An Eigenvector Approach for Analysing Linear Feedback Systems

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Abstract

Formal analysis plays an important role in understanding how feedback structures drive dynamical behaviour. As we know the state behaviour is determined by a linear combination of behaviour modes (associated with eigenvalues). The weight of each mode is a product of a coefficient and a right eigenvector component. An emerging technique in eigen-based analysis focuses on the behaviour mode weight, together with the behaviour mode (eigenvalue), to identify the dominant feedback structure. The purpose of incorporating the weight analysis is to conduct an overall assessment of how feedback structure influences on the state behaviour. This paper revises the conventional eigensolution to the state trajectory by alternating the behaviour mode coefficient to be a product of the normalized left eigenvector, and the system initial conditions. Therefore, the overall behaviour changes due to the changes in a system element (a link or a pathway) can be fully assessed by calculating the eigenvalue, right and left eigenvector sensitivities. Through studying the eigenvector sensitivity, we observe that the right and left eigenvector sensitivities associated with the same mode cannot be evaluated separately. We present an analytical approach to the eigenvector-related sensitivity computation, i.e., a linear combination of the right and left eigenvector sensitivity.

1 Introduction

System dynamics is a study of the behaviour over time of variables of interest. This can be achieved by means of ordinary differential equations. Dynamical phenomena can be investigated by analyzing these mathematical representations.

Formal analysis is a way of investigating how feedback structure determine the behaviour. The objective is to gain insight into which loops dominant over time, so that policy actions can be developed to improve the system performance. A wide range of methods aimed at resolving this issue has been developed: behavioural method (Ford, 1999), pathway participation method (Mojtahedzadeh, 2001; Mojtahedzadeh et al., 2004), eigen-based method (Forrest, 1983; Goncalves, 2006; Saleh et al., 2009) which is the focus of this paper.

Eigenvalue elasticity analysis (EEA) is an approach for identifying the dominant feedback structure of a dynamic system. The core idea of EEA is addressed as follows: (1) decompose the behaviour of the variable of interest into a linear combination of behaviour modes, each of which is characterized by an eigenvalue; (2) eigenvalue elasticity with respect to various system elements (a link, a pathway or a feedback loop) is computed to assess the influence of that element; (3) the one yields the biggest elasticity is regarded as the dominant structure.

Eigenvector analysis is introduced in recent years. This paper presents a new approach on eigenvector analysis. First, we decompose the linear state behaviour into a finer detail, i.e., the coefficient of each behaviour mode is actually a product of the left eigenvector (associate with particular mode) and the initial conditions of the system. Then, based on the new representation of the solution, we can assess the overall change of the variable behaviour due to the change of a system element by taking the change from eigenvector into consideration. *Sensitivity* is adopted to measure the influence to the state due to the alternation of a certain element here. During the procedure of computing the eigenvector sensitivity, we find the right and left eigenvector sensitivities have to be calculated simultaneously. Furthermore, we present a formula to compute eigenvector sensitivity, making the computation without much effort.

This paper is organized as follows: first we will give a brief introduction to related researches and outline the background of linear systems. Then we will show a new solution adopted here to solve the linear systems and explore it by a linear system. Next, the following two sections deal with the overall state sensitivity and describe the procedure of calculating the eigenvector sensitivity with respect to different system elements (a link or a pathway) respectively. In addition, the previous linear example will be utilized again to clarify the approach of eigenvector sensitivity computation. Finally, we end the paper with the conclusion and future work.

2 Background research

2.1 Literature review

Formal analysis eigenvalue and eigenvector approaches are both derived from the linear system theory, where the state behaviour can be decomposed into the linear combination of behaviour modes. However, they focus on different parts of the solution.

Eigenvalue analysis was first introduced into feedback loop analysis by Forrester (1982). Many papers discussed EEA approach and applied it in both linear and linearized models (Kampmann, 1996; Goncalves et al., 2000; Saleh and Davidsen, 2000; AbdelGawad et al., 2005). One drawback over the conventional EEA is that it is used to identify the dominant feedback structure at the level of the model but fails to relate it to any selected variable of interest (Ford, 1999).

Saleh et al. (2006) went beyond the eigenvalue to the behaviour mode related weight, which is named as DDW, shorthand for dynamic decomposition weight. Moreover they exemplified the DDW approach by two simple business cycle models, in which both links and parameters are given small perturbance and the weight elasticity to parameters are assessed. As a consequence, the leverage points, e.g., the most influential parameters and links are identified. A good reason for using DDW analysis is that focusing on the weights, rather than on the eigenvalues, may be a more efficient way to develop policy recommendations. However, the DDW approach is conducted in a numerical way.

Meanwhile, other researchers (Goncalves, 2006) looked directly at the eigenvector and clearly proposed an analytical method to incorporate eigenvector analysis to the EEA. They demonstrated the eigenvector analysis by a linear inventory-workforce model. In the example, the eigenvalues and eigenvectors of the system are first represented by the loop gains and constants, then the eigenvector sensitivity with respect to the loop gain can be obtained by simply differentiating the equation on both sides. The second step of this method can be performed without much effort, yet the difficulty lies in the first step that transforming the eigenvalue and eigenvector to the representation of the loop gains is extremely hard especially when the system is of bigger size. A more crucial problem in this approach is that the eigenvector is not fixed, therefore it is impossible to have a unique representation by the loop gain, so does the eigenvector elasticity.

2.2 Background in analyzing linear dynamic system

Broadly speaking, we say a phenomenon represented by a stimulus-response mechanism is **linear** if, to a given change in the intensity of the stimulus, there corresponds a proportional change in the response. As concerns dynamical systems, a continuous system can be written as:

$$\frac{dx}{dt} = \dot{x} = f(x) \qquad x \in R^n \quad t \in R$$

The system is linear if function $f: R^n \to R^n$ satisfies:

$$f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$$

for any $\alpha, \beta \in R$ and $v, w \in R^n$. A linear system with n equations can be put in a compact matrix form, as

$$\dot{x} = Ax \qquad x(t_0) = x(0)$$

where $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \dot{x}_n \end{bmatrix}$, a vector of first time derivatives of the state variables $x(t)$; $x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix}$, a vector of state variables $x(t)$; $x(0)$ is also an n-by-one column vector representing the initial condition of the system; A is the compact gain matrix of this linear system, denoting by Eq. (1). Each entry represents the partial derivative of the net change of a state variable with respect to another state variable. In a linear system, all of its entries are constants .

$$A = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \cdot & \cdot & \frac{\partial \dot{x}_1}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial \dot{x}_n}{\partial x_1} & \cdot & \cdot & \frac{\partial \dot{x}_n}{\partial x_n} \end{bmatrix} \tag{1}$$

Several important concepts associated with matrix A are introduced here. **Eigenvalues** λs are a special set of scalars that each of which satisfies the following equation:

$$|A - \lambda I| = 0$$

where I is an n-by-n identity matrix. It is known as the **characteristic equation**. We assume the gain matrix A has n distinct eigenvalues. Thus, it has n independent **right eigenvectors** and **left eigenvectors** respectively. The right eigenvector r_i is an n-by-one column vector while the left eigenvector ℓ_i^H is an one-by-n row vector (the superscript H denotes the *conjugate transpose* for the case of the complex numbers), both of which are associated with eigenvalue λ_i and henceforth, we call these two vectors as an **eigenpair**.

