; |s| \leq 1$$

we can transform (7) to an equation for f(s,t):

$$\frac{\partial f}{\partial t} = -a_{11}(s-1)\frac{\partial f}{\partial s} + a(s-1)f \qquad (8)$$

At steady state,

$$(s-1)\left[\frac{\partial f}{\partial s} - \frac{a}{a_{11}}.f\right] = 0 \qquad (9)$$

so,that

$$f_{ss}(s) = f(0)\exp(as/a_{11})$$

and since

$$f(1,t) = \sum_{x_1=0}^{\infty} P(x_1,t) = 1$$

we have

$$f_{ss}(s) = \exp(a(s-1)/a_{11})$$

so that

$$P_{ss}(x_1,t) = \frac{(a/a_{11})^{x_1}\exp(-a/a_{11})}{x_1!} \qquad (10)$$

The corresponding deterministic model is simply

$$\dot{x}_1 = a - a_{11}x_1 \qquad (11)$$

so that the steady-state distribution of the number of units in stock, x_1, given by (10) is seen to be Poisson with a peak at the average value a/a_{11} which is the steady-state solution of the deterministic (or'macroscopic') model. This result and its derivation and generalization to

more complex linear systems are classical results in the chemical kinetics literature(13,14) and their only interest here is to exemplify a new way of looking at system dynamics models[8].

The real power of this approach derives from more recent results for nonlinear systems. A formal expansion of the Master Equation (6) is performed in a consistent manner, recognizing that stochastic effects may be 'small' by introducing a scale parameter $\varepsilon = \Omega^{-1}$, where Ω is a measure of the size of the system relative to the stochastic fluctuations (11,15,16)(for example, if the total number of units in a stock is large compared to the variations in order quantities). The 'local' nature of fluctuations is represented in the Master Equation by rewriting (6) as

$$\frac{\partial P}{\partial t} = \int \left[W(\underline{X} - \Delta\underline{X}, \Delta\underline{X}) P(\underline{X} - \Delta\underline{X}, t) - W(\underline{X}, \Delta\underline{X}) P(\underline{X}, t) \right] d\Delta\underline{X} \qquad (12)$$

Introducing scaled variables by means of the relations

$$\begin{aligned} \underline{X} &= \Omega(\underline{y} + \Omega^{-1/2}\underline{x}) \\ \Omega^d P(\underline{X}, t) &= \Psi(\underline{y} + \Omega^{-1/2}\underline{x}, t) \quad [d = \text{dimension of state-space}] \\ p(\underline{x}, t) &= \Omega^{-1/2}\Psi \\ W(\underline{X}, \Delta\underline{X}) &= \Omega w(\underline{x}, \Delta\underline{X}) \end{aligned} \qquad (13)$$

we can derive the expansion(15):

$$\frac{\partial p(\underline{x}, t)}{\partial t} - \Omega^{1/2}\dot{\underline{Y}} \frac{\partial p}{\partial \underline{x}} = \Omega \sum_{n=1}^{\infty} \Omega^{-n/2} \frac{1}{n!} \left(-\frac{\partial}{\partial \underline{x}} \right)^n c_n(\underline{y} + \Omega^{-1/2}\underline{x}) p(\underline{x}, t) \qquad (14)$$

where $c_n(\underline{z}) = \int d\Delta\underline{X} w(\underline{z}, \Delta\underline{X}) (\Delta\underline{X})^n$ (15)

