Insights into Income Policy for Enhancing Employment and Stability of Capital Accumulation

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Abstract. This paper reveals how the deterministic non-linear Lordon – Goodwin model (LGM-I) endogenously generates cycles of absolute and relative over-accumulation of capital similar in important aspects to the Marx industrial cycle. This model hyperbolizes acuteness of accumulation crises. By correcting functional relations for implicit rate of capital accumulation, the present paper transforms LGM-I into LGM-II. Scenario I of sluggish stabilization at a low level of employment is based on a refined model LGM-II. Scenario II is a futile attempt to achieve hastily a substantially higher level of employment than in scenario I by policy optimization within the same improper social structure of accumulation. This paper revises original equations for profit sharing and bargained wage terms and offers an upgraded model of capital accumulation LGM-III where a growth rate of total profit depends positively on a gap between the target and current employment ratios. The latter policy rule does not cause over-shooting of profit and under-shooting of wage in satisfying scenario III unlike ill-defined policy rule in scenario II rooted in LGM-I and LGM-II. The Structural Control Theory has helped in the policy design phase. A qualitative analysis of local stability for non-linear models is extended by exposing transients to distant attractors.

Introduction

Soaring unemployment, shrinking GDP, reduced share of wages in total income and lack of credit getting through to businesses and households, widened inequalities are manifestations of the ongoing crisis in Eurozone (Crisis and debt in Europe 2010; The Financial Times March 1, 2012). The disappointing macroeconomic performance, the acute necessity to set up the conditions for sustainable development and to bring about a sharp fall in unemployment have rekindled interest in the possible macroeconomic role for profit sharing and/or excess income levy.

"Profit sharing refers to definite arrangements under which workers regularly receive, in addition to their wages and salaries, a share on some predetermined basis, in the profits of the undertaking, the sum allocated to workers varying with the level of profits". This is the official definition adopted at an International Congress on Profit Sharing held in Paris in 1889 (Cynog-Jones 1956).

A significant number of empirical studies have shown that profit sharing schemes have a positive impact, increasing labour productivity and reducing monitoring costs, with mixed evidence pertaining to wage flexibility; profit sharing – if well-designed – not only ensures a fairer distribution of income, but has been shown also to improve productivity and growth (ILO 2011: 45–47).

To be effective, profit sharing measures must be part of an overall wage-determination process. Otherwise, pro-cyclical measures of this nature run the risk of reducing employees' incomes in times of crisis, potentially intensifying income inequalities (Teulings and Hartog 1998).

The author would like to recall a paradox that profit is the decisive factor of long waves under capitalism and could be the key for smoothing them. A more efficient social control over the oscillatory macroeconomic system requires a substantial reshaping of primary income distribution that takes into explicit account this dual characteristic of profit. The present paper elaborates this duality by focusing on accumulation cycles with a period within one decade.

The Lordon model, named for brevity LGM-I, is an intellectual achievement with a great deal of enticing mathematical sophistication. It remains one of the best two-dimensional Goodwinian models as explanation of endogenous cycle of capital accumulation similar in important aspects to the Marx industrial cycle. Although this model helps to carry out extreme conditions tests of structural robustness of policy rules developed by the author of present paper for more sophisticated and realistic models, serious drawbacks of LGM-I must be revealed and possibly removed for the benefit of social and economic modelling.

The Structural Control Theory is to be applied as a main tool for conceiving efficient and robust stabilization policy aimed at pro-employment shifts in primary distribution of income. The Classical Optimal Control Theory and modal methods, although useful and efficient for particular tasks, do not suffice in the present case for the following reasons.

In an optimal control problem, an objective function is usually expressed in a form of an integral over a time period, and a system of first-order differential equation governs evolution of state variables (Pontryagin 1976). It was recognised later that a prerequisite of successful control is a profound structural perfection that could not be usually provided by optimization within an initial structure.

In the modal methods (Mohapatra and Sharma 1985), a closed loop control is designed by moving the eigenvalues of a linearized model and is expressed as a function of the level variables. Still the modal methods are ill-suited for "brutal" non-linearity in functional forms.

The Structural Control Theory has helped the present research in the policy design phase. Target employment ratio as the control parameter (in conjunction with a growth rate of total profit) plays the pivotal role in policy design for the oscillatory macroeconomic system.

As written by R.M. Goodwin, "…there must be a successful policy of forcing or persuading employers and trade unions to forgo raising real wages in consequence of tightness of the labour market, in exchange for high and stable employment along with rising output" (Goodwin 1990: 110). In view of the present author, successful policy requires strengthening elements of feed-forward control over capital accumulation and primary income distribution. Feed-forward control, as known, changes variables according to expected future states of the economy. For solving this Goodwin's conjecture, a refined profit sharing is synthesised with an advocated policy rule (a negative relationship between the growth rate of total profit and employment ratio for properly defined boundaries).

This paper demonstrates how the system dynamics methodology extends opportunities offered by a qualitative mathematical analysis of local stability for non-linear models. The paper proves that local stability analysis is not sufficient for policy recommendations being not powerful enough for explicit presentation of transients to attractors shaped as fixed-point or closed orbits.

Section 1 analyzes LGM-I based on its own assumptions. This model endogenously generates absolute and relative over-accumulation of capital. Section 2 is a constructive critique of this model that suggests a modified successive model LGM-II putting more pragmatic restrictions on the rate of capital accumulation. Section 3 revises key assumptions and functional forms of these two models and transforms LGM-II into LGM-III where a growth rate of total profit depends positively on a gap between a target and current employment ratios.

Sections 2 and 3 also suggest three scenarios of capital accumulation. Scenario I of a sluggish stabilization at a low level of employment is essentially taken from the original paper as a benchmark; it is based on a refined model LGM-II. Scenario II is a futile attempt to achieve faster stabilization at a substantially higher level of employment than in scenario I by policy optimization within the same improper social structure of accumulation. Upgrading this structure in scenario III based upon LGM-III removes excessive over-accumulation of capital in relative and absolute terms typical for scenarios I and II.

Words of caution are not to be forgotten. In the following three models local stability of a stationary state is fully determined by the "distribution block". This conclusion is proved for the models that abstract from important properties of real economy. In a more realistic setting with an endogenous capital-output ratio as a factor of a rate of capital accumulation even the suggested overt closed loop control is not sufficient for complete eradicating industrial cycles (Ryzhenkov 2010).

"Another major problem", – has written a team of supporters of the social structure of accumulation theory (McDonough et.al. 2010: 13) – "will be responding to the high levels of inequality that have developed... A permanent and substantial transfer of present and future income from the top to the middle and bottom of the income distribution can solve this problem. But such a change is highly unlikely to take place solely as a response to economic considerations. This change may only come as a response to an effective popular movement that demands it. Building such a movement out of the legacy of the long period of retrenchment and labor weakness will most likely be a lengthy task."

Similarly to the report (ILO 2012) and to the author's earlier articles, this paper emphasises the importance of placing jobs at the top of the policy agenda. The gained insights into income policy, coherent with employment policy, are helpful for moving out of the austerity trap.

1 The Lordon – Goodwin model (LGM-I)

1.1 The model assumptions, structure and equations

The model extensive form

The model is formulated in continuous time. Time derivatives are denoted by a dot, while a hat will indicate growth rates. Let P, K, L, N, w, $v = L/N$, $a = P/L$, $s = K/P$ and $u = wL/P$ denote real output, fixed capital, employment, labour supply, the real wage rate, employment rate, output per worker, capital-output ratio and the wage share in the national income (net output), respectively. All variables are in real terms and net of depreciation.[1]

In this paper, as in Marxian economic theory generally, the wage share (broadly defined) is interpreted as unit value of labour power. K. Marx saw use-value of this commodity in its unique ability to produce surplus value.

Theoretically profit is money form of surplus value, in the present context it is measured by surplus product. Profit sharing means workers' appropriation of a part of net output that would be otherwise appropriated by capitalists. In real practice, profit sharing may take different forms, for example, a variety of excess income levy (Ryzhenkov 2007).

It is assumed that the society invest a part of net output in fixed capital without material delay, all wages are consumed, the product market is always in equilibrium and all savings serve as internal finance for investment purposes. Besides that capital-output ratio (s) and labour supply (N) are fixed at the chosen level of abstraction, so s = const and $\hat{N} = n = 0$, respectively. Owing to the assumed constancy of capital-output ratio, the growth rate of capital intensity and of that of output per worker are the same: $\widehat{K/L} = \hat{a}$.

The profit ratio is determined in a standard manner as a ratio of unit profit to capital-output ratio

$$r = \frac{1-u}{s}, \qquad (1)$$

$r > 0$ for $0 < u < 1$.

The surplus labour and surplus value is $S = (1 - u)L$. The surplus product and profit is $M = (1 - u)P = Sa$.

The growth rates of fixed capital and net output equals each other owing to the assumed constancy of the capital-output ratio and defined as a continuous monotonically decreasing function of relative wage in the following (rather sophisticated) equation

$$\hat{K} = \hat{P} = g(u), \qquad (2)$$

where the negative derivative $g'_u < 0$ for $0 < u < 1$[2], $g''_u < 0$ for $0 < u < 1$, $\exists\ 0 < u_c < 1$, $g(u_c) = 0$,

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[1] Labour income includes not only the wage but also payroll taxes and other social contributions paid by firms. In empirical research, statistical series associated with the labour share are adjusted for self-employment.

The labour share long term decline is a puzzle that is largely unsolved by "neoclassical" economics (Blanchard 2006: 43). Deeper insights can be found in the social structure of accumulation theory (McDonough et.al. 2010, Ryzhenkov 2010). See also footnote 7.

[2] In the original text there is inaccuracy: $0 \leq u \leq 1$. The substitution of the segment [0, 1] by the inter-

$\exists \lim_{u\to 1} g(u) = a,\ -\infty \le a < 0$. Notice that it is possible that $g(u)$ is far below zero for a very high relative wage. Postulating that $g(u)$ can be infinitely lower than –1 means, in the author's opinion, a possibility of the Atlántida-scale catastrophe that does not prohibit recovery after that epic meltdown. Here hyperbolising comes to its dizzy heights. The laws of conservation of matter and energy could be violated by this hyperbolising that is not allowed by sound economic reasoning. So a more cautious equation (2a) as a profound modification of the equation (2) is proposed in section 1.3.

The dynamics of the real wage sum ups the evolution of two terms

$$\hat{w} = \hat{w}^m + \hat{w}^b. \qquad (3)$$

A reduction in the reserve army of unemployed facilitates workers' bargaining strength and therefore promotes the growth in real wage. Besides that profitability facilitates this growth as well. Consequently, the first term in the equation (3) is the "usual" Phillips component that follows from the wage-bargaining equation

$$\hat{w}^m = \gamma(v) - \pi_b, \qquad (4)$$

for $v \in (0,1)$ the first and second derivatives are positive,[3] i.e., $\gamma'_v > 0$ and $\gamma''_v > 0$, $\exists\ 1 > v_n > 0$ such that $\gamma(v_n) = 0$ and, finally, $\exists \lim_{v\to 1}\gamma(v) = b \le +\infty$, $\pi_b \ge 0$;

the second term is a manifestation of the profit sharing effect

$$\hat{w}^b = \eta(r) = \delta(u) + \pi_b, \qquad (5)$$

where the derivatives have the opposite signs: $\eta'_r > 0$ and $\delta'_u < 0$. A magnitude of the constant π_b is to be chosen from a continuum of possible magnitudes for maintaining both non-negative stationary $\hat{w}^{m*} \ge 0$ and $\hat{w}^{b*} \ge 0$ such that $\hat{w}^{m*} + \hat{w}^{b*} = g^* > 0$ (see a concrete magnitude chosen for simulation runs in Table 2 in section 2).[4]

The constant π_b is added by the present author in refining the original model. This element does not guarantee absence of violent movements of the both wage terms on a transient to a stationary state.

The author recalls again the constancy of capital-output ratio that maintains the qualitative equivalence of $\eta(r)$ and $\delta(u)$ in the equation (5). The apparently 'innocent' properties of the function (4) hide a Pandora box of instability aggravated by absence of labourers' competition for jobs as well by implicit negative dependence of a growth rate of capital intensity on relative wage in this model.

The equation for the growth rate of output per worker relies on a Kaldor – Verdoorn *empirical* regularity, called in the cited papers (Lordon 1995, 1997) *a law of technical change* (the obvious exaggeration). Originally considered as a linear relationship between growth rates of productivity and production, this regularity is envisaged under a surrogate nonlinear and logistic form that substitutes a growth rate of net output by that of fixed capital allowed purely formally by the constancy of the capital-output ratio (s)

$$\hat{a} = f(g), \qquad (6)$$

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val (0, 1) takes into account that $u = 1$ and $u = 0$ are both incompatible with capitalist production relations: in the first case the labour power has no social utility (the ability to produce a surplus product) and can not be sold, in the second the emptiness of a consumption bundle prohibits reproducing this commodity.

[3] In the original text there is inaccuracy: $0 \le v \le 1$. The substitution of the segment [0, 1] by the interval (0, 1) takes into account that $v = 1$ and $v = 0$ are both incompatible with capitalist production relations: in the first case there is no reserve of labour power vital for the capitalist mode of production, in the second the latter is simply impossible.

[4] A shortcoming of this type of stabilization policy in the original model for implicit $\pi_b = 0$ is possibility of "negative profit sharing" if $\hat{w}^{b*} < 0$. This inaccuracy of the original model is cured to some extent by the present author's refining of the equations for the growth rates of both wage terms.

where $f'_g \geq 0$ for $0 \leq g \leq +\infty$, $f(0) \neq 0$, $\exists \lim_{g \to \infty} f(g)$, $\exists \tilde{g} > 0$ such that $f''(\tilde{g}) = 0$, $f''_g > 0$ when $g < \tilde{g}$ and $f''_g < 0$ when $g > \tilde{g}$.[5]

By substituting g by expression (2) in the equation (6) we get the growth rate of output per worker as a function of the relative wage

$$\hat{a} = f[g(u)] = \phi(u), \tag{7}$$

where the intermediate derivatives have the opposite signs $f'_g > 0$ and $g'_u < 0$, so finally the consequential derivative is negative: $\phi'_u = f'_g\, g'_u < 0$. Here the positive influence of a growth rate of capital intensity on that of labour productivity that is a characteristic of real capitalist economy is left without attention for the detriment of the original theoretical considerations. The lack of a causal link from a growth rate of capital intensity to that of output per worker is hidden by empty identity $\hat{a} = \widehat{K/L}$ for s = const.

