Online Appendix to Accompany the Paper:

MAKING THE NUMBERS?

"SHORT TERMISM" & THE PUZZLE OF ONLY OCCASIONAL DISASTER

S1- The finite horizon optimal allocation

The optimal control problem for maximizing expected revenue in our problem can be written as maximizing the expected revenue subject to the dynamics of the system and the budget constraint:

$$Max \int_{t=0}^{t=T} E(R)\, dt$$

Subject to:

$$\frac{dC}{dt} = e_C \rho - \frac{C}{\tau}$$

$$C(0) = C_0$$

$$\alpha + \beta = 1$$

$$0 \leq u \leq 1$$

To solve this problem we can set up the present value Hamiltonian with the co-state variable λ which represents the shadow price of the capability at any point in time. Noting that environmental shock to the revenue, S, has a mean of zero and is not correlated with the other component of revenue and is zero-mean, we have $E(R) = C^{\alpha} e_R^{\beta}$, and thus we find a rather simple Hamiltonian function:

$$H(C, u, t) = C^{\alpha} e_R^{\beta} + \lambda \left(e_C \rho - \frac{C}{\tau} \right)$$

The necessary conditions for finding the optimal allocation policy, u, is:

$$\frac{\partial H}{\partial u} = 0$$

$$\frac{\partial H}{\partial C} = -\frac{d\lambda}{dt}$$

$$\lambda(T)C(T) = 0$$

These conditions are also sufficient because the Hamiltonian is concave with respect to u and C for feasible values of C and u. Solving the first constraint we find the following optimal allocation fraction:

$$u_{Dyn} = Min\left(1, \left(\frac{\beta}{\lambda h\rho}\right)^{\frac{1}{1-\beta}} C\right)$$

After replacing the optimum allocation in the second condition, the dynamics of co-state variable is described by the following differential equation:

$$\frac{d\lambda}{dt} = \frac{\lambda}{\tau} - (1-\beta)\left(\frac{\beta}{\lambda h\rho}\right)^{\frac{\beta}{1-\beta}}$$

Solving this differential equation (using Bernoulli method) we get the following time trajectory for the shadow price of capability, λ:

$$\lambda(t) = \left(\frac{\tau(1-\beta)}{\left(\frac{h\rho}{\beta}\right)^{\frac{\beta}{1-\beta}}} + Ke^{\frac{t}{\tau(1-\beta)}}\right)^{1-\beta}$$

Using the end state condition (λ(T)=0) we can solve for the constant K which gives the analytical expression for the time trajectory of λ for any time horizon and combination of parameters.

$$K = \frac{-\tau(1-\beta)}{\left(\frac{h\rho}{\beta}\right)^{\frac{\beta}{1-\beta}} e^{\frac{T}{\tau(1-\beta)}}}$$

Inspecting the results, we note that this constant term is negative, and very small as long as T>τ(1-β), that is, the time to end of horizon is appreciably smaller than the time constant for the erosion of capability. Therefore λ is almost constant until we get fairly close (relative to τ) to the end of investment horizon (T), at which time the shadow price starts to decline precipitously (note the exponential term in equation for λ), leading to increasing allocation of resources to revenue generation and a decline of capability, until at exactly time T the shadow price and capability stocks both become zero.

Therefore, assuming $T>\tau(1-\beta)$, we can find a constant shadow price of capability that applies for a large section of our time horizon:

$$\frac{d\lambda}{dt} = 0 \Rightarrow \lambda = \left(\frac{\tau(1-\beta)}{\left(\frac{h\rho}{\beta}\right)^{\frac{\beta}{1-\beta}}} \right)^{1-\beta}$$

Replacing λ with this steady state value in equation for u_{Dyn} and simplifying the equations we get the following expression for the approximate optimal control allocation:

$$u = Min\left(1, u^* \frac{C}{C^*}\right)$$

This simple expression suggests that optimal allocation in the dynamics case is 1) consistent with the steady state allocation, that is, if capability is at the steady-state optimal level, the allocation will be the same as steady state. 2) Variations of the optimal path in capability are compensated for by linear shifts in the allocation fraction: when capability falls short of the optimal steady state value, the allocation favors capability investment, while too much capability (relative to steady state) will lead to more effort (than steady state optimal) being allocated to revenue generation. Note that the heuristic used in our paper simplifies to this function when $\gamma = 0$.