By choosing the function $\underline{y}$(which is so far undetermined) so that terms of order $\Omega^{1/2}$ are cancelled we get

$$\dot{\underline{Y}} = c_1(\underline{y}) \qquad (16)$$

which is the generalization to nonlinear systems of the deterministic

equation, here representing the motion along the most probable path. Thus it is only in the limit $\Omega \to \infty$ that the motion of the system can be adequately represented by a (possibly nonlinear) state-space model. It is evident from equation (15) that knowledge of a model for the most probable path is not sufficient to specify the stochastic dynamics ($w(\underline{z}, \Delta\underline{X})$) uniquely[9]. However, the expansion does allow us to see how new dynamic behavior may arise by providing an expression for the distribution of fluctuations about the most probable path and conditioned by it. To the next order in Ω(i.e. Ω^0), we have the Fokker-Planck equation for $p(\underline{x}, t)$:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \underline{x}} K(\underline{y}) \underline{x} p(\underline{x}, t) + \frac{1}{2} \frac{\partial}{\partial \underline{x}} D(\underline{y}) \frac{\partial}{\partial \underline{x}} p(\underline{x}, t) \qquad (17)$$

where $K_{ij}(\underline{y}) = \frac{\partial}{\partial y_j} c_1^i(\underline{y})$

$$D_{ij}(\underline{y}) = c_2^{ij}(\underline{y}) \qquad (18)$$

which give the drift regression and diffusion of the distribution of fluctuations in state-space. From this approximate Master Equation we can derive equations for the motion of a fluctuation $\delta\underline{Y}$

$$\frac{d}{dt} \delta\underline{Y} = \underline{\underline{K}}(\underline{y}) \delta\underline{Y} \quad , \delta\underline{Y}(0) \neq \underline{0} \qquad (19)$$

and for the first and second moments of the fluctuation distribution

$$\underline{\mu} = \int d\underline{x} p(\underline{x}, t) \underline{x} \quad ; \quad \underline{\underline{\sigma}} = \int d\underline{x} p(\underline{x}, t) \underline{x}\, \underline{x}^T$$

namely $\frac{d}{dt} \underline{\mu} = \underline{\underline{K}}(\underline{y}) \underline{\mu}$

$$\frac{d}{dt} \underline{\underline{\sigma}} = \underline{\underline{K}}\, \underline{\underline{\sigma}} + (\underline{\underline{K}}\, \underline{\underline{\sigma}})^T + \underline{\underline{D}} \qquad (20)$$

In equation (20), $\underline{\mu}(0) = \underline{0}$ in our approximation, however as equation (19) shows, if there is a fluctuation then it evolves according to $\underline{\underline{K}}(\underline{y})$[10]. In particular, equation (19) may show limit-cycle behavior and equation

(17) including diffusion terms may show development of 'state-spatial' structures due to a balancing of diffusion ($\underline{D}$) and convection ($\underline{K}$) terms. Both sorts of behavior have been seen in physical systems and studied by the Brussels group (Prigogine, Nicolis and colleagues) under the general heading of "dissipative structures"[11].

One important result of this work has been to show that the variance of the fluctuations ($\underline{\sigma}$) may diverge and that relaxation to the steady-state may be slower than exponential. Both effects depend on the behavior of eigenvalues of $\underline{K}$ near critical points and are not due to the stochasticity(25). More recently several authors have proposed similar stochastic models for the diffusion of technology(17) and the extension of the principle of balancing 'convective' and 'diffusive' behavior to establish a hierarchy of models of large systems(18). These remain to be investigated in more detail.

The above discussion has shown that a proper treatment of stochastic effects can lead to

i) deterministic system dynamics models for the most probable evolution of the state variables as the lowest order approximation in the limit of large system size, equation (16),

ii) equations for the evolution of fluctuations around the most probable state, equation (19),

iii) equations for dissipative structures capable of generating new behavior not contained in the deterministic model, by means of equation (17).

The Master Equation-Reservations

The previous results must be treated with the usual caution due any modeling effort. Just as there is no unique state-space model for

a system, there is no unique stochastic model for a system. The stochastic argument cannot select which model is best in a given situation, however it can occasionally provide a means to determine from observations and analysis the values of some parameters and the structural form of relationships(23).

A more serious reservation is the justification for the Markovian assumption. The most general form of the Master Equation is non-Markovian for reasons suggested in the discussion of external and internal sources of stochasticity, namely finite autocorrelation times for real processes(19). However, aggregation (or equivalently, smoothing-in-time in many systems) permits the use of the Markovian assumption for the analysis of micro-level behaviour whose time-scale is longer than the autocorrelation time of 'sub-micro'-level events. Thus the Markovian assumption is valid for analyzing interactions of people as consumers in the expression of their utility functions but not the processes of formation of these utility functions.

