Using Difference Equation to Model Discrete-time Behavior in System Dynamics Modeling

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Abstract. In system dynamics modeling, differential equations have been used as the basic mathematical operator. Using difference equation to build system dynamics models instead of differential equation, can be insightful for studying small organizations or systems with micro behavior. In this paper we explain how we can use difference equations to build system dynamics models. We illustrate the use of this method through its application to a case study in supply chain management.

1 Introduction

In the field of social sciences, simulation is accepted as a powerful tool that helps researchers to get more insight into the system, especially in cases where practical experiments are not feasible. However, depending on which approach and tool we select to model a system, the quantitative and qualitative results of the simulation may vary.

Dealing with time is one of the main issues that every modeler should think of, before selecting a tool for simulation. Some researchers see social systems as continuous-time systems and therefore use differential equation-based tools to simulate a system. In the contrary, other researchers consider social systems as discrete time systems. Therefore, they select discrete-time simulation tools.

System dynamics modeling (SDM) takes a continuous-time approach and constructs models with differential equations. While flows that get in or out of stocks can be represented continuously or in discrete points of time [14], system dynamic modelers argue that it is an “acceptable approximation” to consider individual items as continuous streams that can be divided infinitely [14]. For instance, in a organization, people are individuals and are hired in discrete manner, but system dynamic modelers assume that the flows of people are continuously divisible.

Besides the approximation that is caused by assuming discrete flows as continuous streams, another source of approximation is using average delay instead of pure delay. For instance, system dynamic modelers assume that since in a post office with a large number of letters, all the letter are not delivered at once and there is a distribution around the average delivery time, it is an acceptable approximation to use average delay to model such cases.

System dynamics modeling is aimed to study long term behavior of systems at the macro level. This modeling approach is commonly used to study large organization or global phenomena. Therefore, these approximations are acceptable. However, the concepts of stock and flow have the potential to be used for studying systems at the micro level and to explore behavior of small organizations or phenomena (e.g, hiring system in a small organization, a supply chain system constructed by a few people). However, the problem is that the main characteristic of these kinds of system is that their flows are discontinuous and most of the time there are not many items in delay. Therefore, using differential equation to construct these kinds of system can lead to inaccurate quantitative results. In order to avoid these inappropriate approximation, we propose using difference equations instead of differential equations as the basic mathematical operator that determines the relation between a stock and its flows. This method of constructing system dynamics models allows us to model discrete flows and pure delay.

The structure of this paper is as follows. In section 2, we look into difference equation and differential equation modeling. In section 3 we present a example case which we will be using to illustrate difference between the quantitative result of both approaches. In section 4, we propose our method. In section 5, we rebuild the working example with the help of the method. In section 6, we compare the quantitative results of the working example. In section 7, we will finish with some concluding remarks.

2 Background

2.1 Difference Equations and Differential Equations

Control theory classifies dynamical systems, whose state varies during time, into two subdivisions: continuous-time (CT) dynamical system and discrete-time(DT) dynamical system. In CT-systems, the state of the system changes after every infinite short interval of time while in DT-systems the state of the system varies at distinct points in time.

Differential equation is the basic operator for modeling continuous time systems. A simple population model with the growth rate r in the CT approach is modeled by the differential equation(integral) as following:

$$\int_0^t r \times p(t)dt \tag{1}$$

Difference equation is the basic operator for modeling discrete time systems. The simple population model that we already represent with differential equation can be modeled by a difference equation in discrete time approach as following.

$$p_n = (1 + r) \times p_{n-1} \tag{2}$$

Population in time n is equal to population in previous time p_{n-1} plus $r * p_{n-1}$.

Difference equation as the main operator of discrete-time modeling has been used recently, especially after developing digital computers[10].

2.2 Discrete-time modeling in Literature

System dynamics literature rarely addresses discrete-time modeling. [14] emphasize that instead of first order systems in continuous time modeling which can not generate oscillated behavior, first order systems in discreet-time approach can oscillate or even generate chaotic behavior. [2] points out that the equations which construct a system dynamics model can be either differential equations or difference equations. [2] argues that although a system dynamics model can be continuous, discrete or hybrid, in practice, SD takes discrete systems as continuous system since continuous-discrete hybrid model can be cumbersome to build and analyze. [3,4] point out that by replacing dt of a system dynamics model with 1 we can have a discrete-time version of system dynamics.