S2-The infinite horizon optimal allocation

The infinite horizon optimal control problem with discounted revenue can be written as:

$$Max \int_{t=0}^{t=\infty} e^{-rt} E(R)\, dt$$

Subject to:

$$\frac{dC}{dt} = e_C \rho - \frac{C}{\tau}$$

$$C(0) = C_0$$

$$\alpha + \beta = 1$$

$$0 \le u \le 1$$

Here *r* is the continuous time discount rate. To solve this problem we set up the current value Hamiltonian with the transformed co-state variable ψ and follow the regular steps:

$$H(C, u, t) = C^{\alpha} e_R^{\beta} + \psi \left(e_C \rho - \frac{C}{\tau} \right)$$

To find the optimal allocation policy, u, we solve the following equations:

$$\frac{\partial H}{\partial u} = 0$$

$$\frac{\partial H}{\partial C} = -\frac{d\psi}{dt} + r\psi$$

$$\lim_{T \to \infty} e^{-rt} \lambda(T) \ge 0, \lim_{T \to \infty} e^{-rt} \psi(T) C(T) = 0$$

Solving the first constraint, we find the following optimal allocation fraction which is similar to the finite horizon case:

$$u_{Dyn} = Min \left(1, \left(\frac{\beta}{\psi h \rho} \right)^{\frac{1}{1-\beta}} C \right)$$

After replacing the optimum allocation in the second condition, the dynamics of co-state variable is described by the following equation:

$$\frac{d\psi}{dt} = \psi(r + \frac{1}{\tau}) - (1 - \beta) \left(\frac{\beta}{\psi h \rho} \right)^{\frac{\beta}{1-\beta}}$$

In this case, we observe that the equilibrium ψ value satisfies the terminal conditions, and thus provides the following solutions for the optimal co-state trajectory and allocation:

$$\frac{d\psi}{dt} = 0 \Rightarrow \psi = \left(\frac{\tau(1-\beta)}{(1+r\tau)\left(\frac{h\rho}{\beta}\right)^{\frac{\beta}{1-\beta}}} \right)^{1-\beta}$$

$$u = Min\left(1, u^* \frac{C}{C^*}(1+r\tau)\right)$$

In the infinite horizon discounted case the optimal allocation differs from the finite horizon, undiscounted, case with a factor of (1+rτ): if capabilities are slow to erode (large τ) and if discount rate is high, the baseline allocation favors revenue generation beyond the steady state optimal allocation. Moreover, the infinite horizon case does not include the precipitous decline in the value of capability at the end of time horizon (because there is no end to the time horizon).

S3- The effort allocation function and characteristic of the resulting phase diagram

Variables and model definition (reproduced from the paper)

Revenue Function:	$R = C^{\alpha} e_R^{\beta}(1+S)$
Allocated effort to revenue	$e_R = uh$
Allocated effort to capability	$e_C = (1-u)h$
System's dynamics	$\frac{dC}{dt} = e_C\rho - \frac{C}{\tau}$
Optimal steady state allocation policy	$u^* = \frac{\beta}{\alpha+\beta}$
Allocation heuristic used in this study	$u = Min\left(1, u^*\left(\frac{R^T}{R_{u^*}}\right)^{\gamma}\left(\frac{C}{C^*}\right)^{1-\gamma\beta}\right)$
Target revenue	$R^T = {C^*}^{\alpha} {e_R^*}^{\beta}$

Expected revenue using optimal steady state policy $R_{u^*} = C^{\alpha} e_R^{*\,\beta}(1+S)$

Effort to revenue under optimal steady state policy $e_R^* = u^* h$

Capability using optimal steady state policy $C^* = (1-u^*)h\rho\tau$

Throughout the rest of the document it is assumed that we are using a constant return to scale production function (α+β=1).