3 General solution to linear systems: a second order example

We will present a general solution to linear systems which distinguishes itself from the conventional formula (below) adopted in eigen-based approaches (Güneralp, 2005; Goncalves, 2009).

$$x_i(t) = e^{t\lambda_1}\, r_{1i}\alpha_1^0 + e^{t\lambda_2}\, r_{2i}\alpha_2^0 + ... + e^{t\lambda_n}\, r_{ni}\alpha_n^0 \tag{2}$$

If we can normalize the eigenpairs of the system gain matrix in such a way that:

$$\ell_i^H r_j \begin{cases} = 1 & : \quad i = j \\ = 0 & : \quad i \neq j \end{cases} \tag{3}$$

The new solution of a linear system can be generated in Eq. (4). For the details of how to derive the solution, please see appendix A.

$$x_i(t) \quad = \quad e^{t\lambda_1} r_{1i}\, \ell_1^H x(0) + e^{t\lambda_2} r_{2i}\, \ell_2^H x(0) + ... + e^{t\lambda_n} r_{ni}\, \ell_n^H x(0) \tag{4}$$

Within each behaviour mode in the above solution, we find the term $\ell_i^H x(0)$ is a number generated via the vector multiplication, which is called the **coefficient** associated with that behaviour mode. Furthermore, it is interesting to notice that for the same behaviour mode in different state variables, the coefficient does not change while right eigenvector switches to its corresponding component. A note to the above solution is that all the eigenvectors are normalized to satisfy Eq. (3).

Compared with Eq. (2), the only difference lies in the two solutions is that Eq. (4) manages to decompose the mode coefficient α_j^0 in Eq. (2) into the left eigenvector ℓ_j^H and the system initial condition $x(0)$. We will see this decomposition provides us with an opportunity to assess the overall state sensitivity with respect to a certain system element.

To get a better understanding of the new decomposition method, we use a linear system from Strogatz (2000) to illustrate it. The linear system has only two state variables and no auxiliaries and its stock-flow diagram is shown in Figure 1.

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= 4x - 2y; \qquad (x_0, y_0) = (2, -3) \end{aligned}$$

The gain matrix can be obtained by definition:

$$A = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$$

Therefore, the characteristic equation of this linear system is:

$$0 = |A - \lambda I| = \begin{vmatrix} 1 - \lambda & 1 \\ 4 & -2 - \lambda \end{vmatrix} = (1 - \lambda)(-2 - \lambda) - 4 \times 1 = (\lambda + 3)(\lambda - 2)$$

hence the eigenvalues $\lambda_{1,2} = 2, -3$. As for the eigenvectors, they can be computed from the defini-

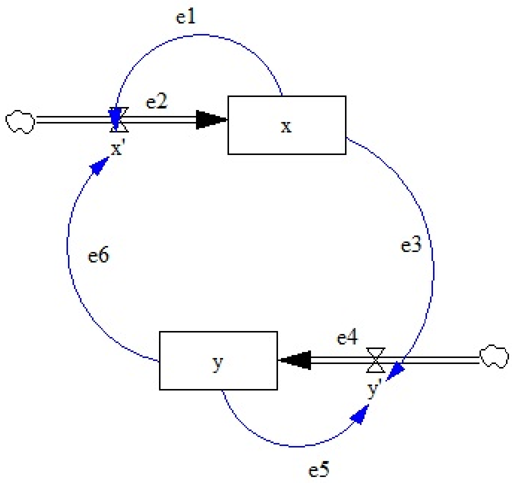

Figure 1: Stock and flow diagram of the linear system

tions using different eigenvalues.

$$\begin{aligned} Ar_i &= \lambda_i r_i \\ \ell_i^H A &= \lambda_i \ell_i^H \end{aligned}$$

We compute the eigenpair associated with $\lambda_1 = 2$ to clarify the eigenvector normalization.

$$\begin{aligned} 0 &= (A - \lambda_1 I) r_1 \\ &= \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} \end{aligned}$$

As two rows in $(A - \lambda_1 I)$ are proportional, to solve this equation is equivalent to solve: $-r_{11} + r_{12} = 0$, so the solution to the right eigenvector:

$$r_1 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where α is a nonzero scalar. The left eigenvector is obtained by following the same procedure.

$$\begin{aligned} 0 &= \ell_1^H (A - \lambda_1 I) \\ &= \begin{pmatrix} \ell_{11}^H & \ell_{12}^H \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \end{aligned}$$

As two rows in $(A - \lambda_1 I)$ are proportional, we only need to solve $-\ell_{11}^H + 4\ell_{12}^H = 0$. Therefore, the left eigenvector is computed:

$$\ell_1^H = \beta \begin{pmatrix} 4, & 1 \end{pmatrix}$$

where β is a nonzero scalar. Finally, we have to adjust the eigenpair to make them satisfy the the normalization condition. An easier way to accomplish is to let $\alpha{=}\frac{1}{\beta}$, the normalized eigenpair are turned out to be:

$$
\begin{aligned}
r_1 &= \alpha(1,1)^H \\
\ell_1^H &= \frac{1}{\alpha}(4/5, 1/5)
\end{aligned}
$$

Table 1 lists the eigenvalues and normalized eigenvectors generated by Matlab. Note that every eigenpair shown here is one of the infinite possible values.

Eigenvalue	Right eigenvector	Left eigenvector
$\lambda_1 = 2$	$r_1 = [0.7071,\ 0.7071]^H$	$\ell_1^H = [1.1314,\ 0.2828]$
$\lambda_2 = -3$	$r_2 = [-0.2425,\ 0.9701]^H$	$\ell_2^H = [-0.8246,\ 0.8246]$

Table 1: The eigenvalues and eigenvectors of the linear system

Constructing the solution by Eq. (4), produces:

$$
\begin{aligned}
x(t) &= e^{\lambda_1 t} r_{11} \ell_1^H x(0) + e^{\lambda_2 t} r_{21} \ell_2^H x(0) \\
&= e^{2t} * 0.7071 * (1.1314, 0.2828) \begin{pmatrix} 2 \\ -3 \end{pmatrix} + e^{-3t} * (-0.2425) * (-0.8246, 0.8246) \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\
&= e^{2t} + e^{-3t} \\
y(t) &= e^{\lambda_1 t} r_{12} \ell_1^H x(0) + e^{\lambda_2 t} r_{22} \ell_2^H x(0) \\
&= e^{2t} * 0.7071 * (1.1314, 0.2828) \begin{pmatrix} 2 \\ -3 \end{pmatrix} + e^{-3t} * 0.9701 * (-0.8246, 0.8246) \begin{pmatrix} 2 \\ -3 \end{pmatrix} \\
&= e^{2t} - 4e^{-3t}
\end{aligned}
$$