Besides system dynamics literature, social simulation literature addresses different approaches for modeling discrete time systems with the help of difference equations. Inspiring from control theory studies, some researches use Z-transform to build and analyze discrete-time models. [5] analyze a discrete-time model of a four echelon supply chain system with the help of Z-transform. [7] study the dynamic stability of a vendor managed inventory supply chain by constructing a discrete transfer function of the system.

Besides the Z-transform approach to study discrete-time systems, some researcher use mathematical representations and state space approaches. Mathematical representations of a system is mostly used when the system is constructed by one equation. [6] develop a mathematical model of a simple population model called Logistic model. [1] develop a discrete-time version of epidemic model. [9] study pattern formation of the discrete-time predator prey model. The state-space approach is a mathematical representation which is used where the system is constructed by a few number of difference equations. [13] applied the state-space approach to study bullwhip effect in a simplified supply chain. [8] represent a generalized order-up-to policy in supply chains using the state-space approach.

Discrete event simulations (DES) may also be considered as discrete time methods, as they are suitable for modeling the systems in which variables change in discrete-times [12]. DES view systems as discrete sequence of events in time. In other words, DES is an event-based modeling approach that is different from the other mentioned approaches that are equation-based.

So far, we mentioned changing dt of system dynamics modeling, z-transform, mathematical and state-space approach as the main approaches for discrete-time modeling. We will later explain that although the first approach: changing $dt = 1$ in system dynamics modeling, seems a acceptable way to model a system with discrete time, it may result in inaccurate behavior of system. Using Z-transform and mathematical representation of discrete time system are suitable for linear system. However, constructing system dynamics models with difference equations can be used for both linear and nonlinear system. Besides, our proposed method takes the advantage of the diagramming aspect of the formal system dynamics modeling which can be more powerful than Z-transform or other mathematical approaches for participatory modeling and for giving insights to the clients.

3 Working Example

As a running example, we take a one-echelon supply chain adopted from beer game distribution [14,15] to present the innovative aspects of our method. The reason for selecting this example is due to the fact that in the beer game distribution model we study the micro behavior of a small group of individuals. Therefore, we can show how difference equation can benefit the result of simulation. Figure 1 depicts the stock and flow diagram of this case.

Description of System The system under study is a typical cascade production-distribution system consisting of one retailer. Customer demand is exogenous and retailer must supply the amount of product requested by the customer. If there is insufficient product in stock, the retailer will keep surplus order in backlog until delivery can be made. All the retailer's orders will be fully satisfied after delay (4 weeks) and there is no limitation for the wholesaler to supply the retailer.

Order Policy The retailer tries to keep the level of inventory at the desired level (1.5 times of order received). Every order is determined by the number of orders that the retailer has received and two adjustments (correction) for inventory and supply line.

$$\begin{aligned}&OrderPlacedRate = MAX(OrderReceivedRate + DesEffectiveInventory\\&Correction + DesOrderPlacedCorrection, 0)\end{aligned} \tag{3}$$

$$\begin{aligned}&DesEffectiveInventoryCorrection = (DesiredInventory - Effective\\&Inventory)/DelTime\\&DesiredInventory = OrderReceivedRate \times InventoryCoverage\\&EffectiveInventory = Inventory - Backlog\end{aligned} \tag{4}$$

$$\begin{aligned}&DesOrderPlacedCorrection = WeightOnSupplyLine \times (DesiredOrderPlac\\&ed - OrderPlaced)/DelTime\\&DesiredOrderPlaced = OrderReceivedRate \times DelTime\end{aligned} \tag{5}$$