A system dynamics coding of production relations applies sequences of *causal* links between relevant economic categories (Рыженков 2012). Figure 1 represents the detailed LGM-I causal-loop structure. The stock and flow variables are joined by arrows. An arrow, going from one variable to another, means that the latter variable is a function of the former. The arrows, differently shaped, represent either integration of the flows into stocks or information linkages. A sign of a respective ordinary or partial derivative determines a direct link polarity. A polarity of a feedback loop is determined by a chain differentiation rule for a complex function and, especially, by a number of negative derivatives (positive – for even, negative – for odd).

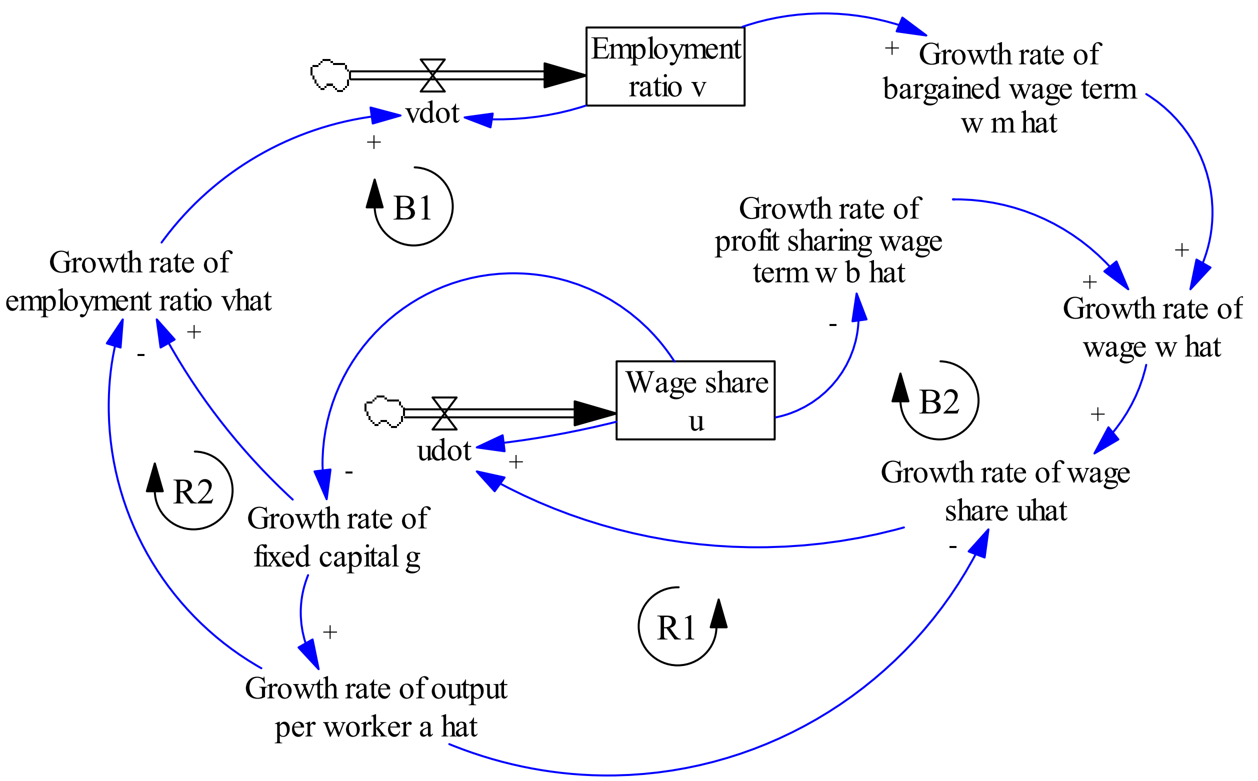

Figure 1 – The extensive causal loop structure of the LGM-I with implicit rate of capital accumulation (c)

[5] Unfortunately, the cited Lordon papers do not explore accuracy of this technical substitution (of $\hat{P}$ by $\hat{K} = g$) for models with variable capital-output ratio where it creates logical circles in definitions. Moreover, empirical support for this substitution is not given there or elsewhere, to the knowledge of the present author.

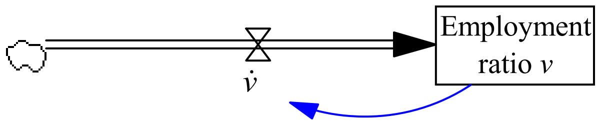

Figure 2 – The first loop involving the relative wage u with a potentially alternating sign $[g(u) - \phi(u)]$

It becomes clear from the inspecting of the Figures 2 – 7 that the extensive form LGM-I has four main and two little feedback loops. Because of alternating polarity the little feedback loops can be balancing or reinforcing, so signs of their polarity are skipped. Whereas the loops with alternating polarity and analogue of negative loop B1 (Figure 5) are also the characteristics of the original Goodwin model, the positive loops R1 and R2 have close analogues in the Goodwinian model later proposed by L. Boggio (see a critique in Ryzhenkov 2009).

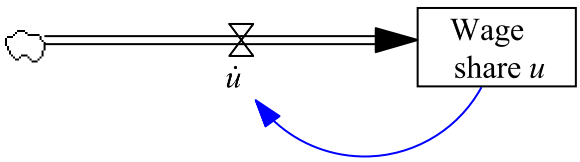

Figure 3 – The second loop involving the employment ratio v with a potentially alternating sign $[\gamma(v)+\delta(u)-\phi(u)]$

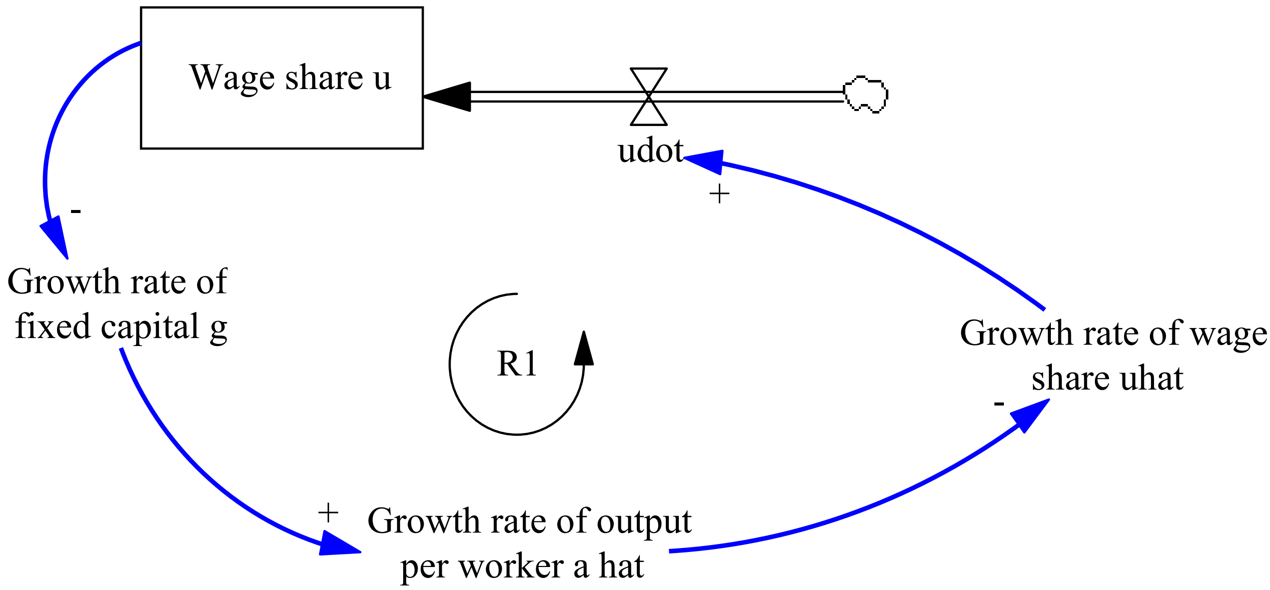

Figure 4 – The first order positive loop R1 involving the relative wage u and growth rate of output per worker $\hat{a}$

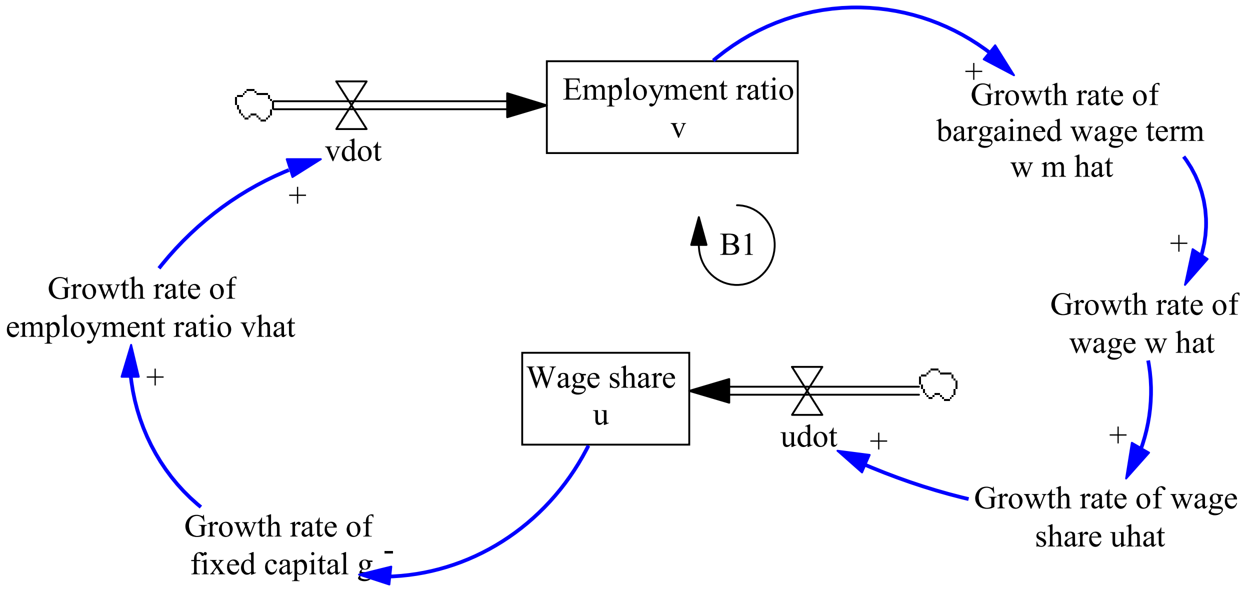

Figure 5 – The second order negative loop B1 involving the relative wage *u* and employment ratio *v*

LGM-I contains no direct or reinforcing roundabout increasing returns. Still it includes reinforcing roundabout decreasing returns reflected by the loop R2 (Ryzhenkov 2009). The loop R2 facilitates local instability of a stationary state.

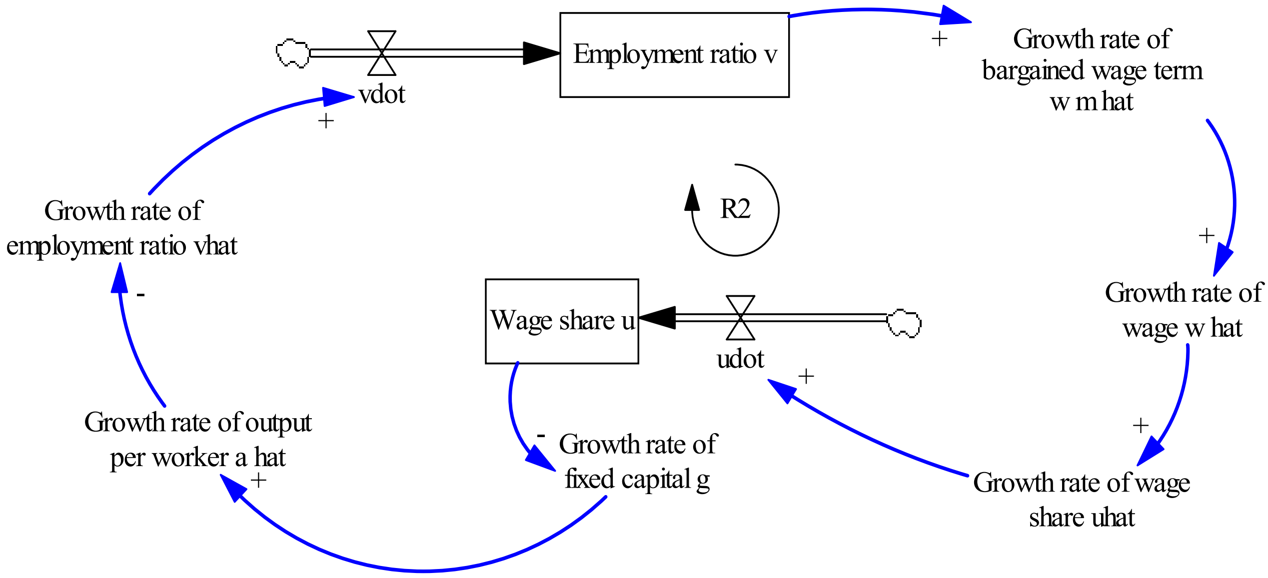

Figure 6 – The second order positive loop R2 involving the relative wage *u* and employment ratio *v*

The loop B2 seemingly the most innovative element of the LGM-I is an element of a particular form of a simple model of capital accumulation presented earlier (Ryzhenkov 1993). That paper prompts including of the growth rate of output per worker in the profit sharing term left unnoticed in (Lordon 1995, 1997). Even from pure formal point of view this would be an advantage that simplifies a stability analysis (see Figure 13 in section 3).

The forms of proportional and derivative feedback control, used in the model economy, are not sufficient to eliminate deviations from a non-trivial stationary state and tend to cause fluctuations. The next section explains rigorously existence and local stability of a non-trivial stationary state.