4 Sensitivity analysis of state behaviour

The state behaviour solution in Eq. (4) is the starting point for the sensitivity analysis. Our first step is to relate the system behaviour to the compact link. A compact link is an information connection from a state variable, e.g., x_q, to a net rate of a state, e.g., $\dot{x_p}$, whose gain is defined:

$$
a_{pq} = \frac{\partial \dot{x_p}}{\partial x_q} = A(p,q)
$$

From a broader sense, except the link relating a flow to a state, whose gain is by default to be 1 (inflow) or -1 (outflow) (Kampmann, 1996), the gain of any link connects two variables v_q to v_p is defined by taking partial derivative of the head, v_p with respect to the partial derivative of the tail, v_q:

$$
g_{pq} = \frac{\partial v_p}{\partial v_q}
$$

We can see the compact link gains are the same as entries in the gain matrix A, a perturbance of any entry of A can vary eigenvalues and eigenvectors. This would also change the state behaviour. To assess the impact to the state behaviour (Eq. (4)) due to the change in a compact link gain, sensitivity is introduced to measure this influence, i.e., taking partial derivative of the state x_i with respect to the compact link gain a_{pq}:

$$\begin{aligned}
x_i(t) &= e^{t\lambda_1} r_{1i}\, \ell_1^H x(0) + e^{t\lambda_2} r_{2i}\, \ell_2^H x(0) + \ldots + e^{t\lambda_n} r_{ni}\, \ell_n^H x(0) \\
&= \sum_{j=1}^{n} e^{t\lambda_j} r_{ji}\, \ell_j^H x(0) \\
\frac{\partial x_i(t)}{\partial a_{pq}} &= \sum_{j=1}^{n} \left(\frac{\partial e^{t\lambda_j}}{\partial a_{pq}} r_{ji}\, \ell_j^H\, x(0) + e^{t\lambda_j} \frac{\partial r_{ji}}{\partial a_{pq}}\, \ell_j^H\, x(0) + e^{t\lambda_1} r_{ji}\, \frac{\partial \ell_j^H}{\partial a_{pq}} x(0) \right) \\
&= \sum_{j=1}^{n} \left(\frac{\partial e^{t\lambda_j}}{\partial \lambda_j} \frac{\partial \lambda_j}{\partial a_{pq}} r_{ji}\, \ell_j^H\, x(0) + e^{t\lambda_j} \frac{\partial r_{ji}}{\partial a_{pq}}\, \ell_j^H\, x(0) + e^{t\lambda_j} r_{ji}\, \frac{\partial \ell_j^H}{\partial a_{pq}} x(0) \right) \qquad (5)
\end{aligned}$$

In a linear system, the gain matrix A is time-invariant and constant, so:

$$\partial e^{t\lambda_j} / \partial \lambda_j = t e^{t\lambda_j}$$

Substituting it to Eq. (5) yields:

$$\frac{\partial x_i(t)}{\partial a_{pq}} = \sum_{j=1}^{n} \left(t e^{t\lambda_j} \frac{\partial \lambda_j}{\partial a_{pq}} r_{ji}\, \ell_j^H\, x(0) + e^{t\lambda_j} \frac{\partial r_{ji}}{\partial a_{pq}}\, \ell_j^H\, x(0) + e^{t\lambda_j} r_{ji} \frac{\partial \ell_j^H}{\partial a_{pq}} x(0) \right) \qquad (6)$$

Eq. (6) evaluates the overall state sensitivity with respect to a compact link gain. It indicates the variation in the behaviour of $x_i(t)$ due to the change of link gain a_{pq} can be attributed to three sources:

1. $\partial \lambda_j / \partial a_{pq}$, eigenvalue sensitivity. The first term in equation captures the influence to a behaviour mode owing to the eigenvalue. The eigenvalue sensitivity with respect to the link gain a_{pq} is defined and calculated:

$$S_{a_{pq}}^{\lambda_j} = \frac{\partial \lambda_j}{\partial a_{pq}} = \ell_{jp} \times r_{jq} \qquad (7)$$

   where ℓ_{jp} is *pth* component of left eigenvector ℓ_j^H and r_{jq} is *qth* component of right eigenvector r_j. The proof of computing the eigenvalue sensitivity with respect to the compact link gain is in appendix B.

2. $\partial r_{ji} / \partial a_{pq}$, the ratio between the change of *ith* component in the right eigenvector r_j and the change in the compact link gain a_{pq}. It is an element in the entire right eigenvector sensitivity,

which is defined as follows:

$$S^{r_j}_{a_{pq}} = \frac{\partial r_j}{\partial a_{pq}} = \begin{bmatrix} \frac{\partial r_{j1}}{\partial a_{pq}} \\ . \\ . \\ \frac{\partial r_{jn}}{\partial a_{pq}} \end{bmatrix}$$

3. $\partial \ell^H_j / \partial a_{pq}$, the ratio between the change of the left eigenvector and the change of the compact link gain. Analogously, it is the left eigenvector sensitivity:

$$S^{\ell^H_j}_{a_{pq}} = \frac{\partial \ell^H_j}{\partial a_{pq}} = \begin{bmatrix} \frac{\partial \ell^H_{j1}}{\partial a_{pq}} & . & . & \frac{\partial \ell^H_{jn}}{\partial a_{pq}} \end{bmatrix}$$

Eq. (6) also suggests as the time increase, the influence of eigenvalue alternation gradually dominates the change of the behaviour mode. We use these new definitions to rewrite Eq. (6):

$$\frac{\partial x_i(t)}{\partial a_{pq}} = \sum_{j=1}^{n} e^{t\lambda_j} \left\{ tS^{\lambda_j}_{a_{pq}}\, r_{ji}\, \ell^H_j\, x(0) + S^{r_{ji}}_{a_{pq}}\, \ell^H_j\, x(0) + r_{ji}\, S^{\ell^H_j}_{a_{pq}} x(0) \right\} \tag{8}$$

Compared with the work from Goncalves (2009), we take the coefficient sensitivity (the last term in the above equation) into consideration as well. It is actually equivalent to the left eigenvector sensitivity multiplied by the initial conditions. We see that the decomposition of the coefficient makes the overall state sensitivity analysis possible as the state sensitivity is decomposed into a linear combination of the eigenvalue sensitivity and eigenvector sensitivity.

5 An analytical approach to the eigenvector sensitivity computation

Our focus now is to solve Eq. (8) analytically. Its first term, $tS^{\lambda_j}_{a_{pq}}\, r_{ji}\, \ell^H_j\, x(0)$, can be solved as the eigenvalue sensitivity is obtained by Eq. (7). However, a challenge remaining is to calculate the right and left eigenvector sensitivity.

Computing the derivatives of matrix eigenvectors is studied in the domain of structural optimization. It is important because these eigensolutions characterize the normal modes of vibration for structure modeled. The method proposed by Nelson (1976) is generally accepted as the most efficient and exact method for eigenvector sensitivity analysis. Since the eigenvector is not unique by a scalar, this method draws on the eigenvector normalization conditions to generate the eigenvector sensitivity. Nevertheless we will find it is hardly applicable in this particular eigenvector sensitivity problem.