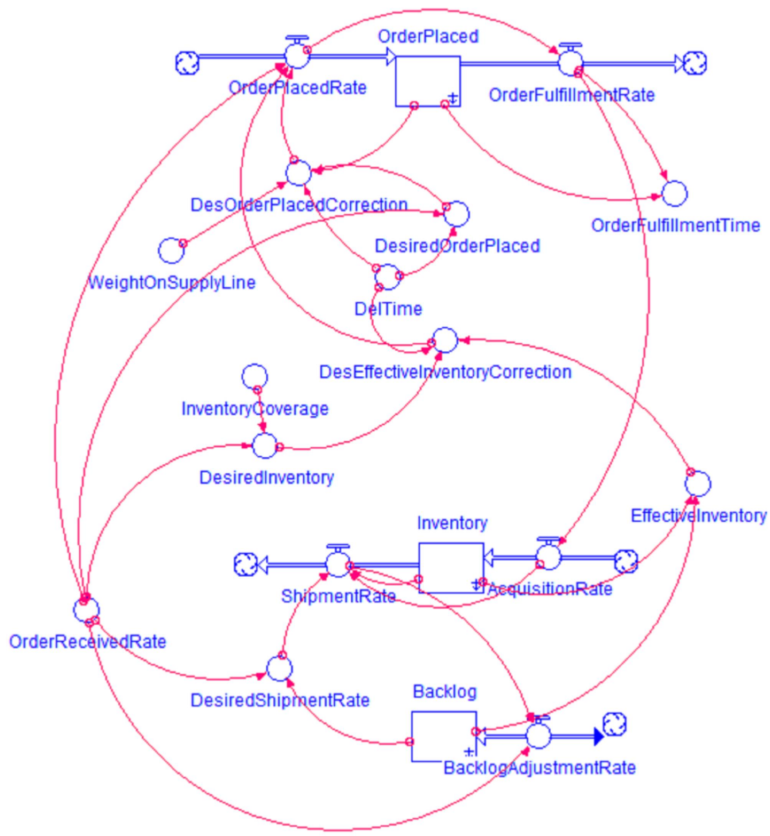

Fig. 1. Beer Game Example

Shipment and Demand Policy The desired shipment rate is the accumulation of the backlog and the order that the retailer has received. Due to limitation in inventory, it is not always possible to satisfy desire shipment. Shipment rate is determined by the minimum of desired shipment rate and inventory. It means if inventory is lower than the desires shipment rate, the retailer will support a part of order and backlog equal to the level of inventory. Otherwise, he can satisfy all the order and backlog.

In order to put the model in the steady-state, we set $OrderReceivedRate$ to 4 and $InventoryCoverage$ to 1.5. The initial amount of inventory is 6 equal to $DesiredInventory$. The initial amount of $OrderPlaced$ would be 16. At time 4 we increase $OrderReceivedRate$ to 8 in order to analyze the behavior of model.

$$ShipmentRatemin(DesiredShipmentRate, (Inventory/dt) + AcquisitionRate) \tag{6}$$

4 System Dynamics Modeling with Difference Equation

Although, differential equation is traditionally used as the mathematical operation that determines the relation between stocks and flows, difference equation is also compatible with the concept of stock and flow [11]. Therefore, it can also be used as the basic operator of SDM.

Using differential equation (Formula 7) in order to study the micro level behavior of systems or the short term behavior of small organization in which flows are not changed in every instance of time, renders the model far from reality and leads to in inaccurate quantitative results of the simulation. For instance, when modeling the process of 'making orders by retailers', if there are many retailers in the model, assuming that an order takes place every instance of time is reasonable. However, when there is only one retailer in the system (or a limited number of them), making the assumption that an order is taking place in every instance of time is unrealistic. Therefore, for such cases, using difference equation is a more reasonable option.

Time delays often play an important role in the dynamics of systems. How we model delay in systems is very crucial in determining the behavior of models. Using average delay (Formula 8) instead of pure delay to model a system at the micro level or study short term behavior of a small system or organization can bring some inaccuracy in quantitative result of simulation. For instance, in our working example, as we are modeling the behavior of a individual retailer, there is no distribution of delay time. The retailer will receive his product after a constant delay of time. Therefore, it would not be appropriate to use average delay in these kind of cases as it is far from reality.

$$stock(t) = \int_0^t (inflow(t) - outflow(t))\,dt + stock(0) \tag{7}$$

$$outflow(t) = \frac{stock(t)}{D} \tag{8}$$

In order to avoid the mentioned inappropriate approximation, we propose to use difference equations to construct system dynamics models. In this approach, the amount of stock is calculated by Formula 9 which calculates the amount of stock based on the inflow and the outflow and the previous amount of stock in every discrete point of time. In order to model D step time pure delay depicted in Formula 10, the amount of delayed outflow is equal to amount of inflow in time $t - D$. The amount of the stock that is linked to the delayed outflow is equal to summation of all inflow which are in queue to become outflow.

$$stock[t] = stock[t-1] + [inflow - outflow] \tag{9}$$

$$\begin{aligned} outflow(t) &= inflow[t-D] \\ stock[t] &= \sum_{t=t-D}^{t-1} inflow(t) \end{aligned} \tag{10}$$

Although, in practice, all simulation software use difference equation to calculate differential equation with the help of Rungg Kutta or Euler numerical method, it does not mean that we can change a system dynamics model constructed by differential equation to the difference equation version by setting dt to 1. Due to the fact that changing the sequence between events in discrete time modeling changes the numerical result of models, setting $dt = 1$ may result in chaotic behavior of formal system dynamics model as we do not consider the sequence between events during the steps of time. Even if we set dt to 1 in formal system dynamics models because we cannot model pure delay the numerical result of our model would be different from the difference equation version of that model.