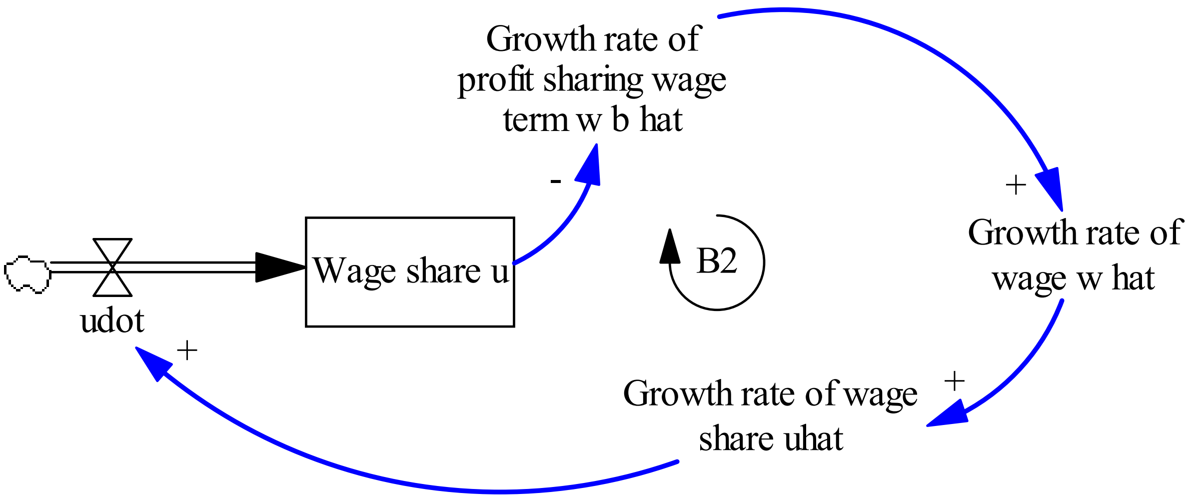

Figure 7 – The first order negative loop B2
involving the relative wage u and profit sharing term $\hat{w}^b$

The model intensive form

The rate of change of the employment ratio is given by

$$\hat{v} = \hat{K} - K\hat{/}L - n = \hat{K} - (\hat{s} + \hat{a})$$
$$= g(u) - \phi(u), \qquad (8)$$

where $K\hat{/}L = \hat{a}$ and $\hat{N} = n = 0$ as assumed by F. Lordon. The reader sees that the growth rate of the employment ratio depends only on the relative wage. The equation (8) represents a simple form of a proportional and derivative control, whereby a magnitude of an implicit adjustment coefficient equals one. The obvious loss of generality that results from the rather excessive simplifications is a 'price' for these simplifications. The equality $\hat{v} = g - \hat{a}$ in the equation (8) could be easily derived as a consequence of a linear Kaldorian technical progress function for the special assumptions under considerations.

The equation (9) defines the rate of change of the relative wage

$$\hat{u} = \hat{w} - \hat{a}$$
$$= \gamma(v) + \delta(u) - \phi(u). \qquad (9)$$

The equation (9) implies a similar form of a proportional and derivative control as well. Still the growth rate of the relative wage depends not only on the relative wage itself but on the employment ratio too.

The intensive deterministic form of the LGM-I for the initial system of economic growth, endogenous technical change, class struggle and social partnership is the system of two non-linear ODEs (10) and (11) (the equation (10) follows from the equation (8), the equation (11) – from (9)).

$$\dot{v} = [g(u) - \phi(u)]v \qquad (10)$$
$$\dot{u} = [\gamma(v) + \delta(u) - \phi(u)]u \qquad (11)$$

Taking into account u_c from the definition of the equation (2) and v_n from the definition of the equation (4), a single non-trivial stationary state is found as a solution of the following system arisen on a basis of the equations (10) and (11)

$\dot{v} = 0$, i.e., $g(u) = \phi(u) \leftrightarrow 0 < u^* < u_c < 1$,[6]

[6] Given the logistic form assumed for the productivity function, it is possible to get a graphic "resolution" for this equation. It reveals that there exists a single stationary growth rate, $g^* > 0$ (Lordon 1995: 1414). In view of the present author, a Taylor expansion of function $\phi(u)$ including its first and second derivatives must suffice for sufficient precision in finding g^* and u^* as a typical numerical example demonstrates (skipped in the present paper).

and

$\dot{u}=0$, i.e., $\gamma(v)=\phi(u)-\delta(u)\leftrightarrow 0<v_n<v^*<1$.

The system of equations (10) and (11) is used straight away for defining Jacoby matrix. It includes all the partial derivatives of the two phase variables (v and u):

$J=$

$\hat{v}$	$[g'_u(1-f'_g)]v<0$
$\gamma_v' u>0$	$\hat{u}-[-\delta'_u+f'_g\, g'_u]u$

. (12)

The property $\frac{\partial \dot{v}}{\partial u}<0$ is proved in (Lordon 1995) for a non-trivial stationary state $x^*=(v^*, u^*)$ only based on a graphical consideration. Computer simulations with a typical special case of LGM-I validate $\frac{\partial \dot{v}}{\partial u}<0$ (and $\frac{\partial \hat{v}}{\partial u}<0$) not only for a non-trivial stationary state and its vicinity but for wider intervals too.

Correspondingly, the intensive form of LGM-I has two previous little feedback loops (Figures 2 and 3) and the main negative loop (Figure 8). These three loops are a well-known common property of the Goodwinian predator-prey models. Integration of $\dot{v}$ and $\dot{u}$ creates the implicit delays fostering fluctuations.

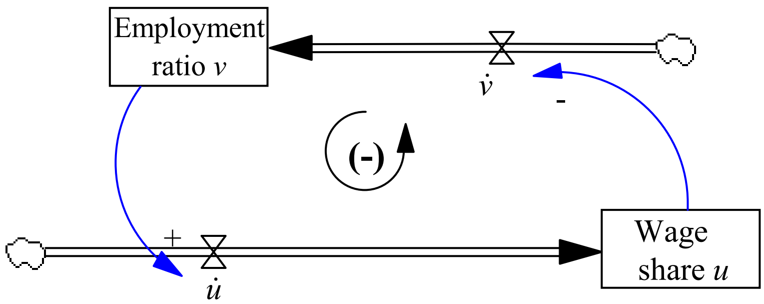

Figure 8 – The second order intensive negative loop involving the relative wage u and employment ratio v

1.2. Relative and absolute (I and II) over-accumulation of capital

Relative over-accumulation of capital takes place when growing capital (K) is accompanied by declining rate of profit (r). In LGM-I, a declining rate of profit always means growing relative wage. Details are also important.

Relative over-accumulation of capital in LGM-I implies in particular: for constant and monotonously growing employment ratio $v\geq v^*$, $\hat{K}=g>0$, $\dot{u}>0$ and $\hat{r}<0$, still $g'_u<0$ for $0<u<1$; after arrival at the point u_c where $\hat{K}=g=0$ a further increment of u courses $\hat{P}_{min}=\hat{K}_{min}=g_{min}<0<\hat{a}_{min}$ for $u=u_{max}$ and $v=v^*$ as in a Marx crisis of industrial cycle. The employment ratio falls from v^* to v_{min} whereas the rate of profit increases from its minimum to r^*. Notice fixed capital (K) declines whenever $1>u>u_c$.

Briefly recall two forms (I) and (II) of absolute excess of capital.

I) If the fall in the rate of profit is not compensated through the mass of profit (M), when the increased capital produced just as much, or even less, profit than it did before its increase: for $\hat{P}=\hat{K}=g>0$, $\hat{M}=\hat{P}-\frac{\dot{u}}{1-u}\leq 0$ or for $s=\text{const}$

$$0 < g \le \frac{\dot{u}}{1-u} \ . \qquad (13)$$

Absolute over-accumulation of type I is sufficient for relative over-accumulation of capital. Indeed, 0 $< g \le \frac{\dot{u}}{1-u} \Rightarrow \dot{r} < 0$.

Let relative over-accumulation of type I takes place: $\hat{P} \le \hat{K} + \frac{\dot{u}}{1-u}$ or $\frac{\dot{u}}{1-u} \ge 0$. Then a violation of the definition I is possible if $g > \frac{\dot{u}}{1-u} > 0$. It means that relative over-accumulation of capital is not necessarily accompanied by absolute over-accumulation of capital of type I, or the former is not sufficient for the latter.

II) The fall in the profit share (unit surplus value) is not compensated through the mass of surplus labour, when the increased capital produced just as much, or even less, surplus value than it did before its increase.

For LGM-I this means that for $\hat{K} = g > 0$ (when $0 < u < u_c$) there is

$$\hat{S} \le 0 \equiv \hat{L} - \frac{\dot{u}}{1-u} \le 0. \qquad (14)$$

With N = const, this condition turns into

$$\hat{S} \le 0 \equiv \hat{v} - \frac{\dot{u}}{1-u} \le 0. \qquad (15)$$

If the condition (13) is satisfied and $\hat{a} \ge 0$, the condition (15) is also fulfilled: if $g \le \frac{\dot{u}}{1-u}$ then $\hat{v} = g - \hat{a} \le g \le \frac{\dot{u}}{1-u}$. In other words, if the first definition of absolute over-accumulation of capital of type I is true, the second one of type II is also true.

Consider three cases appropriate for the definition of type II grasped as the condition (15).

A. $\hat{K} = \hat{a} > 0$. In LGM-I it is true only for $u = u^*$ when $\hat{v} = 0$. Relative over-accumulation means $\frac{\dot{u}}{1-u} > 0$ or $0 = \hat{v} < \frac{\dot{u}}{1-u}$. Therefore relative over-accumulation implies here absolute over-accumulation (II) and vice versa.

B. $\hat{a} > \hat{K} > 0$. In LGM-I it is true only for $u > u^*$ when $\hat{v} < 0$. Let there is relative over-accumulation, $0 < \frac{\dot{u}}{1-u}$. So $\hat{S} \equiv \hat{v} - \frac{\dot{u}}{1-u} < 0$. Therefore relative over-accumulation imply absolute over-accumulation (II) again.

If u increases for $u \in (u^*, u_{\max})$ there is $\hat{v} < 0$ and the condition (14) holds – it simply turns into $v_{\max} > v > v^*$ (the actual employment ratio is lower than the maximal employment ratio and higher than stationary one) for the increased capital. We see that this secondary definition of *absolute* over-accumulation of capital (II) means *relative* over-accumulation of capital as profit rate (r) declines.

C. $0 < \hat{a} < \hat{K} = g$. In LGM-I with constant labour supply it is true only for $u < u^*$ when $\hat{v} > 0$. Let relative over-accumulation takes place when u moves from local minimum to u^* (respectively, employment ratio v from v^* to local maximum) then there is, first, no absolute over-accumulation II if (for u substantially lower than u^*) $\hat{v} > \frac{\dot{u}}{1-u} > 0$, second, no absolute over-accumulation I if II if (for u lower than u^*) $\hat{v} + \hat{a} > \frac{\dot{u}}{1-u} > 0$.

Since in the considered model $\hat{a} > 0$ during a cycle, the requirements for absolute over-accumulation of type I are stronger than for type II: whenever the first takes place the second also happen, but the opposite is not always true. Besides that, a period of relative over-accumulation is not much longer that that of absolute over-accumulation of type II in the same model due to constancy of the labour supply. On the other hand, owing to growing output per worker, a period of simultaneous relative and absolute over-accumulation of type I is shorter than period of simultaneous relative and absolute over-accumulation of type II.

We have established that in LGM-I, *relative* over-accumulation of capital is not a sufficient condition of *absolute* over-accumulation of capital of type I, although the latter is sufficient for the former and for *absolute* over-accumulation of capital of type II. In LGM-I, *relative* over-accumulation of capital is a necessary condition for *absolute* over-accumulation of capital of types I and II.

LGM-I (with s = const) excludes an increase of capital-output ratio (s) as an immediate course of declining profitability that systematically happens in reality. The other shortcoming – the rate of capital accumulation ($c = g/r$) is not explicit. The paper (Рыженков 2012) reports on a decoding of this key category (principle endogenous variable) as a step forward in understanding the social "genome".

Consider the chain of accumulation processes as a time consequence with leading, lagging and coinciding indicators. Table 1 presents a detailed picture of a cycle of capital accumulation on the progressively increasing scale. The ground for phase delineating in Table 1 is v ($\hat{v}$, $\hat{P}$, g, r and $\hat{a}$ are synchronic with each other and with $1 - u$, $\hat{u}$ is synchronic with v). A local minimum of net output P roughly corresponds to $\hat{v}_{\min}$, $\hat{v}$ leads v by one phase or a quarter of a cycle. The information in a box in this Table means that a magnitude of a particular variable is "close to or equals to the given."

Table 1 – Extrema and intermediate magnitudes of the main variables during a capital accumulation cycle in vicinity of a limit cycle for $\delta \approx \delta_0$ and $\pi = \delta u^*$

Variable	Crisis	Depression	Recovery	Boom
$\hat{S}$	0	max	0	min
$\hat{M}$	g^*	max	g^*	min
r	min	r^*	max	r^*
g	min	g^*	max	g^*
c	min	c^*	max	c^*
S	min	higher than minimum	max	lower than maximum
v	v^*	min	v^*	max
$\hat{w}^m$	$\hat{w}^{m*}$	min	$\hat{w}^{m*}$	max
$\hat{v}$	min	0	max	0
$\hat{a}$	min	g^*	max	g^*
$\hat{w}^b$	min	$\hat{w}^{b*}$	max	$\hat{w}^{b*}$
$\hat{u}$	0	min	0	max
u	max	u^*	min	u^*
$\hat{w}$	g^*	min	g^*	max

Notes. The rate of capital accumulation $c = g/r$; δ_0 is a critical magnitude for Andronov – Hopf bifurcation in LGM-I and LGM-II (see section 1.3), parameter π is introduced in section 1.3; a full cycle takes about 6,875 years for the chosen parameters' magnitudes.

Relative accumulation of capital pre-empts absolute II over-accumulation. Similarly, absolute II over-accumulation pre-empts absolute I over-accumulation. Just after achieving a maximum of surplus value (S), absolute II over-accumulation starts hidden by energized movement to local maximums of employment ratio, profit and net output (v, M, P) at the final stages of the boom. As a reminder that the true barrier of capital accumulation is capital itself the crisis suddenly becomes apparent with transition to a more severe I form of absolute over-accumulation.

Some typical properties of LGM-1 are explained additionally in Appendix, where Figures A.1 and A.2 shed light on capital over-accumulation and capital destruction afterwards near the limit cycle.

A falling to local minimums of profitability and profit finally goes together with a partial destruction of fixed capital that ends relative and absolute I over-accumulation. After that the economy sinks to local minimums of net output and fixed capital that ends absolute II over-accumulation. Even before the local minimum of fixed capital is attained, both profit and profitability already increase still accompanied by declining surplus value, employment and employment rate. Turnaround in profit and surplus value forestalls that of employment ratio. Adequate destruction of obsolete fixed capital, reduction of relative wage and resumption of capital accumulation – these interwoven processes altogether give rise to recovery and repeated movement to local maximums of profitability, surplus value, profit and employment ratio… Every such accumulation cycle promotes a spiral of capital accumulation.

LGM-I does not guarantee capitalist reproduction on a constant or increasing scale. We can sum up the worst case scenario in LGM-I when the positive feedback loops R1 and R2 (Figures 4 and 6) prevail and become unchecked (for $\delta < \delta_0$ – see section 1.3). At first, the worst case scenario is characterised by escalating absolute (I, II) and relative over-accumulation of capital when u moves from u^* to 1. Later, after u attains u_c and exceeds this threshold, destruction of fixed capital takes place accompanied by ever growing unemployment and declining net output ($\hat{P} = g < 0$ and $\hat{v} < 0$). In comparison with the local capital accumulation cycle (for $\delta \approx \delta_0$) described above this is mayhem.