5.1 Problems in the right eigenvector sensitivity

We first attempt to solve the right eigenvector sensitivity. Let A be a compact gain matrix of a linear system with n state variables and assume A has n distinct eigenvalues. In the following discussion, we will explicitly incorporate the eigenvector scalar to indicate its non-uniqueness. From the definition of eigenvector and eigenvalue, we get:

$$A\,(c_i r_i) = \lambda_i\,(c_i r_i)$$

where c_i is a scalar. Rearranging the above equation yields:

$$0 \;=\; (A - \lambda_i\, I)\, c_i r_i$$

In order to obtain the right eigenvector sensitivity with respect to the compact link gain, we take partial derivative of both sides in the above equation with regard to an entry in A, e.g., $A(p,q)$ and use the product rule:

$$\begin{aligned} 0 &= \frac{\partial (A - \lambda_i\, I)\, c_i r_i}{\partial A(p,q)} \\ &= c_i\,\frac{\partial (A - \lambda_i\, I)}{\partial A(p,q)}\, r_i + (A - \lambda_i\, I)\,\frac{\partial (c_i r_i)}{\partial A(p,q)} \\ &= c_i\,\frac{\partial A}{\partial A(p,q)}\, r_i - c_i\,\frac{\partial (\lambda_i\, I)}{\partial A(p,q)}\, r_i + (A - \lambda_i\, I)\,\frac{\partial (c_i r_i)}{\partial A(p,q)} \end{aligned} \tag{9}$$

For the first partial derivative in above equation, since each entry in A is independent of the others, so the result yields:

$$\frac{\partial A}{\partial A(p,q)} = \begin{pmatrix} \frac{\partial A(1,1)}{\partial A(p,q)} & & & \cdot \\ & \cdot & \frac{\partial A(p,q)}{\partial A(p,q)} & \\ & & \cdot & \\ \cdot & & & \frac{\partial A(n,n)}{\partial A(p,q)} \end{pmatrix} = \begin{pmatrix} 0 & & 0 \\ & \cdot\;\; 1 & \\ & \cdot & \\ 0 & & 0 \end{pmatrix}$$

The second term is a diagonal matrix with *ith* eigenvalue sensitivity to the link gain A(p,q) at its diagonal. In the last term, the partial derivative of $c_i r_i$ is what we are looking for, i.e., *ith* right eigenvector sensitivity. Expanding the above equation to see more in details:

$$\begin{aligned} 0 &= c_i \begin{bmatrix} 0 \\ \cdot \\ r_{iq} \\ \cdot \\ 0 \end{bmatrix} - c_i \begin{bmatrix} \frac{\partial \lambda_i}{\partial A(p,q)} & & 0 \\ & \cdot & \\ 0 & & \frac{\partial \lambda_i}{\partial A(p,q)} \end{bmatrix} \begin{bmatrix} r_{i1} \\ \cdot \\ r_{in} \end{bmatrix} + (A - \lambda_i\, I) \begin{bmatrix} \frac{\partial (c_i r_{i1})}{\partial A(p,q)} \\ \cdot \\ \cdot \\ \frac{\partial (c_i r_{in})}{\partial A(p,q)} \end{bmatrix} \\ &= c_i \begin{bmatrix} -S^{\lambda_i}_{a_{pq}}\, r_{i1} \\ \cdot \\ -S^{\lambda_i}_{a_{pq}}\, r_{ip} + r_{iq} \\ \cdot \\ -S^{\lambda_i}_{a_{pq}}\, r_{in} \end{bmatrix} + (A - \lambda_i\, I) \begin{bmatrix} S^{r_{i1}}_{a_{pq}} \\ \cdot \\ S^{r_{ip}}_{a_{pq}} \\ \cdot \\ S^{r_{in}}_{a_{pq}} \end{bmatrix} \end{aligned} \tag{10}$$

Unfortunately, Eq. (10) cannot be solved for $S^{r_i}_{a_{pq}}$ as the matrix $(A - \lambda_i I)$ has no inverse (it is singular). Moreover, the mathematical computation reflects a fundamental problem of our eigenvector sensitivity analysis: our definition of the eigenvector sensitivity is not precise enough. The definition of the eigenvector sensitivity:

$$S_{a_{pq}}^{r_i} = \frac{\partial c_i r_i}{\partial A(p,q)} = \frac{c_i^* r_i^* - c_i r_i}{A(p,q)^* - A(p,q)} = \frac{\Delta r_i}{\Delta A(p,q)}$$

where $A(p,q)^*$ is the new link gain and $c_i^* r_i^*$ is the eigenvector of the new matrix. The problem is neither $c_i r_i$ nor $c_i^* r_i^*$ is unique due to their scalars. We resort to the idea of normalization we have chosen in Eq. (3), however this does not provide us sufficient information to make the eigenpair unique: the normalization can only determine one eigenvector provided the other is given.

As we have identified the eigenvector sensitivity is not deterministic due to the non-uniqueness of the eigenvector itself, a straightforward solution can be adding another normalization to generate unique eigenvector. There are various normalizations available: $\|r_i\|_2 = 1$ (its norm is 1), $r_{ij} = 1$ (define a component of r_i to be 1) and so on. Nevertheless, these normalizations are not necessary here. The normalization provided in Eq. (3) is sufficient and satisfactorily expresses the eigensolution to the linear system. Before we proceed further to solve this problem, we would also have to look at the left eigenvector sensitivity.

5.2 The left eigenvector sensitivity

The left eigenvectors sensitivity can be derived following the same way as the right eigenvector sensitivity. From the definition of the left eigenvector, we have: $(\ell_i^H/c_i)A = \lambda_i(\ell_i^H/c_i)$. Differentiating it with respect to the compact link gain $A(p,q)$ on both sides and bringing them to the right hand side:

$$\begin{aligned} 0 &= \frac{\partial(\ell_i^H/c_i)}{\partial A(p,q)}(A - \lambda_i I) + \frac{\ell_i^H}{c_i}\frac{\partial(A - \lambda_i I)}{\partial A(p,q)} \\ &= S_{a_{pq}}^{\ell_i^H}(A - \lambda_i I) + \frac{\ell_i^H}{c_i}\left(\frac{\partial A}{\partial A(p,q)} - S_{a_{pq}}^{\lambda_i} I\right) \end{aligned} \tag{11}$$

Analogously, the row vector $S_{a_{pq}}^{\ell_i^H}$ is not unique and has one free variable in its solution. At this point, we have to make use of the normalization condition between the eigenpair:

$$(\ell_i^H/c_i)\, c_i r_i = 1 \tag{12}$$

which provides an extra constraint to determine that free variable. Taking partial derivative of the above equation with respect to the link gain $A(p,q)$, renders:

$$\begin{aligned} 0 &= \frac{\partial(\ell_i^H/c_i)}{\partial A(p,q)}(c_i r_i) + \frac{\ell_i^H}{c_i}\frac{\partial(c_i r_i)}{\partial A(p,q)} \\ &= S_{a_{pq}}^{\ell_i^H}(c_i r_i) + \frac{\ell_i^H}{c_i} S_{a_{pq}}^{r_i} \end{aligned} \tag{13}$$

By virtue of Eq. (11) and (13), the left eigenvector sensitivity can be solved on the condition that c_i and $S_{a_{pq}}^{r_i}$ are determined.