Phase diagram

The phase diagram for the system reflects the changes in capability (dC/dt) as a function of capability. Specifically, replacing the equation for allocation into the system's dynamics, we get:

$$\frac{dC}{dt} = (1-u)h\rho - \frac{C}{\tau} = \left(1 - Min\left(1, u^*\left(\frac{R^T}{R_{u^*}}\right)^{\gamma}\left(\frac{C}{C^*}\right)^{1-\gamma\beta}\right)\right)h\rho - \frac{C}{\tau}$$

after replacement and simplification, we get:

$$\dot{C} = \frac{dC}{dt} = h\rho\left(1 - Min\left(1, u^*\left(\frac{C}{C^*}\right)^{1-\gamma}\frac{1}{(1+S)^{\gamma}}\right)\right) - \frac{C}{\tau}$$

The allocation function thus has one adjustment factor that responds to the capability level, and another that responds to environmental shocks. The latter component reduces variability in response to the environmental shocks and the former either smoothes revenue (γ>1) or fixes capability shortfalls (γ<1) in response to deviations of capability. The response to capability level is thus the result of two competing forces, one which attempts to align the capability level with the optimal trajectory based on the optimal control policy, and another which compensates for falling capability by increasing allocation to revenue generation, thus smoothing the revenue trajectory. These forces are at balance when γ=1, capability renewal tendencies win for smaller γ and revenue smoothing dominates for $\gamma \geq 1$.

For simplifying the analysis of the system, we focus on the deterministic version of the equation, where the impact of environmental noise is excluded from calculations of capability change:

$$\dot{C} = h\rho\left(1 - Min\left(1, u^*\left(\frac{C}{C^*}\right)^{1-\gamma}\right)\right) - \frac{C}{\tau}$$

By equating this equation to zero, we find that it always has a fixed point at C^* where capability equals the optimal steady state capability. The existence and number of other fixed points depends on γ.

1) $\gamma \leq 1$:

For $0 \leq C \leq C^*$ we have:

$$\dot{C} = h\rho\left(1 - u^*\left(\frac{C}{C^*}\right)^{1-\gamma}\right) - \frac{C}{\tau} \geq h\rho(1 - u^*) - \frac{C^*}{\tau}$$

Yet the right hand side of inequality is by definition zero, so $\dot{C} \geq 0$.

Using a similar argument, it is easy to see that for $C \geq C^*$ the net rate of change in capability is always negative. Therefore for $\gamma \leq 1$ the phase diagram includes a single equilibrium at C^* and no other fixed points, the system will always move back towards this equilibrium.

2) $\gamma > 1$:

Calling $C^*\left(\frac{1}{u^*}\right)^{\frac{1}{1-\gamma}} = C^s$, the net flow equation has two ranges:

$$\dot{C} = -\frac{C}{\tau} \quad if \quad C < C^s$$
$$\dot{C} = h\rho\left(1 - u^*\left(\frac{C}{C^*}\right)^{1-\gamma}\right) - \frac{C}{\tau} \quad if \quad C \geq C^s$$

Therefore at least one additional fixed point exists at *C=0*. We now focus on the behavior of $\dot{C}$ when $C \geq C^s$.

First, we observe that $\dot{C}$ is a continuous function in C that at C^s takes the $-\frac{C}{\tau}$ value and at C^* is zero. The extremum for $\dot{C}$ can be found by equating its derivative with respect to C to zero which provides the following unique solution:

$$\frac{\partial \dot{C}}{\partial C} = 0 \Rightarrow C = \left((\gamma - 1)h\rho\tau C^{*\gamma-1}\right)^{\frac{1}{\gamma}}$$

On the other hand:

$$\frac{\partial^2 \dot{C}}{\partial C^2} = -(\gamma - 1)\gamma r u^* C^{*\gamma-1} C^{-\gamma-1}$$

All the terms in the equation are positive, except for the one negative sign, therefore the second derivative of capability flow with respect to capability is always negative in this region. As a result, the extremum found above is the only maximum for the net capability flow function which should be above zero (given that $\dot{C} = 0$ at C^*) and thus there is a single other point at which $\dot{C} = 0$. This point can be found numerically[1] by solving the rate-level equation for zero capability change rate. Given the positive first derivative of $\dot{C}$ with respect to C at this point, it is also the only tipping point for the system.

To recap, when $\gamma \leq 1$, the system includes a single unique equilibrium at $C = C^*$. For $\gamma > 1$ the system includes exactly three fixed points: two are stable equilibria at $C = 0,\ C = C^*$ and one is a tipping point located in between.

[1] No general analytical solution exists for the location of tipping point.