1.3 Local stability of the non-trivial stationary state in LGM-I

The unforced continuous-time non-linear system include the equations (10) and (11). Consider the local behaviour of its solution near the stationary state $E^* = (v^*, u^*)$ in R^2 applying the linear system

$$\dot{y} = J^* y, \qquad (16)$$

where J^* is the Jacoby matrix, that can be solved explicitly. The equation (17) defines the Jacoby matrix for the stationary state $E^* = (v^*, u^*)$:

$$J^* = \begin{array}{|c|c|} \hline 0 & \left[g_u'(1-f_g')\right]v^* < 0 \\ \hline \gamma_v' u^* > 0 & -\left[-\delta'_u(u^*) + f'_g(u^*)g'_u(u^*)\right]u^* \\ \hline \end{array}\;. \qquad (17)$$

Notice, first, that it is not shown explicitly that the all partial derivatives are calculated for the stationary state, second, the determinant of this matrix is

$$|J^*| = -\left[g'_u(1-f'_g)\right]v^* \gamma'_v u^* > 0. \qquad (18)$$

We see a peculiarity of this particular heterodox macroeconomic model: since $\hat{v}$ depends only on u, u^* does not depend on v^*, although v^* depends on u^*. It creates an illusion of a purely technological nature of the stationary relative wage. This illusion was also common to the infant neoclassical economics that went even further by assuming heroically full employment ($v = v^* = 1$).[7]

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[7] Modern neoclassical economics applies the notion of natural rate of unemployment that is a denial of full employment equilibrium. Still even a prominent representative of the neoclassical economics conceded (Blanchard 2006: 8): "One might have hoped that, with 30 years of data, with clear differences in the evolution of unemployment rates and policies across countries, we would now have an operational theory of unemployment. I do not think that we do. Many theories have come and – partly – gone. Each has added a layer to our knowledge, but our knowledge remains very incomplete. To use a well worn formula, we have learned a lot, but we still have a lot to learn."

The present author would add that the system dynamics approach to the Marx theory of capitalist reproduction is very helpful not only for operational theory of unemployment but for filling gaps in knowledge of the economic school that pretends to be neoclassical (with implied emphasis on *classical*) as well.

Proposition 1. The stationary state $E^* = (v^*, u^*)$ is locally stable provided the profit sharing term absolutely dominates the endogenous technical change term: the trace of the matrix J^*, $\mathrm{Tr}(J^*) < 0$, i.e., $0 < \delta'_u(u^*) < \phi'_u(u^*)$; as both are negative, the meaning of an equivalent stability condition is absolute dominance, $|\delta'_u(u^*)| > |\phi'_u(u^*)|$ (Lordon 1995: 1413).

Proposition 2. If the stationary state $x^* = (v^*, u^*)$ is locally stable it is locally stable asymptotically. The proof by the present author uses the fact that the roots of the characteristic equation have negative real parts, it is omitted for brevity.

A genesis of cycles in the phase space in LGM-I

F. Lordon (1995) defines the profit sharing term in a typical example as a linear function of the relative wage

$$\delta(u) = -\delta u + \pi, \tag{19}$$

where $\delta'_u = -\delta < 0$.

Then the conclusion on Andronov – Hopf bifurcation in LGM-I can be easily proved (cf. Lordon 1995: 1415–1416). Consider the stationary state of the system (10) and (11) as dependent on the control parameter δ:

$$\dot{x} = 0 = F(x, \delta). \tag{20}$$

The determinant of the Jacoby matrix (J^*) differs from zero in our case for any possible non-trivial stationary state (x, δ) as $|J^*| > 0$. The implicit function theorem ensures that for every δ in a neighbourhood $Br\,(\delta_0) \in R$ of the parameter value δ_0 a unique stationary state x^* exists.

The stationary relative wage depends neither on π nor on δ. For fixed π the stationary employment ratio depends on this control parameter positively $\frac{\partial v^*}{\partial \delta} = (\gamma^{-1})'_v\, u^* = \frac{u^*}{\gamma'_v(v^*)} > 0$. Likewise the stationary employment ratio depends on π for fixed δ negatively $\frac{\partial v^*}{\partial \pi} = -(\gamma^{-1})'_v = -\frac{1}{\gamma'_v(v^*)} < 0$. Both these influences neutralize each other when $\pi = \delta u^*$ then v^* does not depend on both (δ and π).

Now we are well prepared to a number of important propositions. See *Appendix* (with equations numbers filling gaps in the preceding text) for their proofs.

Proposition 3. The requirements of the Hopf theorem is satisfied for the system (10) and (11) at x^* for

$$\delta_0 = -\phi'(u^*) = -f'_g(u^*)g'_u(u^*) > 0. \tag{21}$$

Then, according to the Hopf theorem, there exists some periodic solution bifurcating from $x^*(\delta_0)$ at $\delta = \delta_0$ and the period of fluctuations is about $2\pi/\beta_0$ ($\beta_0 = \lambda_1(\delta_0)/i = -\lambda_2(\delta_0)/i$). If a closed orbit is an attractor, it is usually called a *limit cycle*.

A period of the cycle in result of Andronov – Hopf bifurcation at $x^* = (v^*, u^*)$ for δ_0 is

$$T_c \approx 2\pi/\sqrt{|J(x^*)|} = 2\pi/\sqrt{-[g'_u(1-f'_g)]v^*\gamma'_v u^*}\,. \tag{25}$$

If the characteristic equation has a pair of complex conjugate roots, the period of fluctuations in vicinity of the stationary state is

$$T_c = 2\pi/\sqrt{-[g'_u(1-f'_g)]v^*\gamma'_v u^* - \frac{[(\delta + f'_g g'_u)u^*]^2}{4}}\,. \tag{26}$$

With rare exceptions, subjectivism and superficiality, due to known socio-economic reasons, mostly prevail over scientific search for hidden truth in writings of self-proclaimed neoclassical school.

Besides δ_0 two other critical magnitudes for the same control parameter are defined: δ_1 and δ_2 when the expression in the root is zero.

Proposition 4. If $\delta \in (\delta_2, \delta_1)$, the roots of the characteristic equation are complex conjugate (see Appendix).

In relation to δ_1 and δ_2, it is reasonable to consider two cases:

a) the stationary employment ratio is v^*, $\pi = \delta u^*$, the stationary profit sharing term $\hat{w}^{b*} = 0$;

b) the stationary employment ratio is X, $\pi \neq \delta u^*$, $\pi = \pi_X = g^* - \gamma(X) + \delta u^*$.

In both cases a) and b) the stationary employment ratios (v^*, X) do not depend on δ. In case b), the stationary profit sharing term $\hat{w}^{b*} = g^* - \gamma(X) < 0$ if $X > v^*$ and $\pi = \pi_X < \delta u^*$.

For $\delta \geq \delta_1$ there is no cycles on a transient to a stable node at all. The next proposition is straightforward.

Proposition 5. For $\delta_1 > \delta > \delta_0$ and $\pi \neq \delta u^*$ converging fluctuations to new stable focus x^* have a shorter period in vicinity of the stationary state than that of closed orbits in result of the Andronov – Hopf bifurcation at x^*: $T_\delta < T_c$. For diverging fluctuations around an unstable focus (when $\delta_2 < 0 < \delta_0$ and $\pi \neq \delta u^*$) a decrease in the control parameter δ in relation to δ_0 lengthens a period of a cycle in vicinity of the stationary state, so $T_\delta > T_c$.

For $\pi = \delta u^*$, an infinitesimal alteration of the control parameter δ in relation to δ_0 does not affect the period of fluctuations in vicinity of the stationary state that remains unchanged. Clearly, in agreement with the expressions (23) and (26) for $\pi = \delta u^*$ when v^* do not depend on δ, bigger increase in the control parameter promotes a period of converging fluctuations on a transient to the locally asymptotically stable stationary state (focus). Notice, that a period of the cycle is *ceteris paribus* longer for $\pi = \delta u^*$ as that for $\pi = 0$.

The influence of δ on amplitude of converging fluctuations on a transient to a focus x^* have been revealed by the present author through computer simulations for concrete functional forms suggested in the original paper (Lordon 1995). It has been found out that if an initial displacement from a stationary state is not minuscule a rate of growth of fixed capital (g) and other variables behave erratically and leave a region of economic viability.

Let us turn to the crux of the whole system dynamics story.

1.4 The modified model

The original papers written on stabilization of cycles of capital accumulation strangely missed the point of intimate link of stabilization policy with target employment ratio. The papers (Lordon 1995, 1997) pay no attention to dependence of the stationary employment ratio on the control parameters π and δ, similarly amplitude of transients to the fixed-point attractor is not mentioned altogether. This research is an attempt to fill these and other disturbing gaps.

Consider the possible definitions of both control parameters depending on a target employment ratio X. Particularly valuable is recognition of their contradictory influences. For simplicity, it is supposed that $\pi_b = 0$ in the equations (4) and (5) as implicitly in (Lordon 1995).

Case 1.

The stationary growth rate of profit sharing element is zero:

$\hat{w}^b = -\delta(u - u^*)$, $\hat{w}^{b*} = 0$, i.e., $-\delta u^* + \pi = 0$, $\pi = \delta u^*$, $v^* = \gamma^{-1}(g^*)$ for all δ; the stationary employment ratio (v^*) does not depend on δ.

Here a conformist stabilization policy strives to achieve a fixed stationary employment ratio that determines the target ($X = v^*$) instead of being determined by that (a motto is: “Avoid strong intervention in the natural course of market events”).

Case 2.

A fight back stabilization policy ("Bring change to Washington, D.C.") in which a target determines a stationary employment ratio $v^* = X$ either for fixed δ or for fixed π, thereby two possibilities follow from equivalence of the stationary growth rates of wage and output per worker

a) $\hat{w}^* = \gamma(X) - \delta u^* + \pi_X = g^*$

then

$$\pi_X = g^* - \gamma(X) + \delta u^*;$$

hereby the higher $v^* = X$, the lower $\pi = \pi_X$ for fixed δ and $\hat{w}^{b*} < 0$ for $\pi = \pi_X < \delta u^*$ as explained above;

b) $\hat{w}^* = \gamma(X) - \delta_X u^* + \pi = g^*$

then

$$\delta_X = \frac{\gamma(X) + \pi - g^*}{u^*},$$

hereby the higher $v^* = X$, the higher $\delta = \delta_X$ for fixed π.

Deliberate changes of magnitudes of the two control parameters have contradictory macroeconomic consequences. Policy optimization will be added for finding their best constellations.

The original model generates erratic movement if the initial state differs from a stationary state by 5-10 per cent or even less. A remedy for establishing more realistic behavioural patterns is putting sufficient restrictions from below and above on an implicit rate of capital accumulation and subsequently on a growth rate of fixed capital (g).

The implicit rate of capital accumulation in LGM-I is $c = g/r$ according to the equations (1) and (2) whereas a stationary implicit rate of capital accumulation is such that $1 \geq c^* \geq 0$ It is assumed that this definition and accompanied specifications overlooking the necessity of a proposed amendment are modified according to the compound equation (29) in LGM-II as the author's modification of LGM-I:

$c = 0$ if $g(u)/r < 0$ (29a)

$c = 1$ if $g(u)/r > 1$ (29b)

$c = g(u)/r$ if $1 \geq g(u)/r \geq 0$. (29c)

A growth rate of fixed capital is determined now by the equation (2a) that substitutes (2)

$$g = cr. \tag{2a}$$

A growth rate of fixed capital is thus restricted from below and above: the floor is zero, the ceiling is a profit rate. This modification does not alter most of the above Lordon propositions and does not affect a stationary state. Still period and amplitude of fluctuations outside vicinity of a stationary state is, as a rule, changed by this modification. A transient to a stationary state becomes more realistic thanks to stronger viability of capitalist reproduction.

Notice that after turning the rate of accumulation in the explicit auxiliary variable changing according to the equations (29a) – (29c) the causal loops structure of LGM-II differs from that of LGM-I. Still the current paper skips it for brevity.

LGM-II is a foundation for scenario I that is with the exception of the equation (29) completely based on the numerical example of sluggish stabilization of capital accumulation (for $\delta = 1{,}3 > \delta_0$) from (Lordon 1995). For brevity, detailed descriptions of this basal scenario and two other scenarios are given together in next sections (Tables 3 and 4, Figures 9–11, 16–17).

2. Optimization for improper structure in scenario II based on LGM-II

Consider dynamic policy optimization for LGM-II over parameters π and δ in a Vensim model. An optimization domain is set with rather wide rectangular boundaries facilitating an attaining of a high target employment ratio with the gained knowledge of this target connection with the both control parameters.

Let us define the optimization problem for LGM-II. The specific functional forms and initial magnitudes of the original parameters are also (as in scenario I) taken from the paper (Lordon 1995: 1421–1422). Numerical results follow.

$$Maximise\left[-\int_{T_0}^{T}|v - X|dt\right] \qquad (30)$$

subject to
$X = 0{,}95$,
$\dot{x} = f_{restricted}[x(t), \delta, \pi]$,
given initial $x_0 = (v_0, u_0)$, a_0, s_0, N_0,
$0 \le \delta \le 10 << \delta_1$, $-10 \le \pi \le 0$, $T_0 = 1$, $T = 16$.

The following magnitudes of parameters are found for scenario II compared with the other two (Table 2). Magnitudes in Table 2, including those for Lordon's δ= 1,3, are also calculated by the author of this paper himself. Tables 3 and 4, Figures 9–11, 16–18 present additional substantial features of the simulation runs framed as three scenarios of capital accumulation.

Table 2 – Magnitudes of parameters and variables in three models of capital accumulation

Parameter	LGM-II without optimization (scenario I)	LGM-II with optimization (scenario II)	LGM-III (satisfying in scenario III)
c_2			0,398
c_1			0,098 (set as g*)
π	0	–0,534	
δ	1,300	10	
π_b	0,916	7,253	
δ_1	8,236	39,476	
δ_2	–6,049	–37,289	
$\mathrm{Re}\lambda_{1,2}$	–0,069	–2,970	–0,4507
T_c	2,639	0,505	infinity
v^*	0,760	0,950	0,950
u^*	0,667	0,667	0,667
g^*	0,098	0,098	0,098
$\hat{w}^{b*}$	0,049	0,049	0,049
$\hat{w}^{m*}$	0,049	0,049	0,049

Notice: the Andronov – Hopf bifurcation takes place near the stationary state in LGM-I for $\delta_0 \approx 1{,}095$.