5.3 The implicit constraint on the eigenvector sensitivity

Let us return to the problem of seeking for a unique solution to the right eigenvector sensitivity, although we fail to solve it by the common way (drawing upon the normalization), we find out an implicit constraint in this specific application context, ***the state behaviour sensitivity with respect to the compact link gain in Eq. (8), which implies this value*** $(\partial x_i/\partial a_{pq})$ ***is a constant in spite of the non-unique value of the eigenvector sensitivity***. Let us change the state sensitivity in Eq. (8) to gather the eigenvector-related sensitivity. Since the behaviour modes are independent of each other, we can consider them separately. Within each behaviour mode, the term with the eigenvalue sensitivity is solvable and yields a constant value, so we can ignore it in the eigenvector-related sensitivity computation. The remaining terms contain eigenvector sensitivity, we will drop $e^{t\lambda_j}$ and the initial condition $x(0)$ as they do not affect the eigenvector either. For clarity, we use the formulas to demonstrate these steps below.

$$\begin{aligned}
\frac{\partial x_i(t)}{\partial a_{pq}} &= \sum_{j=1}^{n} e^{t\lambda_j}\left\{tS_{a_{pq}}^{\lambda_j}\, r_{ji}\, \ell_j^H\, x(0) + S_{a_{pq}}^{r_{ji}}\, \ell_j^H\, x(0) + r_{ji}\, S_{a_{pq}}^{\ell_j^H} x(0)\right\} \\
&\Longrightarrow\; e^{t\lambda_j}\left(tS_{a_{pq}}^{\lambda_j}\, r_{ji}\, \ell_j^H\, x(0) + S_{a_{pq}}^{r_{ji}}\, \ell_j^H\, x(0) + r_{ji} S_{a_{pq}}^{\ell_j^H} x(0)\right) \\
&\Longrightarrow\; e^{t\lambda_j}\left(S_{a_{pq}}^{r_{ji}}\, \ell_j^H\, x(0) + r_{ji} S_{a_{pq}}^{\ell_j^H} x(0)\right) \\
&\Longrightarrow\; S_{a_{pq}}^{r_{ji}}\, \ell_j^H\; +\; r_{ji} S_{a_{pq}}^{\ell_j^H}
\end{aligned} \tag{14}$$

The scalar is explicitly added to Eq. (14) and this is shown in Eq. (15).

$$\begin{aligned}
C_{ij} &= \frac{\partial (c_i r_{ij})}{\partial a_{pq}}\,\frac{\ell_i^H}{c_i} + c_i r_{ij} \frac{\partial (\ell_i^H / c_i)}{\partial a_{pq}} \\
&= S_{a_{pq}}^{r_{ij}}\, \frac{\ell_i^H}{c_i} + (c_i r_{ij})\, S_{a_{pq}}^{\ell_i^H}
\end{aligned} \tag{15}$$

where C_{ij} is a $1 \times n$ vector, and the notation is swapped as subscript i refers to the mode and j refers to the state variable x_j. Owing to this implicit constraint, we know that C_{ij} is a vector with constants, which will yield a constant value after multiplying $x(0)$. From previous analysis, we are aware that the left eigenvector sensitivity can be solved by knowing the associated right eigenvector sensitivity while the right eigenvector sensitivity has only one unknown. Therefore, we are going to use that unknown to express the eigenpair sensitivity, substitute them to Eq. (15) and examine the outcome. The previous example will be used again to explain the procedure. Without loss of generality, a general linear system will also be adopted when addressing this problem.

In the following, we deal with the dominant mode sensitivities (associated with $\lambda_1 = 2$) with respect to the compact link gain a_{11} in the concrete example and link a_{pq} of mode i in the general linear system. First, let us look at the solution to the right eigenvector sensitivity. Start with the Eq. (10). As we assume A has distinct eigenvalues, Eq. (10) is a linear system with $n - 1$ independent functions and n unknowns. Thus, there is one free variable in the solution. We choose one unknown as the free variable to represent the solution. Row reduction is used to produce the outcome as follows:

$$
\begin{aligned}
S^{r_{ik}} &: unknown \\
S^{r_{i1}} &= \alpha_{i1} S^{r_{ik}} + \beta_{i1} c_i \\
\dots & \\
S^{r_{in}} &= \alpha_{in} S^{r_{ik}} + \beta_{in} c_i
\end{aligned} \tag{16}
$$

where $S^{r_{ik}}$ is the *kth* component of *ith* eigenvector sensitivity, and α_i and β_i are both constant vectors. For the computation convenience, we use another eigenpair value associated with 1^{st} mode:

$$
r_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \ell_1^H = \left(\frac{4}{5}, \frac{1}{5}\right)
$$

The counterpart in the concrete example:

$$
\begin{aligned}
S^{r_{11}} &: unknown \\
S^{r_{12}} &= S^{r_{11}} - \tfrac{1}{5} c_1
\end{aligned}
$$

At this point, we start to evaluate the left eigenvector sensitivity. By utilizing Eq. (11), we can follow the similar calculation as in the right eigenvector sensitivity and the solution is given below:

$$
\begin{aligned}
& S^{\ell_{ik}^H} \\
S^{\ell_{i1}^H} &= u_{i1} S^{\ell_{ik}^H} + v_{i1}/c_i \\
\dots & \\
S^{\ell_{in}^H} &= u_{in} S^{\ell_{ik}^H} + v_{in}/c_i
\end{aligned} \tag{17}
$$

where $S^{\ell_{ik}^H}$ is the *kth* component in the *ith* left eigenvector sensitivity, u_i and v_i are constant vectors. It is easy to get the counterpart in the example:

$$
\begin{aligned}
& S^{\ell_{11}^H} \\
S^{\ell_{12}^H} &= \tfrac{1}{4} S^{\ell_{11}^H} - \frac{1}{25 c_1}
\end{aligned}
$$

It is time to make use of the eigenpair normalization. Let us substitute Eq. (16) and (17) for S^{r_i} and $S^{\ell_i^H}$ in Eq. (13). The solution to the left eigenvector sensitivity can be described by the unknown, i.e., a component in the right eigenvector sensitivity:

$$
\begin{aligned}
S^{\ell_{i1}^H} &= g_{i1} S^{r_{ik}}/c_i^2 + h_{i1}/c_i \\
\dots & \\
S^{\ell_{in}^H} &= g_{in} S^{r_{ik}}/c_i^2 + h_{in}/c_i
\end{aligned} \tag{18}
$$

where $g_i = \begin{bmatrix} g_{i1} \\ . \\ g_{in} \end{bmatrix}$, $h_i = \begin{bmatrix} h_{i1} \\ . \\ h_{in} \end{bmatrix}$, and are both constant vectors. Analogously, our example yields a similar outcome:

$$
\begin{aligned}
S^{\ell^H_{11}} &= -\frac{4}{5} S^{r_{11}}/c_1^2 + \frac{8}{125c_1} \\
S^{\ell^H_{12}} &= -\frac{1}{5} S^{r_{11}}/c_1^2 - \frac{3}{125c_1}
\end{aligned}
$$

Finally, recall the constraint in Eq. (15), if we replace the eigenpair sensitivities with their solutions, the equation reduces to be:

$$
\begin{aligned}
C_{ij} &= S^{r_{ij}}_{a_{pq}} \frac{\ell^H_i}{c_i} + (c_i r_{ij}) S^{\ell^H_i}_{a_{pq}} \\
&= (\alpha_{ij} S^{r_{ik}} + c_i \beta_{ij}) \frac{\ell^H_i}{c_i} + c_i r_{ij} (\frac{S^{r_{ik}}}{c_i^2} g^H_i + \frac{h^H_i}{c_i}) \\
&= \frac{S^{r_{ik}}}{c_i} (\alpha_{ij} \ell^H_i + r_{ij} g^H_i) + \beta_{ij} \ell^H_i + r_{ij} h^H_i \\
&= \mu^H_i \frac{S^{r_{ik}}}{c_i} + \sigma^H_i
\end{aligned} \tag{19}
$$

where $\mu_i = \begin{bmatrix} \alpha_{ij} \ell_{i1} + r_{ij} g_{i1} \\ . \\ \alpha_{ij} \ell_{in} + r_{ij} g_{in} \end{bmatrix}$ and $\sigma_i \begin{bmatrix} \beta_{ij} \ell_{i1} + r_{ij} h_{i1} \\ . \\ \beta_{ij} \ell_{in} + r_{ij} h_{in} \end{bmatrix}$. Meanwhile, we apply the same procedure to the example and obtain:

$$
\begin{aligned}
C_{11} &= S^{r_{11}} \frac{\ell^H_1}{c_1} + c_1 r_{11} S^{\ell^H_1} \\
&= S^{r_{11}} (\frac{4}{5c_1}, \frac{1}{5c_1}) + c_1 \times 1 \times (\frac{-4S^{r_{11}}}{5c_1^2} + \frac{8}{125}, \frac{-S^{r_{11}}}{5c_1^2} - \frac{3}{125}) \\
&= (\frac{8}{125}, \frac{-3}{125})
\end{aligned} \tag{20}
$$