A final reservation about the stochastic approach concerns the possible inconsistencies that may be implicit in the final model when written in the Master Equation form (6). An example is provided by the work of Malek-Mansour and Nicolis(20) who analyzed a simple nonlinear chemical reaction for one intermediate reactant (one state variable, X). They showed that the steady-state solution of the Master Equation was the trivial one X=0, i.e. extinction of the reactant (and the reaction). This solution corresponds to the (unstable) steady-state solution of the macroscopic equations. Yet the Master Equation for the same system near equilibrium (i.e. thermodynamic equilibrium) admits a Poisson distribution as the unique steady-state solution, with mean value given

by the (asymptotically stable) steady-state solution of the macroscopic equation. The problem arises from trying to apply the Master Equation (6) in conditions far from thermodynamic equilibrium where the expansion in equation (17) breaks down. Related difficultiess were shown in a similar case(6) to lead to fluctuations that did not agree with thermodynamic principles. The solution is to use the consistent expansion (17) and recognize explicitly the possibility that in critical regions new behavior may arise that, at a macroscopic level is not consistent with thermodynamic principles, while remaining so at a microscopic level. That is, macroscopic fluctuations need not obey the same entropy constraints as microscopic fluctuations in a system far from equilibrium.

The interest of this argumaent for system dynamics models is to suggest that principles of optimization on a micro-level may be broken on a macro-level and that it is the aggregation of stochastic effects which may permit this 'symmetry-breaking' behavior to show up. This remains a fruitful area for further research in view of the history of development of such aggregate models as the National Model and Coal2.

Other Aggregation Results

The previous discussion was restricted to models in which stochastic effects only of type (i) were considered. A few results exist for the case of type (ii) stochasticity, i.e. when the parameters in the model are stochastic. In a long paper on the n-species Lotka-Volterra equation(21), Goel et al. develop some general results for cases in which the parameters of the interaction between species (the quadratic terms) are random variables with finite or infinitesimal correlation

times. They find that in the first case, the steady-state population distribution is Poisson while in the second case it is not. This result is derived by assuming a Master Equation of Fokker-Planck form and estimating the drift and diffusion coefficients directly. As van Kampen has shown, this procedure is inconsistent in general but the derivation of a consistent result (i.e. the specification of the transition probability, w) remains to be done[12].

In a different vein, a brief paper by Athans et al.(22) discussed the control of a simple system (in discrete time)

$$x(t+1) = a(t)x(t) + b(t)u(t) \qquad (21)$$

in which a and b were assumed to be constants or white noise processes. They found that in the case of $b=\bar{b}$, the Kalman filter factor (used to calculate the cost of control) depends only on the variance of a and hence, even if the mean value of a is less than 1, optimal control over an infinite horizon could be impossible. As the authors remark, this feature is related to stochastic stability with state-dependent noise, i.e. it is related to the proper treatment of stochastic effects due to internal sources as mentioned above[13].

Conclusions

This brief sketch of some aspects of stochastic effects in system dynamics is intended to indicate some of the main results developed in other fields for problems similar to those arising naturally in system dynamics models, namely the proper treatment of stochasticity in the aggregate in dynamic, nonlinear systems. Although these issues have received little attention in the past from the system dynamics community, it is appropriate at this conference to propose a research program that

could build on previous work in system dynamics in an effort to generalize and deepen the foundations of the field.