Tremendous over-shooting for profit, rate of profit and surplus value is the other side of colossal under-shooting for wage due to improper structure of social relations. This type of severe austerity ("short-term" for privileged and not experiencing it) would require draconian means of policy implementation. The described optimization policy leading to socially unsatisfactory outcomes could be rushed by false promises of fast improvements. In result the sound pragmatic idea of profit sharing and /or excess income levy optimization would be undermined or discredited.[8]

[8] There are also skeptical papers on this issue. For instance, the British authors (Blanchflower, Oswald 1987: 16) answered to their own rhetoric question: "Can profit sharing work? There is little empirical evidence that it can, and there are probably better ways of stimulating employment…"

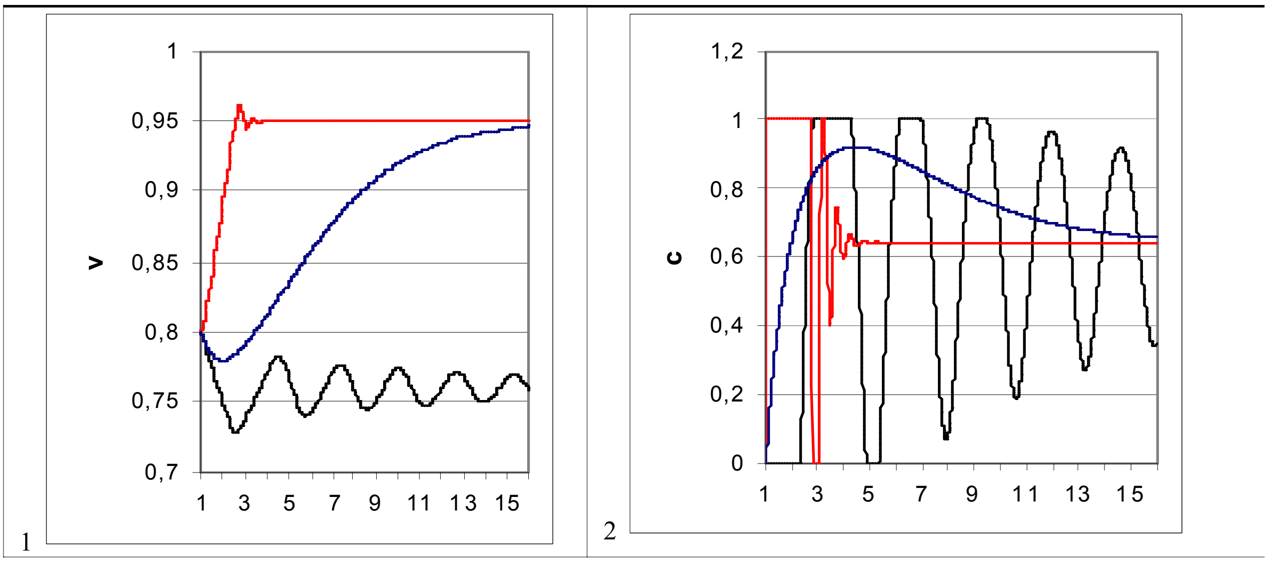


Figure 9 – Evolution in three scenarios over 1–16 (black – I, red – II, blue – III; 1 – employment ratio, v, 2 – rate of capital accumulation, c)

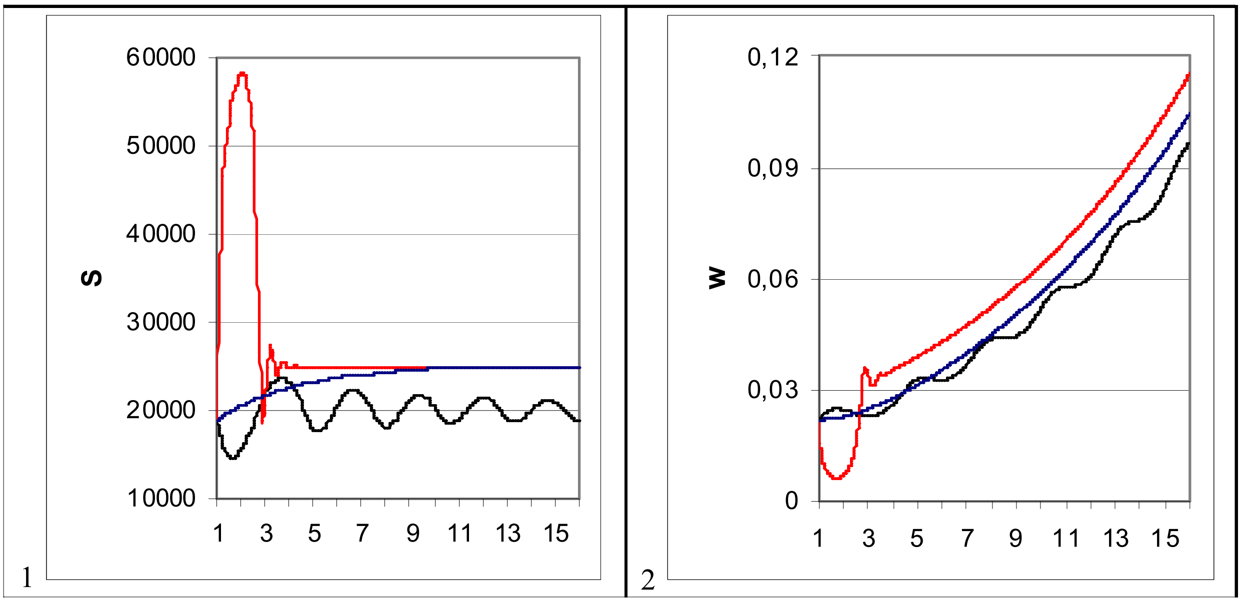
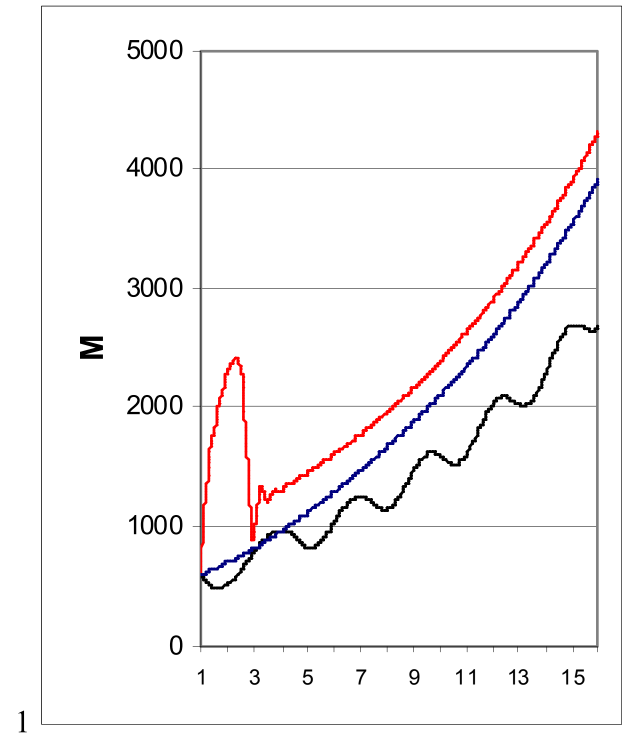


Figure 10 – Evolution in three scenarios over 1–16 (black – I, red – II, blue – III, 1 – surplus value, S, 2 – wage, w)

Table 3 – Summary statistics of surplus value and labour compensation in three scenarios for 1–16

Scenario	Surplus value, $(1 - u)L$				Labour compensation, w			
	Min	Max	Mean	Normalized standard deviation	Min	Max	Mean	Normalized standard deviation
I	14666	23600	19886	0,097	0,022	0,097	0,049	0,434
II	18614	58033	27804	0,312	0,006	0,115	0,058	0,488
III	18905	24889	23667	0,064	0,022	0,105	0,052	0,463



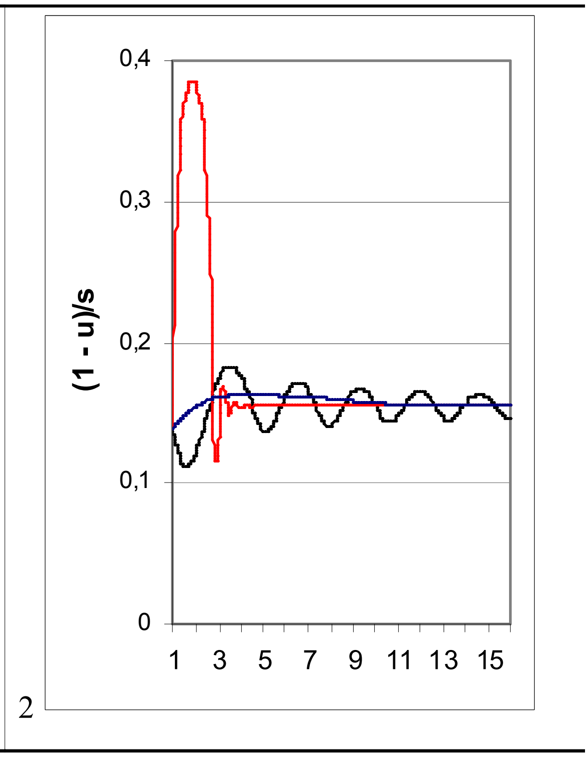


Figure 11 – Evolution in three scenarios over 1–16 (black – I, red – II, blue – III, 1 – profit, M, 2 – profit rate, r)

A recent paper outlines that "an understanding of the main feedback structure of a system, as provided by a small system dynamics model, is essential to effective policy design… effective small system dynamics models must not only be simple, but also include all of the most dominant loops (Ghaffarzadegan, Lyneis and Richardson 2011: 30, 40). Next section is a step forward in this direction.

3. Scenario III based on LGM-III

This section derives LGM-III as a thoughtful transformation of LGM-II aimed at achieving a greater coherence between income and employment policies. We start with designing causal loop diagrams, and after getting deeper insights with their help we write new equations.

It is reasonable to add two new negative feedback loops (Figures 12–13), containing only one level variable, namely relative wage (u), to the structure comprising the initial LGM-I. In the first of them, the growth rate of fixed capital fosters the rate of growth of profit sharing element, in the second of them, the growth rate of output per worker facilitates the rate of growth of profit sharing element as well. These new loops B3 and B4 neutralize first and second order positive FB loops for $\hat{a}$ already given above: B3 compensates R2, B4 compensates R1. They do not include the rate of capital accumulation (c).

The additional first order positive feedback loop provides stronger grip over the relative wage and consequently over profit rate (Figure 14). The rate of change of bargained element becomes positively dependent on relative wage.

Let us assume that the owners of capital, trade-unions representatives and the state officials agree on a desirable growth rate of total profit depending on a difference between an indicated (X_1) and current (v) employment ratios:

$$\hat{M} = c_2(X_1 - v), \qquad (31)$$

where $v < 1 < X_1$; it is assumed that the new control parameter c_2 is *positive*. A simple connection of the indicated (X_1) and target (X) employment ratios is given below – just after definition of a new stationary state by the equation (39).[9] As $\hat{M} = c_2(X_1 - v) > 0$, there is no absolute over-accumulation of capital of type I. Still possibilities of relative and/or absolute over-accumulation of capital of type II remain.

The following equation for the growth rate of total profit is valid for LGM-I and LGM-II, it remains valid for LGM-III as well:

$$\hat{M} = -\frac{\dot{u}}{1-u} + g \,. \tag{32}$$

A new partial dynamic law for the relative wage (unit value of labour power) is determined correspondingly as a consequence of the equations (31) and (32)

$$\dot{u} = (\hat{K} - \hat{M})(1-u) = [g + c_2(v - X_1)](1-u)\,. \tag{33}$$

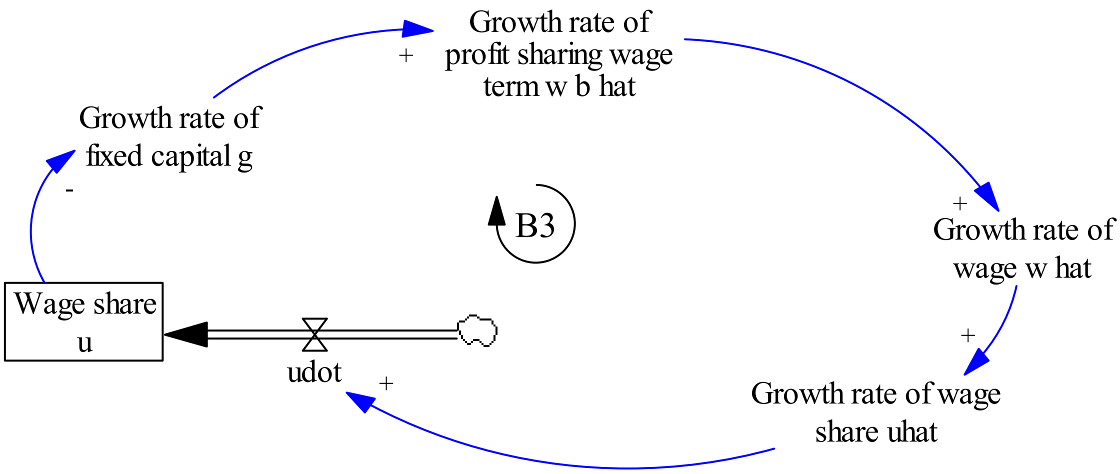


Figure 12 – The first order negative feedback loop B3 controlling unit labour value (u)

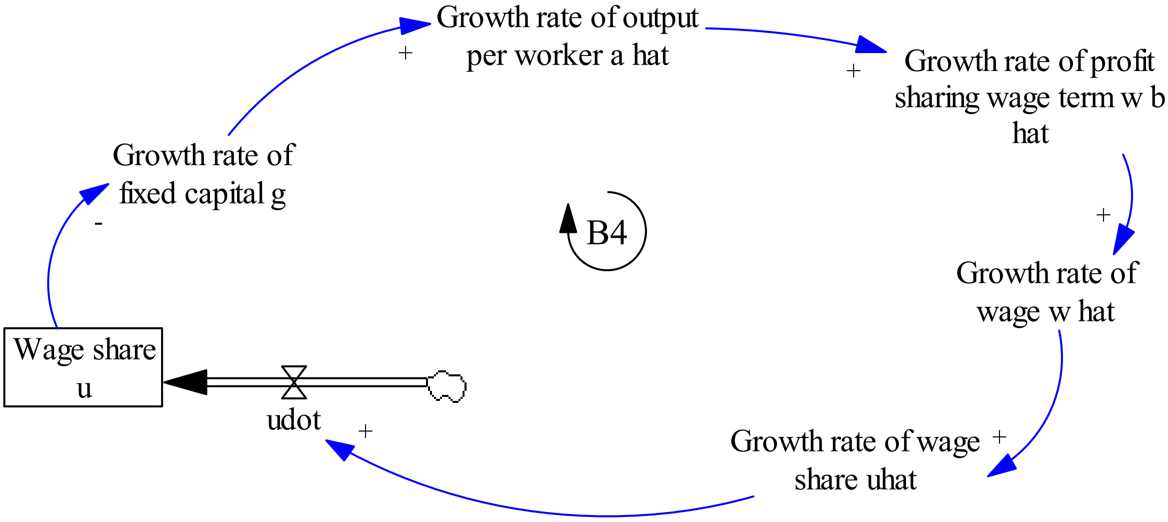


Figure 13 – The first order negative feedback loop B4 controlling unit labour value (u)

[9] The used negative linear dependence of the growth rate of profit on employment ratio (31) can be substituted by a suitable negative non-linear dependence. A corresponding non-linear monotonously declining differentiable function (that is not unique) should not have too high first derivative in absolute term, in other words, this function must not decrease excessively fast (see Figure 18).