Post-multiplying Eq. (19) with the initial conditions $x(0)$ and $e^{t\lambda_i}$ renders the eigenvector-related sensitivity in the state sensitivity in Eq. (8). Furthermore, we have two important conclusions derived from the above equation and analysis procedure:

1. If we choose the scalar c_i in a form as in Eq. (12), the scalar does not impact the eigenvector-related sensitivity in the evaluation of the state sensitivity. It is apparent in the concrete example, whose C_{11} does not involve the scalar c_1. In regard to the general form of the linear system, we can verify it by Eq. (19). The term $\frac{S^{r_{ik}}}{c_i}$ can be rearranged as: $\frac{S^{r_{ik}}}{c_i} = \frac{1}{c_i} \frac{\partial (c_i r_{ik})}{\partial a_{pq}} = \frac{\partial r_{ik}}{\partial a_{pq}}$. Subsequently, we can remove the scalar in Eq. (15):

$$
\begin{aligned}
C_{ij} &= \frac{\partial r_{ij}}{\partial a_{pq}} \ell^H_i + r_{ij} \frac{\partial \ell^H_i}{\partial a_{pq}} \\
&= S^{r_{ij}}_{a_{pq}} \ell^H_i + r_{ij} S^{\ell^H_i}_{a_{pq}}
\end{aligned} \tag{21}
$$

2. The concrete example shows the unknown disappears in Eq. (20) and leads us to believe it is not a coincident but a rule holds in other cases. In the generalized system, it is difficult to observe now. But if we expand all the constants along the eigenvector sensitivity computation process, we can get the same result: the unknown $S^{r_{ik}}$ is canceled out, as μ_i is a zero vector (the computation is complicated and omitted here, we provide another proof in appendix C). We prove that **for a specific behaviour mode i and a state x_j, the eigenpair-related sensitivity in the state sensitivity (in Eq. (15)) yields a constant vector (before multiplying the initial condition) and can be calculated as** *jth* **row in Eq. (22)** (details are presented in appendix C). This conclusion also indicates that the eigenpair sensitivity must be calculated simultaneously as they are dependent of each other.

$$C_i = -(A - \lambda_i I)^{\#} \frac{\partial A}{\partial a_{pq}} r_i \ell_i^H - r_i \ell_i^H \frac{\partial A}{\partial a_{pq}} (A - \lambda_i I)^{\#}, \tag{22}$$

5.4 Eigenvector sensitivity with respect to the pathway and causal link gains

Now let us look at how to develop eigenvector sensitivity with respect to pathway gain. Pathway (Mojtahedzadeh, 1997) is the path that starts from and end with a state variable (both can be the same variables), in addition, it does not contain any other pathways within itself. Let those pathways be P_{pq1}, P_{pq2}, ... , and P_{pqm}, where m is the number of pathways from x_q to x_p. We use g_{pq1}, g_{pq2}, ... , and g_{pqm} to represent those pathway gains respectively. Let the compact link gain from x_q to x_p be a_{pq}. If we sum up the gains of the pathways who share the same starting state and the ending states, they are equivalent to the corresponding compact link gain:

$$a_{pq} = g_{pq1} + ... + g_{pqm} \tag{23}$$

From the definition of the right eigenvector sensitivity with respect to a pathway P_{pqj}, we have:

$$S^{r_i}_{pqj} = \frac{\partial r_i}{\partial g_{pqj}}$$

By utilizing the chain rule and Eq. (23):

$$S^{r_i}_{pq_j} = \frac{\partial r_i}{\partial a_{11}} \frac{\partial a_{11}}{\partial g_{pq_j}} + ... + \frac{\partial r_i}{\partial a_{pq}} \frac{\partial a_{pq}}{\partial g_{pq_j}} + ... + \frac{\partial r_i}{\partial a_{nn}} \frac{\partial a_{nn}}{\partial g_{pq_j}} \tag{24}$$

where n is the number of states in the system. It is known that the partial derivative of the compact link gain to the pathway gain is zero when the pathway does not lie in the compact link, otherwise it is one by Eq. (23). Furthermore, one pathway can only contribute to one compact link. Therefore, Eq. (24) can be simplified as:

$$S^{r_i}_{pq_j} = \frac{\partial r_i}{\partial a_{pq}} \frac{\partial a_{pq}}{\partial g_{pq_j}} = \frac{\partial r_i}{\partial a_{pq}} \times 1 = S^{r_i}_{a_{pq}} \tag{25}$$

The above equation suggests that **the eigenvector sensitivity with respect to the pathway is equal to that with respect to the compact link gain where this pathway lies. Consequently,**

their eigenvector-related sensitivities are also the same.

Let us focus on the causal link. The pathway gain can be computed by multiplying its causal link gains (links between auxiliaries, states and constants):

$$g_{p_u} = \prod_{e_k \in P_u} g_{e_k} \tag{26}$$

We are aware that one causal link can contribute to more than one pathways. By utilizing the chain rule, we can write the eigenvector sensitivity with respect to the causal link gain:

$$S^{r_i}_{e_k} = \frac{\partial r_i}{\partial g_{e_k}} = \frac{\partial r_i}{\partial g_{p_1}} \frac{\partial g_{p_1}}{\partial g_{e_k}} + ... + \frac{\partial r_i}{\partial g_{p_s}} \frac{\partial g_{p_s}}{\partial g_{e_k}} \tag{27}$$

where subscript s denotes the number of total pathways in the system. Eq. (26) suggests:

$$\partial g_{p_u} / \partial g_{e_k} \begin{cases} = g_{p_u} / g_{e_k} & : \quad P_u \in e_k \\ = 0 & : \quad otherwise \end{cases}$$

Therefore, Eq. (27) is rewritten as:

$$S^{r_i}_{e_k} = \sum_{e_k \in p_u} \frac{\partial r_i}{\partial g_{p_u}} \frac{g_{p_u}}{g_{e_k}} = \sum_{e_k \in p_u} \frac{g_{p_u}}{g_{e_k}} S^{r_i}_{p_u} \tag{28}$$

The equation above shows **the eigenvector sensitivity to the causal link gain is equal to the summation of the sensitivity to the pathway which contain the causal link multiplied by their gain ratio**. For the eigenvector-related sensitivity with respect to causal link gain, we can calculate it as follows:

$$\begin{aligned}
C^{(e_k)}_{ij} &= S^{r_{ij}}_{e_k} \ell^H_i + r_{ij} S^{\ell^H_i}_{e_k} \\
&= \left(\sum_{u=1}^{s} \frac{g_{p_u}}{g_{e_k}} S^{r_{ij}}_{p_u} \right) \ell^H_i + r_{ij} \left(\sum_{u=1}^{s} \frac{g_{p_u}}{g_{e_k}} S^{\ell^H_i}_{p_u} \right)_{e_k \in P_u} \\
&= \sum_{u=1}^{s} \frac{g_{p_u}}{g_{e_k}} \left(S^{r_{ij}}_{p_u} \ell^H_i + r_{ij} S^{\ell^H_i}_{p_u} \right) \\
&= \sum_{u=1}^{s} \frac{g_{p_u}}{g_{e_k}} \left(S^{r_{ij}}_{a_{pq}} \ell^H_i + r_{ij} S^{\ell^H_i}_{a_{pq}} \right)_{p_u \in a_{pq}} \\
&= \sum_{u=1}^{s} \frac{g_{p_u}}{g_{e_k}} C^{(a_{pq})}_{ij}
\end{aligned} \tag{29}$$

Eq. (25) and (28) show the relationships between different types of eigenvector sensitivity and Eq. (29) proves the eigenvector-related sensitivity with respect to the causal link gain can be obtained by a series of linear operations on the counterpart with respect to the compact link gain.