We will discuss some Characteristics of proposed method in comparison with formal SDM next.

4.1 Difference Equation Approach Characteristics

Besides the discrete flows and pure delays that make our model closer to reality, using difference equation give modelers some extra opportunity to build more accurate and flexible models.

Logical Statements

In system dynamics models that are constructed by differential equation, logical statements such as if...*then*...*else* bring sharp discontinuities in the model and using them is an issue of debate [14]. However, with our proposed method, we can take advantage of logical statements. Using logical statements makes our model more flexible, especially when we want to model decision making processes as a part of the system.

Memory

System dynamics models constructed using differential equations assume dt as an infinite short step of time cannot represent the notion of previous time for our models. However, in reality, the state of system in a previous time is taken into account to reach a decision. In our working example, since the backlog presents the accumulation value of surplus order during the simulation, we cannot determine how much surplus order has been added to backlog during every step of time. In order to model the decision making process of a retailer who will order whatever product he could not have supplied during the previous time in his next order, we need to know how much surplus was added in previous step of time. SDM constructed by difference equation gives us this opportunity to have the state of system in every time step to make it possible for us to build more flexible models.

Real Data

Data gathering from social systems is commonly conducted in discrete points of time. For instance, the information about the income, profit, and balance sheet of a corporation are booked every month or year. Using these data as inputs for a simulation or reference to examine the behavior of simulation is more compatible by simulating a model in discrete time manner.

In the next section, we will rebuild our working example with the proposed method.

5 Working Example Constructed by Difference Equation

In Section 3, we described our working example which has been developed with differential equation. In this section, we will rebuild this model with the help of difference equation and we will compare the quantitative results of both approaches. In order to apply this new approach, we developed software in Python programming language. As in most SD tools, this software supports graphical definition of equations.

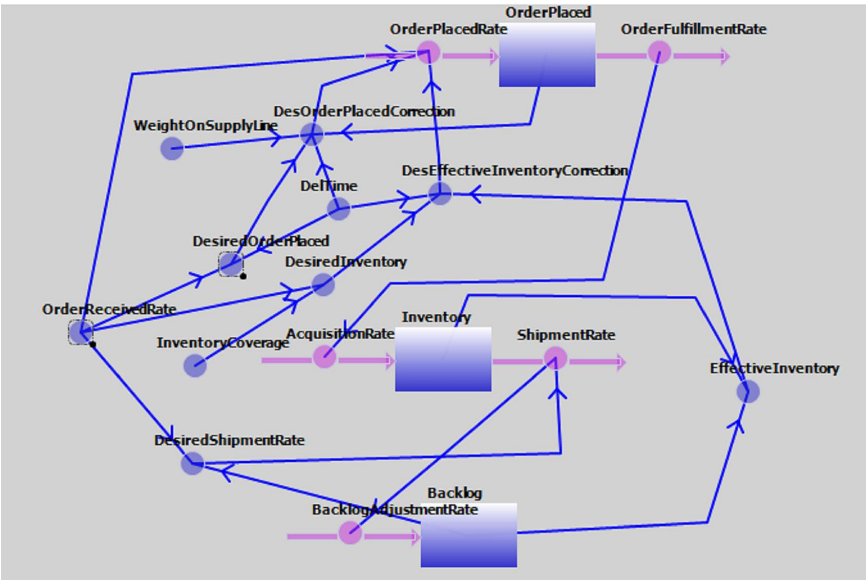

Fig. 2. System Dynamics Supply Chain Example constructed by difference equation

The order, shipment and demand policy of this new model is the same as the differential equation based model. The only difference is the stock and flow relation and delay construction. Since inventory and backlog are both stocks that are not used to model delay, we determine the mathematical relation between these stocks and their flows using Formula 9.

To model the delay between $OrderPlacedRate$ and$OrderFulfillmentRate$, Formula 9 is used to construct pure delay, depicted in Formula 11:

$$\begin{aligned} OrderFulfillmentRate(t) &= OrderPlacedRate[t-4] \\ OrderPlaced[t] &= \sum_{t=t-d}^{t-1} OrderPlacedRate[t] \end{aligned} \tag{11}$$

Besides defining mathematical operations for constructing a model, another issue that is important to determine, is the sequence between the events in every step of time. Depending on how we define the sequence between events during the time steps, the quantitative results of our model will change. For instance, in our case we have three main events: placing new order, shipping and backlogging, receiving previous order. How we arrange the sequence between these events will result in different behaviors in the system. The final issue that must be considered is about the steady state of the system. The steady state of the model must be determined based on the arrangement between events.