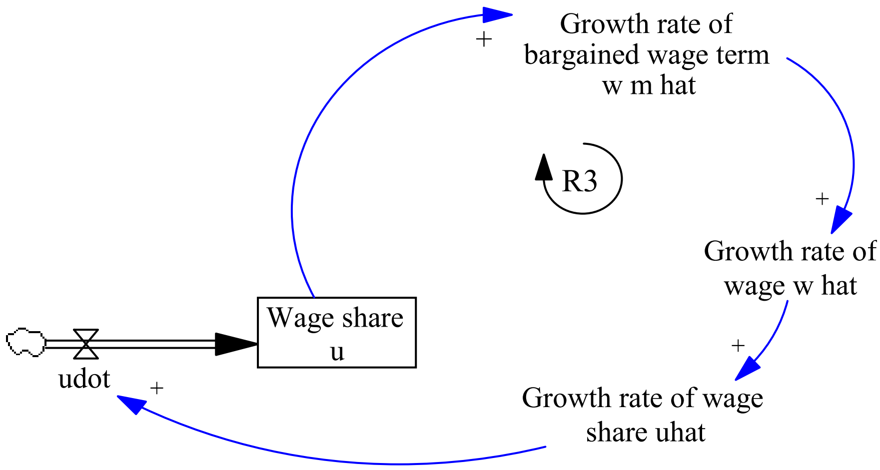


Figure 14 – The first order positive (for $v < X_1$) feedback loop R3 controlling unit labour value (u)

The intensive deterministic form of LGM-III consists of the initial equation (10) and the new equation (33) that substitutes the initial equation (11).

From the equation (3) a new equation for a growth rate of real wage follows:

$$\hat{w} = \hat{u} + \hat{a} = [g + c_2(v - X_1)]\frac{1-u}{u} + \hat{a}\,. \tag{34}$$

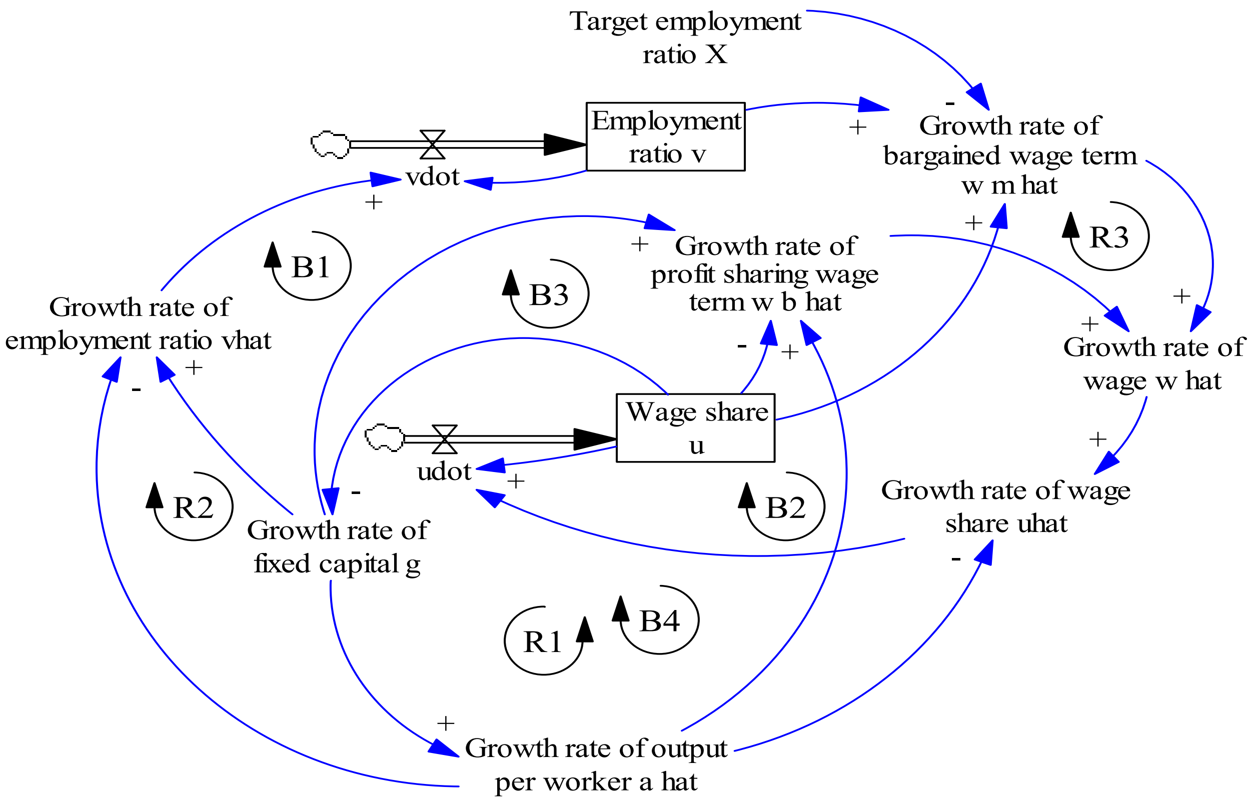


Figure 15 – The extensive causal loop structure of LGM-III with *target* employment ratio X

It is reasonable to detect analogues for two previously defined wage terms. The first is a bargained term

$$\hat{w}^m = \tau(v,u) \quad = \quad c_2(v - X_1)\frac{1-u}{u} + c_1, \tag{35}$$

$c_1 = \text{const} > 0$,
the second is the profit sharing term

$$\hat{w}^b = \hat{a} + g\frac{1-u}{u} - c_1. \qquad (36)$$

Adding an appropriate constant c_1 to $\hat{w}^m$ and deducting it from $\hat{w}^b$ allows getting $\hat{w}^{b*} > 0$ and $\hat{w}^{m*} > 0$ as required. Let $c_1 = g^* = 0{,}098$ for simplicity. The mathematical properties of the newly defined wage terms are explored deeper in Appendix.

Figure 15 presents the extensive causal-loop structure of LGM-III. It includes, in particular, the three additional loops (Figures 12–14).

The intensive deterministic form of LGM-III consisting of the initial equation (10) and new equation (33) serves for defining Jacoby matrix. It includes all the partial derivatives of the two phase variables (v and u):

$J_c =$

$\hat{v}$	$[g'_u(1-f'_g)]v < 0$
$c_2(1-u) > 0$	$-\frac{\dot{u}}{1-u} + g'_u(1-u)$

. (37)

Figures 2, 3 and 8 represent the causal-loop structure of the intensive deterministic form of LGM-III. The devil is in the details that are not visible in these Figures.

The equation (38) defines the Jacoby matrix for the stationary state (39)

$J_c^* =$

0	$[g_u'(1-f_g')]X < 0$
$c_2(1-u^*) > 0$	$g'_u(1-u^*) < 0$

, (38)

$$E_X^* = (X, u^*) \qquad (39)$$

where

$$X = X_1 - \frac{g^*}{c_2},\ \hat{w}^{b*} = g^*\frac{1-u^*}{u^*},\ \hat{w}^{m*} = c_2(X - X_1)\frac{1-u^*}{u^*} + g^* = -g^*\frac{1-u^*}{u^*} + g^* = g^*\frac{2u^*-1}{u^*};$$

for the given target employment ratio the indicated employment ratio is defined as $X_1 = X + \frac{g^*}{c_2}$.[10]

Proposition 6. The stationary state (39) in the upgraded system is always locally stable (for $c_2 > 0$).

Proposition 7. If $0 < c_2 \le c_2^{real}$ both roots of the characteristic equation are real and negative therefore a stationary state is a stable node.

Proposition 8. With the increases of $c_2 > c_2^{real}$ the period of converging fluctuations in vicinity of the stationary state becomes shorter.

If $c_2 > c_2^{real} > 0$ further increases in this control parameter have no influence on a real part of two complex-conjugate eigenvalues of the matrix J_c^*. Still with increases of c_2 over c_2^{real} the complex part becomes more and more important for transients to a stable focus; in simulations fluctuations with shorter and shorter periods become more visible due to growing amplitude.

If $0 < c_2 \le c_2^{real}$ there are no fluctuations at all and it is possible to have asymptotically and monotonously increasing surplus value (S) without absolute over-accumulation II (and without I as well). Satisfy-

[10] It is not shown explicitly that the all partial derivatives are calculated for the stationary state.

ing scenario III chooses $c_2 = c_2^{real}$ for avoiding fluctuations altogether. The upgraded structure and better designed policy lead to better results compared with scenarios I and II even without optimization.

Mohapatra and Sharma paper (1985: 70) suggests: "… it is desired that eigenvalues of the closed-loop system matrix should have negative real parts farther away from zero. Unfortunately, very large negative real parts of eigenvalues make the system highly responsive to input variables, invariably resulting in undesirable overshoots. A rule of thumb that is often followed is to aim for negative real parts of eigenvalues between –1 and –3."

We have seen that the 'receipt' reasonable for linear closed-loop systems becomes disastrous for non-linear closed-loop systems: as shown negative real parts of eigenvalues of a Jacoby matrix for a linearized system is safer to keep between –0,3 and –0,8 instead (Table 2). Besides that the imaginary parts of the eigenvalues of the Jacoby matrix for the linearized system are important determinants for transients to the stationary state.

We remember that for $c_2 > c_2^{real} > 0$ further increases in this control parameter have no influence on a real part of two complex-conjugate eigenvalues of the matrix J_c*, still the imaginary parts of these eigenvalues become more important for a period and amplitude of converging fluctuations to the stationary state.

Stabilization policy requires moderation of demands for wage increases by targeting employment ratio at a high level that substantially exceeds the initial one. At the onset and very beginning of the appropriate stabilization policy, the profit sharing term, initially negative, rises getting more priority whereas the role of non-traditional bargained wage term declines together with employment ratio; later this policy brings into balance both terms by weakening the former and strengthening the latter (Figure 16).

Thus we have confirmed that the Goodwin intuition (Goodwin 1990: 110) is correct: there is a successful moderation of real wages' growth in consequence of tightness of the labour market in exchange for high and stable employment along with rising output: when the growth rate of wage ($\hat{w}$) declines from a maximum to the stationary level (g*) the employment ratio comes closer and closer to the target (X).

Unlike the optimization run based on LGM-II, a transition period within one decade to relatively high employment avoids capital over-accumulation of types I and II completely, although the tendency of general profit rate to fall remains in the mitigated form (Figures 10–11). There are improvements (Table 4, Figures 9–10 and 17) in scenario III (compared with scenario I) of such economic indicators as employment ratio (v), net output (P), output per worker (a), wage (w) and total wage (wL).

The panel 2 of Figure 9 demonstrates that the restrictions on growth rate of fixed capital imposed by the equations (29) and (2a) are much more important for scenarios I and II than for scenario III that is almost unaffected by them. The average rate of capital accumulation (c_{mean}) in scenario III (0,750) is higher than in scenario II (0,676) and in scenario I (0,596). These properties and previous research for a more realistic setting speak for prospective efficiency of this policy in real capitalist economy.

Table 4 – Summary statistics of economic indicators in scenarios I and III for 1–16

Variable	Model	Min	Max	Mean	Normalized standard deviation
a	LGM-II	0,032	0,141	0,073	0,434
	LGM-III	0,032	0,157	0,079	0,457
P	LGM-II	1985	8437	4362	0,436
	LGM-III	1985	11719	5669	0,505
wL	LGM-II	1325	5766	2904	0,436
	LGM-III	1372	7805	3747	0,511

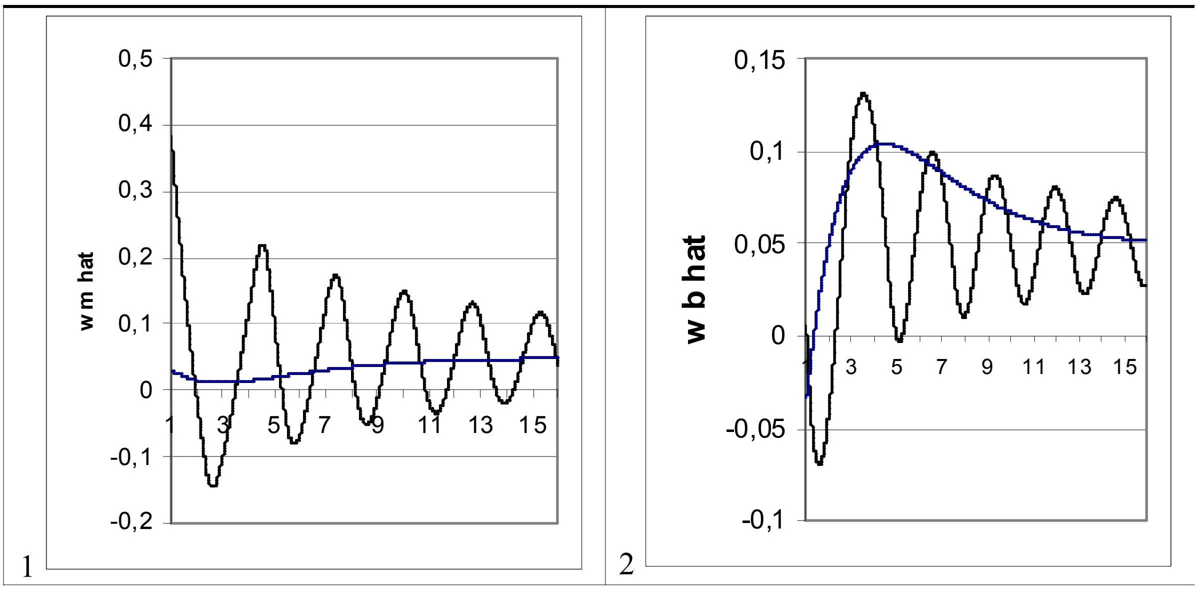


Figure 16 – Evolution in two scenarios over 1–16 (black – I, blue – III, 1 – growth rate of bargained wage term, $\hat{w}^m$ 2 – growth rate of profit sharing term, $\hat{w}^b$)