6 Application to a Linear System

We return to the linear example (see Eq. (30)) for two purposes: elaborate the computation of eigenvector-related sensitivity provided in appendix C (Matlab is used in the computation); demonstrate how to assess the eigenvector-related sensitivity with respect to different system elements. The variable of interest is x and we have obtained its solution in previous discussion (Eq. (31)).

$$\begin{aligned} x &= x + y \\ y &= 4x - 2y; \qquad (x_0, y_0) = (2, -3) \end{aligned} \tag{30}$$

$$x(t) = e^{2t} + e^{-3t} \tag{31}$$

The trajectory of x will be growing exponentially steered by the first mode whereas the second behaviour mode will dissipate as time goes. Thereby we focus on the dominant mode associated with $\lambda_1 = 2$. The stock-flow diagram of this system is presented in Figure 2 to assist our analysis.

Let us begin with constructing the group inverse of $(A - \lambda_1 I)$, $(A - \lambda_1 I)^{\#}$. The eigenvectors are based on Table 1. First we configure matrix P whose columns form an orthonormal basis for $(A - \lambda_1 I)$:

$$P = \begin{bmatrix} -0.2425 \\ 0.9701 \end{bmatrix}$$

Hence it is not difficult to obtain the following results by Matlab. The notations are explained as follows: $[\,|\,]$ isolates columns in a matrix; $[—]$ isolates rows in a matrix; $(\cdot)^{\#}$ represents a group inverse matrix.

$$W = [r_1 | P] = \begin{bmatrix} 0.7071 & -0.2425 \\ 0.7071 & 0.9701 \end{bmatrix}$$

$$W^{-1} = \begin{bmatrix} \ell_1^H \\ \hline P^H(I - r_1 \ell_1^H) \end{bmatrix} = \begin{bmatrix} 1.1314 & 0.2828 \\ -0.8246 & 0.8246 \end{bmatrix}$$

$$\begin{aligned} (A - \lambda_1 I)^{\#} &= W \begin{pmatrix} 0 & 0 \\ 0 & \{P^H(A - \lambda_i I)P\}^{-1} \end{pmatrix} W^{-1} \\ &= \begin{pmatrix} -0.04 & 0.04 \\ 0.16 & -0.16 \end{pmatrix} \end{aligned}$$

Compact link		x→ $\dot{x}_{:A(1,1)}$	y→ $\dot{x}_{:A(1,2)}$	x→ $\dot{y}_{:A(2,1)}$	y → $\dot{y}_{:A(2,2)}$
Eigenvector-related sen.	*before*	$[0.064, -0.024]$	$[-0.096, 0.136]$	$[-0.024, -0.016]$	$[-0.064, 0.024]$
	after	0.2	-0.6	0	-0.2
Eigenvalue sen.		0.8	0.8	0.2	0.2

Table 2: Eigenvector-related and eigenvalue sensitivity of 1^{st} mode

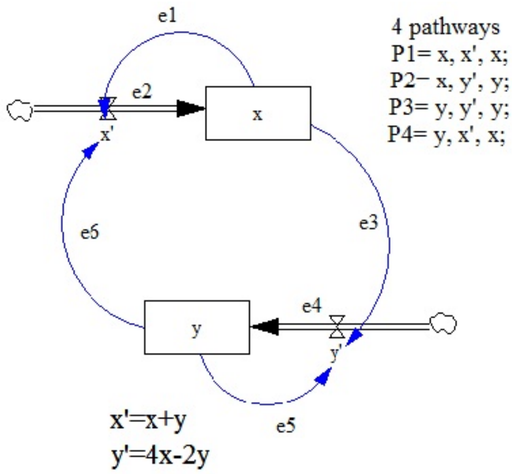

Figure 2: Stock and flow diagram of the linear system

The eigenvector-related sensitivity with respect to a_{11} in terms of 1^{st} mode is given by Eq. (22):

$$
\begin{aligned}
C_1^{(a_{11})} &= -(A-\lambda_1 I)^{\#}\frac{\partial A}{\partial a_{11}} r_1 \ell_1^H - r_1 \ell_1^H \frac{\partial A}{\partial a_{11}}(A-\lambda_1 I)^{\#} \\
&= -\begin{pmatrix} -0.04 & 0.04 \\ 0.16 & -0.16 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix}\begin{pmatrix} 1.1314 & 0.2828 \end{pmatrix} \\
&\quad -\begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix}\begin{pmatrix} 1.1314 & 0.2828 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -0.04 & 0.04 \\ 0.16 & -0.16 \end{pmatrix} \\
&= \begin{pmatrix} 0.064 & -0.024 \\ -0.096 & -0.064 \end{pmatrix}
\end{aligned}
$$

The first row of $C_1^{a_{11}}$ contributes to the eigenvector-related sensitivity in terms of state variable x while the second row corresponds to the counterpart in y. Compare the first row with the result we get in Sec. 5.3 by another method, we obtain identical answer: $[8/125, -3/125]$. The eigenvector-related sensitivity with respect to the compact link gain before and after multiplying by the initial conditions are shown in Table 2, as well as the eigenvalue sensitivity.

From Figure 2, we identify four pathways and lie in four different compact links. Table 3 lists the pathway gains and the compact links they pass through. In addition to the conclusion that the pathway sensitivity is equal to the corresponding compact link sensitivity, so their eigenvector-related sensitivities are also equivalent as depicted in Table 2 and Table 3. The computation of the eigenvalue sensitivity with respect to pathway gain or causal link gain is out of the scope, readers can find more details in AbdelGawad et al. (2005).