6 Results Comparison

We assume that the retailer at the beginning of the week will receive their previous order in the supply line. Then, he will calculate the order that needs to be placed and will satisfy costumer's order by shipping and will adjust the backlog. In order to put system in the steady state, we set the initial value of inventory to 2 based on the assumption that a retailer will ship all $OrderReceived$ during the week and what will be left in inventory would be half of $OrderReceived$ which is equal to 2.

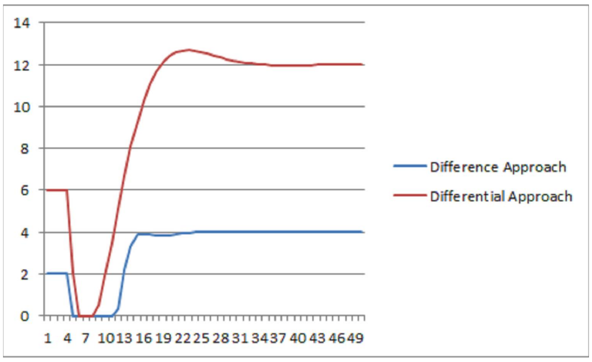

Fig. 3. Inventory in both approaches

Figure 3 shows the difference between the level of Inventory in both approaches. As it is depicted, inventory in the first approach shows a smooth oscillation while in the second approach there is no sign of oscillation.

Examining different $WeightonSupplyLine$ in both approaches shows the fact that in the differential approach, the more $WeightonSupplyLine$ gets close to 1, the more the system presents behavior with lower oscillation. While in the difference approach the more $WeightonSupplyLine$ closer to 0.5, the more the system shows lower oscillation. Figure 4 and Figure 5 present the level of inventory for 3 different values of $WeightonSupplyLine$ parameter of the differential and difference approach models.

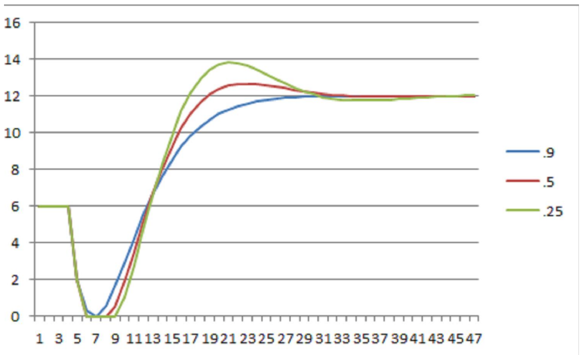

Fig. 4. Inventory with different values for $WeightonSupplyLine$ in the differential approach

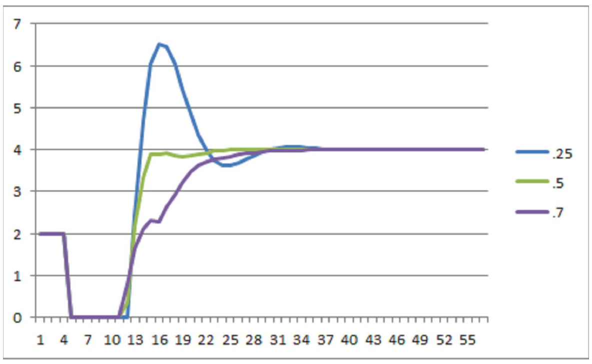

Fig. 5. Inventory with different values for $WeightonSupplyLine$ in the difference approach

Comparing the results of both approaches indicates that with the same $WeightonSupplyLine$ setting, in first approach the retailer is placing order more than what he needs to adjust the inventory and therefore it causes oscillation in inventory behavior. However, in the second approach the oscillation in inventory behavior is not as strong as it is presented in the differential approach which means that the retailer take in to account the existence of delay and the amount of product in delay more than the retailer in the differential approach.

Besides the different quantitative results, we observe qualitative differences especially when defining hypotheses for simulations. For example, as it is depicted in Figure 3, in the difference equation model with the $WeightonSupplyLine$ equal to 0.5, inventory does not oscillate. This fact can challenges the hypothesis that oscillation in the behavior of beer game distribution is because of ignoring the amount of product in supply line.