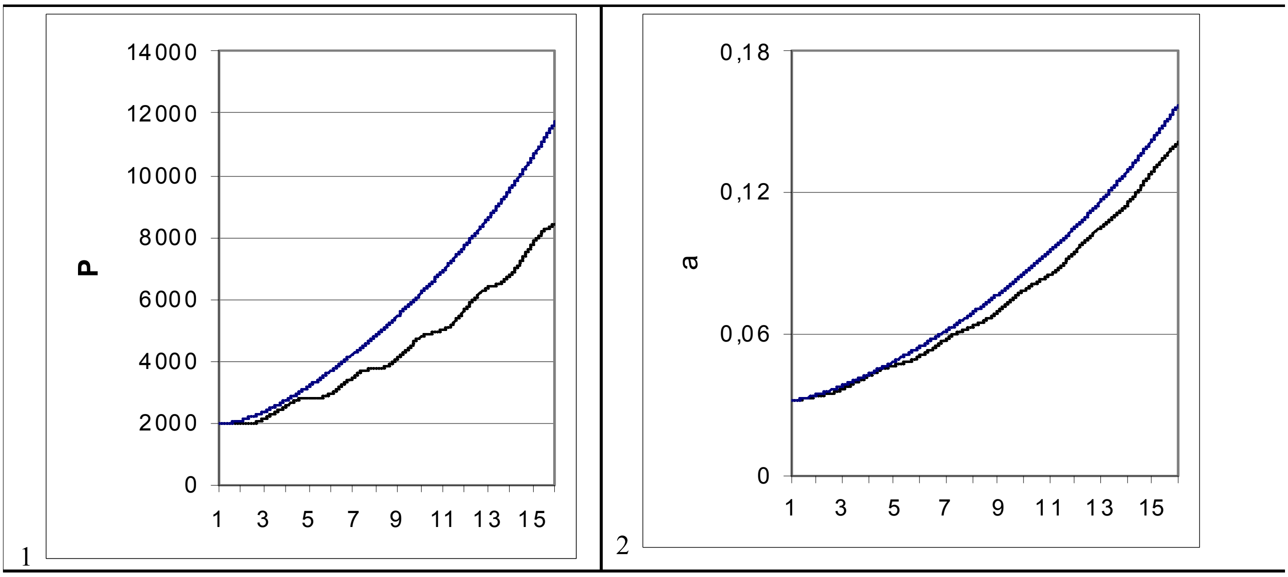


Figure 17 – Evolution in two scenarios over 1–16 (black – I, blue – III,
1 – net output, P, 2 – output per worker, a)

The direct causal links of the employment ratio v and target employment ratio X with growth rate of profit $\hat{M}$ in agreement with the equation (31) characterize only LGM-III and scenario III. These pivotal links are transformed in the links of the employment ratio v, target employment ratio X and wage share u with bargained wage term $\hat{w}^m$ (the equations (35) and (39), Figure 15). Indirect association of the employment ratio and growth rate of profit can be traced for the other two models and corresponding scenarios. Figure 18 displays vividly different shapes of this association for the three scenarios.

For appropriate policy not only the inverse relation between the growth rate of profit and employment ratio is important – at least as important is the slope and the range of the variables. A proper policy based on LGM-III supposes that a magnitude of parameter c_2 is in a suitable interval – especially dangerous would be spurt-type stabilization with $c_2 >> 1$ that will generate initially powerful oscillations combining imperfections of dynamics of scenarios I and II (panels 1 and 2 of Figure 18). Notice that this warning on a magnitude of c_2 is in agreement with the control theory developed for linear dynamical systems: the degree of stability thereby is characterized by the magnitudes of the eigenvalues of the system matrix, and not merely eigenvalues whose real parts are negative (Mohapatra, Sharma 1985: 65). Still the appropriate restrictions on parameter c_2 are to be examined additionally in context of more realistic models.

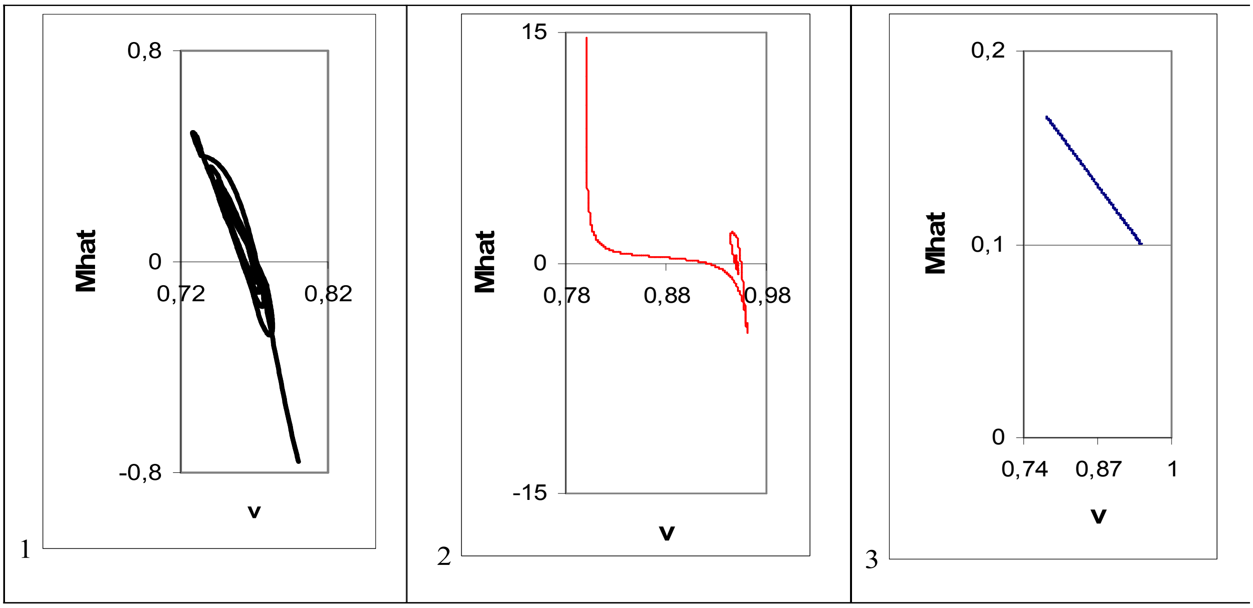


Figure 18 – Employment ratio (v) and growth rate of profit ($\hat{M}$) in three scenarios over 1–16 (1 – scenario I, 2 – scenario II, 3 – scenario III)

Conclusion

By focusing on wage policy issues, the original model of capital accumulation LGM-I (Lordon 1995) develops the embryonic form of stabilization policy neglecting the problem of target employment. This model hyperbolizes acuteness of accumulation crises to inappropriate degree. After correcting implicit basal functional relations for the rate of capital accumulation, LGM-I is transformed into LGM-II.

Applying the Structural Control Theory the present paper uncovers advantages of the overt intelligent closed-loop control that does not cause over-extending of the controlled economy in scenario III unlike ill-defined closed-loop control in scenarios I and II rooted in the original theoretical source. Based on examination of causal linkages, LGM-III is derived as radical improvement of both LGM-I and LGM-II. Not only profit sharing term but traditional bargained wage term as well is redesigned for timely attaining target employment. Transition to the stationary state in LGM-III is not unrealistically violent as in LGM-II or especially as in LGM-I. Austerity prescribed by improper optimization in scenario II based on LGM-II (and even more so if based on LGM-I) becomes unnecessary and avoidable in LGM-III.

LGM-III relies, as demonstrated, not only on negative feedback loops but on positive feedback loops as well. The positive feedback loops are instrumental for the pro-employment stabilization policy that would be hardly possible without them. LGM-III is not free from doubtful relationships common to all three models considered. These models suppose implicitly and wrongly negative dependence of the growth rate of capital intensity on unit labour value; they do not reflect workers' competition for jobs, assume constancy of capital-output ratio and consequently disregard importance of this variable both as a factor of rate of capital accumulation and as a factor of profitability. The assumption of constant labour supply is unwarranted as well. This listing of the "bugs" uncovered in the preceding sections could be continued.

The present author tries to avoid these inaccuracies in his own models. Still, instead of complete revision of the Lordon model (LGM-I), the middle way has been chosen and implemented in this paper: first, mostly confine critique by refining this model on its own assumptions, second, check structural robustness of policy rules developed for more sophisticated and realistic models with a help of that model. The present paper has restricted itself mainly to these tasks reporting on statistically oriented theoretic research in other publications (Ryzhenkov 2007, 2010, 2011).

The author believes that this paper also demonstrates in rather novel way realistic opportunities for moderating substantial dynamic inefficiency in modern capitalism and exposes its potential structural changes for maintaining resilience of the social organism. Efforts on the road ahead will be directed on ascending from abstract to concrete and enhancing empirical refutability of the steadily upgraded models of capital accumulation.

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Appendix

The mathematical properties of LGM-I, LGM-II and LGM-III

The proof of the Proposition 3

We assume the following properties are satisfied:
(a) the components of the function $F(x, \delta)$, corresponding to the system (10) and (11), are analytic (i.e. given by power series);
(b) the Jacoby matrix J^* (16) has a pair of pure imaginary eigenvalues;
(c) the derivative $\frac{d(\mathrm{Re}\lambda_{1,2}(\delta))}{d\delta} < 0$ for $\delta = \delta_0$ (it is the transversality condition);
(d) the stationary state $x^*(\delta_0)$ is asymptotically stable (for $\delta > \delta_0$).

Then, according to the Hopf theorem, there exists some periodic solution bifurcating from $x^*(\delta_0)$ at $\delta = \delta_0$ and the period of fluctuations is about $2\pi/\beta_0$ ($\beta_0 = \lambda_1(\delta_0)/i = -\lambda_2(\delta_0)/i$). If a closed orbit is an attractor, it is usually called a *limit cycle*.

The requirement (a) is obviously satisfied. The requirement (d) for $\delta > \delta_0$ is also true because Propositions 1 and 2 are valid. Consider the remaining two requirements.

The characteristic equation is

$$\lambda^2 + (\delta + f'_g\, g'_u)u^*\lambda - [g'_u(1 - f'_g)]v^*\gamma'_v u^* = 0.^{11} \tag{22}$$

It has two roots:

$$\lambda_{1,2} = \frac{-(\delta + f'_g\, g'_u)u^*}{2} \pm \sqrt{[g'_u(1-f'_g)]v^*\gamma'_v u^* + \frac{[(\delta + f'_g\, g'_u)u^*]^2}{4}}\ . \tag{23}$$

For $\delta = \delta_0$, they are pure imaginary $\lambda_{1,2} = \pm i\sqrt{|J^*|}$ (not divided by 2 – a misprint in Lordon 1995: 1416) because $\delta_0 + f'_g\, g'_u = 0.$ The requirement (b) is satisfied too.

The transversality condition (c) is true as well:

$$\frac{d(\mathrm{Re}\lambda_{1,2}(\delta))}{d\delta} = -\frac{u^*}{2} < 0 \text{ for } \delta = \delta_0. \tag{24}$$

Thus all four requirements are satisfied as stated (Lordon 1995), q.e.d.

The proof of the Proposition 4

The roots of the characteristic equation (22) are real if the expression inside the root in the equation (23) is non-negative. This expression equals zero for

$$-[g'_u(1-f'_g)]v^*\gamma'_v u^* = \frac{[(\delta + f'_g\, g'_u)u^*]^2}{4};$$

[11] It is not shown explicitly in this fragment that the all partial derivatives are calculated for the stationary state x^*.

after rearranging

$$-4\left[g'_u(1-f'_g)\right]v^*\gamma'_v u^* = \left[(\delta + f'_g g'_u)u^*\right]^2,$$

or

$$\frac{-4\left[g'_u(1-f'_g)\right]v^*\gamma'_v}{u^*} = (\delta + f'_g g'_u)^2,$$

finally we get two new critical magnitudes of the control parameter solving the latter quadratic equation

$$\delta_{1,2} = -f'_g g'_u \pm 2\sqrt{-\frac{\left[g'_u(1-f'_g)\right]v^*\gamma'_v}{u^*}}, \qquad (27)$$

where $\delta_1 >> \delta_0 > 0$ and $\delta_2 << -\delta_0 < 0$.

So only if $\delta \in (\delta_2, \delta_1)$, the roots of the characteristic equation (22) are complex conjugate, q.e.d.

The proof of the Proposition 5

The higher is the magnitude of expression inside the square root in the equation (26), the higher is the denominator and the shorter the period of a cycle. Consider dependence of the expression inside the square root on the control parameter δ at δ_0 by calculating a partial derivative (of the same expression):

$$-\frac{\partial\left[g'_u(1-f'_g)\right]v^*\gamma'_v u^* + \partial\left[(\delta + f'_g g'_u)u^*\right]^2/4}{\partial\delta} =$$

$$\left[-g'_u(1-f'_g)\right]\frac{\partial v^*}{\partial\delta}\gamma'_v u^* + \left[-g'_u(1-f'_g)\right]\frac{\partial\gamma'_v}{\partial\delta}v^* u^* - \frac{\left[(\delta + f'_g g'_u)u^*\right]u^*}{2} =$$

$$\left[-g'_u(1-f'_g)\right]\left(\frac{\partial v^*}{\partial\delta}\gamma'_v + \frac{\partial\gamma'_v}{\partial\delta}v^*\right)u^* - \frac{\left[(\delta + f'_g g'_u)u^*\right]u^*}{2}. \qquad (28)$$

Now substitute δ by δ_0 and get the required partial derivative that equals $\left[-g'_u(1-f'_g)\right]\left(\frac{\partial v^*}{\partial\delta}\gamma'_v + \frac{\partial\gamma'_v}{\partial\delta}v^*\right)u^* \geq 0$. When $\pi = \delta u^*$, v^* does not depend on δ therefore a whole partial derivative equals zero in the equation (28). When $\pi \neq \delta u^*$, v^* and γ'_v do depend on δ therefore a partial derivative equals $-g'_u(1-f'_g)u^{*2} + \left[-g'_u(1-f'_g)\right]\frac{\partial\gamma'_v}{\partial\delta}v^* u^* > 0$ as $\gamma'_v > 0$ and $\gamma''_v > 0$, q.e.d.[12]

Some additional explanations of LGM-I typical properties

The growth rate of surplus value equals the sum of growth rates of profitability and employment ratio

$$\hat{S} = \hat{r} + \hat{v}$$

therefore

$$\hat{S} = \hat{r} \Leftrightarrow \hat{v} = 0, \ \hat{S} < \hat{r} \Leftrightarrow \hat{v} < 0 \text{ and } \hat{S} > \hat{r} \Leftrightarrow \hat{v} > 0; \ \hat{r} = 0 \Leftrightarrow \hat{v} = \hat{S};$$

$$\hat{S} = \hat{v} \Leftrightarrow \hat{r} = 0, \ \hat{S} < \hat{v} \Leftrightarrow \hat{r} < 0 \text{ and } \hat{S} > \hat{v} \Leftrightarrow \hat{r} > 0; \ \hat{v} = 0 \Leftrightarrow \hat{r} = \hat{S}.$$

---

[12] The first term in the equation (28) $\left[-g'_u(1-f'_g)\right]\frac{\partial v^*}{\partial\delta}\gamma'_v u^* = -g'_u(1-f'_g)u^{*2} > 0$ as $\frac{\partial v^*}{\partial\delta} = (\gamma^{-1})'_v u^* = \frac{u^*}{\gamma'_v(v^*)}$, $g'_u < 0, f'_g < 1$. The second term is $\left[-g'_u(1-f'_g)\right]\frac{\partial\gamma'_v}{\partial\delta}v^* u^* = \left[-g'_u(1-f'_g)\right]\gamma''_v\frac{\partial v^*}{\partial\delta}u^* > 0$. The third term in the equation (28) for $\delta = \delta_0$ is $-\frac{\left[(\delta_0 + f'_g g'_u)u^*\right]u^*}{2} = 0$ as $\delta_0 = -f'_g g'_u$.