One of the positive attributes of system dynamics is the large variety of models developed to study specific problems. As shown above, it should be possible to derive from these models more general stochastic models which could be studied with a view to establishing more rigorous methods of estimating parameters(23), better understanding of the regions within which some aggregate models are valid(10), and better understanding of the generation of new behavior modes(6). To this end a research strategy would be to identify a number of generic structures gleaned from the experience of the past twenty years and to analyze these structures in depth using the methods sketched here. A preliminary list of candidate models would include:

i) first-order delay (27)
ii) third-order delay (27)
iii) logistic curve or 'S-shaped' growth (27)
iv) two-species Lotka-Volterra model (21)
v) three-species Lotka-Volterra model (21)
vi) two-level Inventory-Workforce model
vii) three-level commodity cycle model (27)
viii) Mass' Inventory-Workforce model (3)
ix) two-level 'World' model

The objective of the analyses would be to give some rigor to the concept of generic structures and to develop some principles for the use of generic structures in different situations, for the aggregation of generic structures and for the analysis of complex systems. Such a strategy, while somewhat abstract compared to the historical strategy of emphasizing model development, could provide a theoretical underpinning unique to system dynamics.

Notes

1. It is a common error to include stochastic effects of type (i) by adding a noise term independent of DT to a rate equation. In the implied limit DT$\rightarrow$0 this procedure is equivalent to using a noise source of zero power. As shown in the text, for certain cases this error is not important to the average or most probable motion.

2. Gaussian white noise refers to the stochastic process e(t) with

   i) $E[e(t)] = 0$
   ii) $E[e(t_1)e(t_2)] = \delta(t_1-t_2)$
   iii) higher moments are given by the rules of Gaussian processes.

   Odd moments are identically zero and even moments are sums of products of terms like (ii)(24).

3. The terminology 'macroscopic' comes from the chemical kinetic literature on the subject (see References 6, 7, 10) in which deterministic equations for chemical reaction rates are assumed to be based on microscopic, inter-molecular reaction equations whose stochastic effects are represented by a Langevin-type term.

4. This is the same as the homoscedasticity assumption in estimation theory.

5. Since the result of this procedure is an asymptotic series with parameter $//\underline{B}//t_c << 1$, the limit does not exist.

6. It is interesting to note that DYNAMO integrates stochastic differential equations according to the (non-physical) Itô rule in which the value of $\underline{B}(\underline{x})$ is that value just before the occurrence of the stochastic pulse. The difference between the two rules is non-trivial conceptually and is the subject of a future Research Note.

7. This 'definition' of W is valid to order $o(\Delta t)$.

8. In Reference 13, it is shown that the generating function at any time is $f(s,t)=\exp(x_1(t)(s-1))$ where $x_1(t)$ is the solution to equation (11) so that the distribution $P(x_1,t)$ is Poisson and its peak (and average

value) moves like $x_1(t)$. Thus in this case we recapture the result based on the Langevin approach where the stochastic input is independent of x_1. In general, the solution differs from the Langevin result, contrary to Portnow and Kitahara's comment(25). These authors cast the asymptotic expansion solution into a Langevin-type equation but fail to specify the necessary interpretation of the resulting stochastic differential equation.

9. We cannot simply add a term to equation (16) as in the Langevin approach since no matter how the resulting stochastic differential equation is integrated, the fluctuations contribute terms of order Ω^{-1} to the 'macroscopic' motion. The higher order fluctuations cannot be modeled by a Langevin approach, and there is no need to since the Master Equation contains the full stochastic model(10).

10. Thus control of fluctuations to order Ω^0 depends on the sensitivity matrix $\underline{K}$ as does estimation, by duality. Note that no assumption has been made about the distribution $w(\underline{x}, \Delta\underline{X})$.

11. We note that 'dissipative structures' arise in critical regions of state space where the expansion breaks down. However existence and location of these regions can be studied by means of equation (17). Recent work by Malek-Mansour and Nicolis has extended the analysis of the critical region by use of renormalization group methods(26).

12. We remark that the case of finite autocorrelation time makes the problem non-Markovian.

13. Note that the 'uncertainty threshold' is due to the fact that the stochasticity is internal since the control u is taken proportional to x. Also note that the problem arises in a discrete-time model and hence has nothing to do with the formal question of how to integrate a stochastic differential equation.

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