7 Conclusion

In this paper, we proposed constructing SDM models with the help of difference equations instead of differential equations. We illustrated this new approach by applying it to a supply chain system.

In SDM, the mathematical relationship between stocks and flows is commonly determined by the differential equation. However,the stock and flow concept is also compatible with difference equation and therefore, we can use difference equation as the basic operator of SDM which leads us to more accurate quantitative result where we study the micro level behavior of a system or small organization.

The proposed approach contributes to SDM in several aspects. Firstly, it provides the opportunity to apply the SDM concept at micro level systems where individual activities change the flows of systems in discrete points in time. Secondly, it can result in more accurate quantitative result for cases that are not large enough to assume their flows as continuous streams. Thirdly, opportunities to use logical statements and memory as explained in Section 4, enhance our ability to model complex systems.

Another contribution of this approach is that since we are constructing discrete-time models with the stock and flow concepts instead of using Z-transform or mathematical representations, it will contribute to the field of discrete-time modeling as it gives an opportunity to modelers to construct a model graphically and allow them to model and analyze nonlinear discrete-time models.

One final contribution of our proposed method is that since our proposed method and discrete event simulation both study the behavior of systems in discrete points in time and use queues to model systems, it may be possible to merge the system dynamics approach with the discrete event modeling. In discrete event simulation, flows of entities that get through the process are determined by random numbers which means that we are dealing with passive entities. However, by merging both concepts, we can develop more deterministic behavior for the flows of entities so that they can be effected by the state of system due to feedback.

As this method is proposed for the first time, it needs more evaluation process to be proved as a reliable approach for constructing system dynamics models. In order to use the ability of this approach for studying the micro behavior of systems a possible option for future work can be merging this method with agent based modeling.

References

1. L. J. Allen. Some discrete-time epidemic models. *Mathematical biosciences*, 124(1):83–105, 1994.
2. Y. Barlas. System dynamics: systemic feedback modeling for policy analysis. *SYSTEM*, 1:59, 2007.
3. Y. Barlas and B. Gunduz. Demand forecasting and sharing strategies to reduce fluctuations and the bullwhip effect in supply chains. *Journal of the Operational Research Society*, 62(3):458–473, 2011.
4. Y. Barlas and M. G. Özevin. Testing the decision rules used in stock management models. In *Proceedings of the 19 th International Conference of the System Dynamics Society 2001*, 2001.
5. J. F. Burns and B. Sivazlian. Dynamic analysis of multi-echelon supply systems. *Computers & Industrial Engineering*, 2(4):181–193, 1978.
6. P. Cvitanovic, R. Artuso, R. Mainieri, G. Tanner, G. Vattay, and N. Whelan. Chaos: classical and quantum. *ChaosBook. org (Niels Bohr Institute, Copenhagen 2005)*, 2005.
7. S. M. Disney and D. R. Towill. A discrete transfer function model to determine the dynamic stability of a vendor managed inventory supply chain. *International Journal of Production Research*, 40(1):179–204, 2002.
8. C. S. Lalwani, S. M. Disney, and D. R. Towill. Controllable, observable and stable state space representations of a generalized order-up-to policy. *International journal of production economics*, 101(1):172–184, 2006.
9. M. G. Neubert, M. Kot, and M. A. Lewis. Dispersal and pattern formation in a discrete-time predator-prey model. *Theoretical Population Biology*, 48(1):7–43, 1995.
10. A. V. Oppenheim, A. S. Willsky, and S. H. Nawab. *Signals and systems*, volume 2. Prentice-Hall Englewood Cliffs, NJ, 1983.
11. G. Ossimitz and M. Mrotzek. The basics of system dynamics: Discrete vs. continuous modelling of time. In *Proceedings of the 26th International Conference of the System Dynamics Society], System Dynamic Society, Wiley-Blackwell (July 2008)*, 2008.
12. O. Özgün and Y. Barlas. Discrete vs. continuous simulation: When does it matter. In *Proceedings of the 27th International Conference of The System Dynamics Society,(06)*, pages 1–22, 2009.
13. C. Papanagnou and G. Halikias. A state-space approach for analysing the bullwhip effect in supply chains. *Proceedings of ICTA*, 5:79–84, 2005.
14. J. Sterman. *Business dynamics*. Irwin-McGraw-Hill, 2000.
15. J. D. Sterman. Modeling managerial behavior: Misperceptions of feedback in a dynamic decision making experiment. *Management science*, 35(3):321–339, 1989.