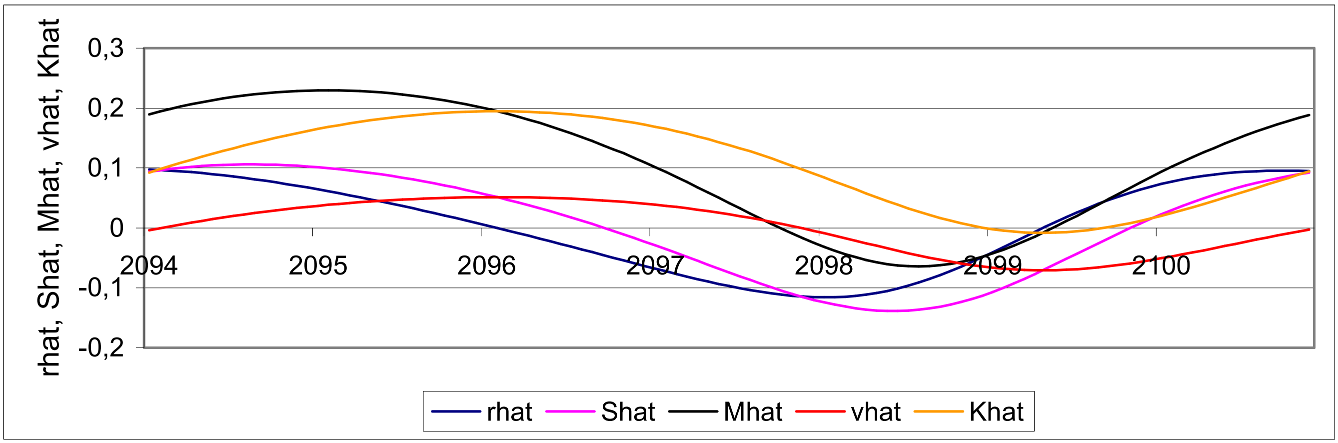

Figure A.1 – Growth rates of profit rate, surplus value, profit, employment ratio and fixed capital over a cycle approximating the limit cycle in LGM-I ($\delta \approx \delta_0$, $\pi = \delta u^*$) with a period of about 6,875 years

The growth rate of profit equals the sum of growth rates of
profitability, employment ratio and output per worker,
surplus value and output per worker,
profitability and fixed capital,
respectively

$$\hat{M} = \hat{r} + \hat{v} + \hat{a} = \hat{S} + \hat{a} = \hat{r} + \hat{K}\ .$$

Therefore (as $\hat{a} > 0$)

$$\hat{M} > \hat{S}\ ;$$

$$\hat{M} = \hat{r} < 0 \Leftrightarrow \hat{K} = g = 0 \ \text{ for } u = u_c = \omega;$$

$$\hat{r} = 0 \Leftrightarrow \hat{M} = \hat{K}\ .$$

Figures A.1 and A.2 illustrate above mentioned properties of capital over-accumulation and capital destruction afterwards in vicinity of the limit cycle. Notice that positive growth rates of profit and fixed capital coincide only at maximal profitability (for $u_{\min}$), whereas negative growth rates of profit and fixed capital – only at minimal profitability (for $u_{\max} > u_c = \omega$).

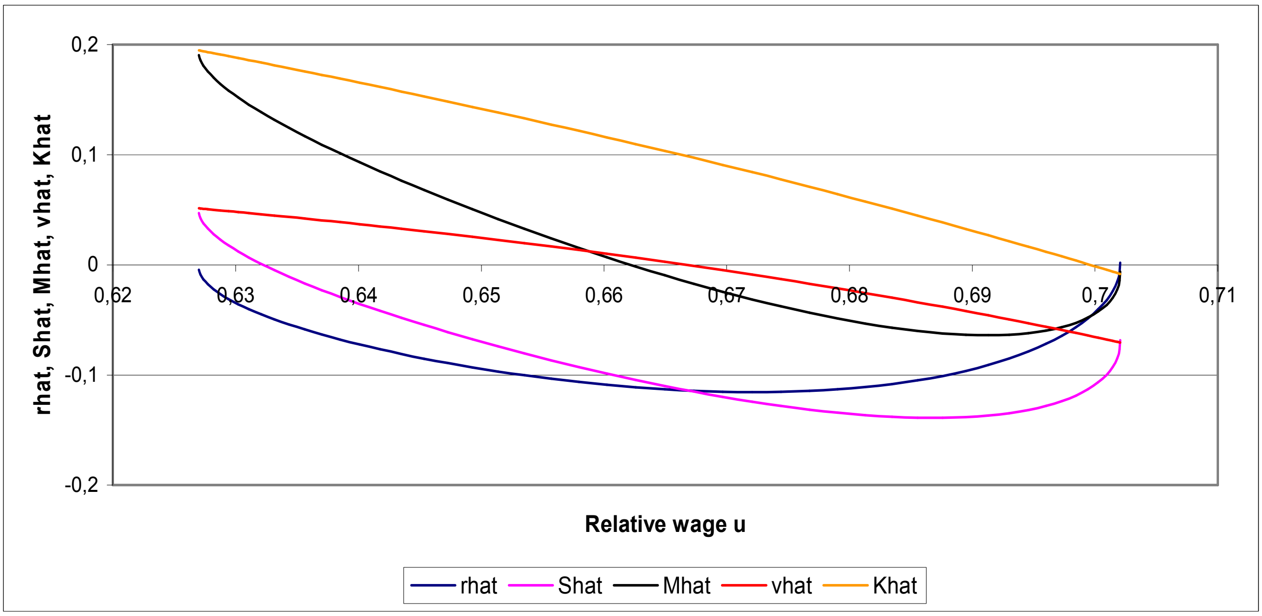

Figure A.2 – Relative → absolute II → absolute I capital over-accumulation near the limit cycle in LGM-I for $\delta \approx \delta_0$, $\pi = \delta u^*$, $u_{\min} \le u \le u_{\max}$, $u_{\max} > u_c = \omega$ (movement from the left to the right during about 47 per cent of the cycle period)

A deeper inspection of these Figures, reflecting computer simulations based on LGM-I, reveals additional properties. For example, for $\hat{M} = \hat{r} < 0$ not only inequalities $\hat{S} < \hat{M}$ and $\hat{S} < \hat{v} < 0$ are true but more specific relation $\hat{S} < \hat{v} < \hat{M}$ is visible. Similarly, for $\hat{M} = \hat{r} > 0$ not only inequalities $\hat{S} < \hat{M}$ and $\hat{S} > \hat{v}$ are true but more specific relation $\hat{v} < \hat{S} < 0 < \hat{M}$ is visible. Analytical proofs of these inferences are not available yet.

The mathematical properties of the wage terms in LGM-III

We have detected two wage terms. The first is a bargained term

$$\hat{w}^m = \tau(v,u) = c_2(v - X_1)\frac{1-u}{u} + c_1 \tag{35}$$

for $\tau'_u = c_2(X_1 - v)\frac{1}{u^2} > 0$ for $v < X_1$, $\tau'_u = c_2(X_1 - v)\frac{1}{u^2} \le 0$ for $v \ge X_1$, $c_1 > 0$, $\tau'_v = c_2\frac{1-u}{u}$, $\tau''_v = 0$, $\exists\ 1 > v_n = X_1 > 0$ such that $\tau(v_n) = 0$ and, finally, $\exists \lim_{v\to 1}\tau(v) = b << +\infty$, $\frac{\partial \hat{w}^m}{\partial u} = \tau'_u(v,u) = -c_2(v - X_1)\frac{1}{u^2} > 0$ for $v < X_1$, $\frac{\partial \hat{w}^m}{\partial v} = \tau'_v(v,u) = c_2\frac{1-u}{u} > 0$.

The second is the profit sharing term

$$\hat{w}^b = \hat{a} + g\frac{1-u}{u} - c_1. \tag{36}$$

Its first derivative for u such that $0 < c < 1$ is negative:

$$\frac{\partial \hat{w}^b}{\partial u} = f'_g\, g'_u + g'_u\frac{1-u}{u} - g\frac{1}{u^2} < 0$$

since $f'_g > 0$, $g'_u < 0$, $f'_g\, g'_u < 0$, $g'_u\frac{1-u}{u} < 0$ and $-g\frac{1}{u^2} < 0$; so there is place for inequality $\left|f'_g\, g'_u + g'_u\frac{1-u}{u}\right| > \left|g\frac{1}{u^2}\right|$.[13]

Notice the property of LGM-I and LGM-II $\gamma''_v > 0$ has gone in LGM-III. The latter has the property $\tau'_u > 0$ for $v < X_1$ that is absent in the two preceding models.

The proof of the Proposition 6

The determinant of the Jacoby matrix (38) for the stationary state (39) is

$$|J_c{*}| = -\lfloor g'_u(1 - f'_g)\rfloor X\ c_2(1 - u^*) > 0, \tag{40}$$

whereas its trace for the same stationary state (39) is

$$\mathrm{Tr}(J_c{*}) = g'_u(1 - u^*) < 0 \tag{41}$$

so the stationary state in the upgraded system of the equations (10) and (33) is always locally stable (for $c_2 > 0$), q.e.d.[14]

The proof of the Proposition 7

The characteristic equation is

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[13] The equations (35) and (36) are given again for the reader's convenience.

[14] A research on global stability of the stationary state (39) in LGM-III and on global stability of stationary state in LGM-I and LGM-II requires substantial additional efforts beyond the scope of this paper.

$$\lambda^2 - g'_u(1-u^*)\lambda - \left[g'_u(1-f'_g)\right]X\ c_2(1-u^*) = 0.^{15} \qquad (42)$$

It has two roots:

$$\lambda_{1,2} = \frac{g'_u(1-u^*)}{2} \pm \sqrt{\left[g'_u(1-f'_g)\right]Xc_2(1-u^*) + \frac{\left[g'_u(1-u^*)\right]^2}{4}}\ . \qquad (43)$$

If the expression in the root in the Eq. (43) is negative two complex-conjugate roots are

$$\lambda_{1,2} = \frac{g'_u(1-u^*)}{2} \pm i\sqrt{-\left[g'_u(1-f'_g)\right]Xc_2(1-u^*) - \frac{\left[g'_u(1-u^*)\right]^2}{4}}\ . \qquad (44)$$

Now they are never purely imaginary contrasting with LGM-I. Using (43) we determine a magnitude of c_2 that turns the expression in the root into zero:

$$c_2^{real} = -\frac{g'_u(1-u^*)}{4(1-f'_g)X}\ . \qquad (46)$$

The roots of the characteristic equation are real for $c_2 \leq c_2^{real}$ and complex conjugate for $c_2 > c_2^{real}$, q.e.d.

The proof of the Proposition 8

If the characteristic equation has a pair of complex conjugate roots, the period of fluctuations in vicinity of the stationary state is

$$T_c = 2\pi / \sqrt{-\left[g'_u(1-f'_g)\right]Xc_2(1-u^*) - \frac{\left[g'_u(1-u^*)\right]^2}{4}}\ . \qquad (47)$$

The higher is the magnitude of expression inside the square root, the higher is the denominator and the shorter the period of a cycle. For example, for $c_2 = 5 > c_2^{real} \approx 0{,}398$ roots are complex conjugate with negative real part; a period of a cycle with diminishing amplitude is 4,1 years.

Consider dependence of the expression inside the square root on the control parameter c_2 by calculating a partial derivative (of the same expression):

$$-\frac{\partial\left[g'_u(1-f'_g)\right]Xc_2(1-u^*) + \partial\left[g'_u(1-u^*)\right]^2/4}{\partial c_2} =$$

$$-\left[g'_u(1-f'_g)\right]X(1-u^*) > 0. \qquad (48)$$

We see that the higher is the control parameter $c_2 > c_2^{real}$, the shorter is the period of converging fluctuations T_c in vicinity of the stationary state, q.e.d.

Errata in Ryzhenkov (2010)*

Page	Is (emphasis added)	Should be (emphasis added)
15	$a_0 \approx$ **0.04521** millions 2005 dollars per person a year, $N_0 \approx$ **80705.1** thousands persons, $P_0 \approx$ **3520.7** billions 2000 dollars a year).	$a_0 \approx$ **0.05135** millions 2005 dollars per person a year, $N_0 \approx$ **80734.0** thousands persons, $P_0 \approx$ **4000** billions 2005 dollars a year).
18	As both $\frac{\partial \hat{w}}{\partial \hat{v}} > 0$ and $\frac{\partial \hat{w}}{\partial n} > 0$, **and** declining growth rates of employment ratio and of labour supply are detrimental for growth rate of real labour compensation if the all other conditions remain the same.	As both $\frac{\partial \hat{w}}{\partial \hat{v}} > 0$ and $\frac{\partial \hat{w}}{\partial n} > 0$, declining growth rates of employment ratio and of labour supply are detrimental for growth rate of real labour compensation if the all other conditions remain the same.

* These misprints have no effect on the remaining text. In particular, the simulations used the proper data.

[15] It is not shown explicitly in this fragment that the all partial derivatives are calculated for the stationary state x^*.