The causal links are marked in Figure 2. Their sensitivities can be assessed by adding up the pathways sensitivity which pass through that link multiplying their gain ratio, so does the eigenvector-related sensitivity which is computed by Eq. (29). For example, the eigenvector-

Pathway	*Gain*	*Compact link*	*Eigenvector−related sen.*	*Eigenvalue sen.*	*State sen.*
$P1$	$g_{e_1} * g_{e_2} = 1$	$x \to \dot{x}$	0.2	0.8	$(0.8t+0.2)e^{2t}$
$P2$	$g_{e_3} * g_{e_4} = 4$	$x \to \dot{y}$	0	0.2	$0.2t\,e^{2t}$
$P3$	$g_{e_4} * g_{e_5} = -2$	$y \to \dot{y}$	−0.2	0.2	$(0.2t-0.2)e^{2t}$
$P4$	$g_{e_6} * g_{e_2} = 1$	$y \to \dot{x}$	−0.6	0.8	$(0.8t-0.6)e^{2t}$

Table 3: Sensitivity with respect to pathway gains of 1^{st} mode

related sensitivity of $e2$ is calculated:

$$\begin{aligned} C_{11}^{(e_2)} &= \frac{g_{p_1}}{g_{e_2}} C_{11}^{(a_{11})} + \frac{g_{p_4}}{g_{e_2}} C_{11}^{(a_{12})} \\ &= \frac{1}{1} * 0.2 + \frac{1}{1} * (-0.6) \\ &= -0.4 \end{aligned}$$

Furthermore, the overall state sensitivity with respect to the causal link gain in terms of the dominant mode is also calculated using Eq. (8). Table 4 exhibits all the information (the eigenvector-related sensitivity is displayed after multiplying the initial condition).

Causal link	*link gain*	*Pathway*	*Eigenvector−related sen.*	*Eigenvalue sen.*	*State sen.*
$e1$	$\partial\dot{x}/\partial x = 1$	$P1$	0.2	0.8	$(0.8t+0.2)e^{2t}$
$e2$	$g(\dot{x} \to x) = 1$	$P1, P4$	−0.4	1.6	$(1.6t-0.4)e^{2t}$
$e3$	$\partial\dot{y}/\partial x = 4$	$P2$	0	0.2	$0.2te^{2t}$
$e4$	$g(\dot{y} \to y) = 1$	$P2, P3$	0.4	0.4	$(0.4t+0.4)e^{2t}$
$e5$	$\partial\dot{y}/\partial y = -2$	$P3$	−0.2	0.2	$(0.2t-0.2)e^{2t}$
$e6$	$\partial\dot{x}/\partial y = 1$	$P4$	−0.6	0.8	$(0.8t-0.6)e^{2t}$

Table 4: Sensitivity with respect to causal link gains of 1^{st} mode

From the above analysis, we can identify the eigenvector-related sensitivity is much affected by the initial condition and remains constant while the influence from the eigenvalue is impacted by the time factor. Table 4 shows b$e6$ has the most significant affect to x in terms of the eigenvector. When we increase the gain of $e6$, the amplitude of x will decrease by 0.6 of that amount from the perspective of the eigenvector. In total, $e2$ plays the most important role in the behaviour of x as it gives rise to the greatest impact in terms of the eigenvalue.

7 Conclusions

This paper describes a mathematical procedure of expressing the state trajectory by an eigensolution in a linear system. The solution differs itself from the conventional solution by decomposing the mode associated coefficient into a product of the corresponding left eigenvector and system initial conditions. An analytical framework of fully evaluating the influence of a certain system (a link or a pathway) element on the state behaviour trajectory is proposed which involves a formula for the eigenvector-related sensitivity in calculating the overall state sensitivity. Furthermore, we demonstrated this overall state sensitivity analysis by a linear model. An important conclusion derived in this paper is that **the right and left eigenvector sensitivities have to be assessed together and simultaneously as the eigenpair are dependent of each other**.

Through studying how the overall changes in behaviour due to changes in link (or pathway) gains, we observe that the derivatives of eigenvectors are closely associated with the short-term impact while the derivatives of eigenvalues are associated with the long-term impact. Therefore, the eigenvector analysis is important in rendering a complete picture of the dynamic change in the behaviour, especially, when we care about its transient behaviour. Furthermore, there is another advantage of eigenvector analysis, it solves a general problem related to the eigen-based methods that they cannot relate the analysis result to the variable of interest. Because eigenvector analysis is associated with the weight of the behaviour mode which varies in different state variables. Finally, it can be used to implement the weight analysis in Saleh et al. (2009) analytically which enables a much more efficient search for leverage policies.

Meanwhile, this approach also has a number of limitations. The proposed overall state sensitivity analysis so far applies only to linear systems, representing a small subset of typical system dynamics models. In addition, we did not present some more complex examples, e.g., oscillatory systems. Finally, we did not relate the eigenvector-related sensitivity analysis to the loop gain. Despite the limitations of the eigenvector related approach, we believe it provides the necessary exploration and beneficial outcomes for extending the application to a wider range of systems. A good practice is to use this approach to analyze the inhomogenous system, which can be transformed into homogeneous systems by translating the system without affecting its qualitative dynamics. Besides, eigenvector analysis in identifying dominant loops is a topic of great interest. These explorations are left as further development and future work.

8 Acknowledgment

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A General solution to linear systems

Let us consider an n-order linear system and assume it has distinct eigenvalues:

$$\begin{aligned} Ax &= \dot{x} \\ x(t_0) &= x(0) \end{aligned} \tag{32}$$

In addition to our previous discussion on this problem, we will utilize the following notations during the process of generating the solution to the linear system:

1. The eigenvalues are placed at the diagonal of a matrix: $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \cdot & \\ 0 & & & \lambda_n \end{bmatrix}$.

2. All the n right eigenvectors of A constitute the right eigenvector matrix $R = \begin{bmatrix} r_1 & . & . & r_n \end{bmatrix}$. If we express the eigenvalues and right eigenvector set of A in a matrix way, it would be as simple as:

$$AR = R\Lambda \tag{33}$$

3. All the n left eigenvectors $\ell_1{}^H, \ldots, \ell_n^H$ make up the left eigenvector matrix $L^H = \begin{bmatrix} \ell_1^H \\ \cdot \\ \cdot \\ \ell_n^H \end{bmatrix}$.

There are two general but important properties that the eigenvectors hold:

1. If v is an eigenvector, then so is γv for any nonzero scalar $\gamma \in C$.

2. The set of right eigenvector R and the set of left eigenvector L^H forms a bi-orthogonal system (Saleh and Davidsen, 2000). This relationship is shown in the formula below:

$$\ell_i^H r_j \begin{cases} \neq 0 & : \quad i = j \\ = 0 & : \quad i \neq j \end{cases}$$

In other words, the left eigenvector is orthogonal to the right eigenvector as long as they are associated with different eigenvalues, otherwise they are not orthogonal. Moreover, with property 1, it is easy to verify that there exist numerous scalars such that the eigenvector normalization satisfies:

$$\ell_i^H r_j \begin{cases} = 1 & : \quad i = j \\ = 0 & : \quad i \neq j \end{cases}$$

or equivalently,

$$L^H R = I \tag{34}$$

The normalization is not unique due to many possible scalars. For example, any nonzero scalar γ can applied to the normalized eigenpair so that $\gamma\,\ell_i^H\,\frac{1}{\gamma} r_i = \ell_i^{H*} r_i^* = 1$, we use $*$ to distinguish the new eigenvector from the old one. After the normalization, post-multiplying Eq. (33) with L^H renders:

$$A = R\Lambda L^H \tag{35}$$

Now we use linear algebra approach to solve Eq. (32). Hence, the solution yields:

$$x(t) = e^{tA}x(0) \tag{36}$$

where e^{tA} is a matrix exponential. Now let us start to decouple the linear system equation. A matrix exponential e^M can be expanded by the power series:

$$e^M = I + M + \frac{(M)^2}{2!} + \frac{(M)^3}{3!} + ... + \frac{(M)^n}{n!} + ... \qquad (n = \infty)$$

where I refers to an n-by-n identity matrix. In conjunction with Eq. (34) and (35), the above equation can be expressed in the following:

$$\begin{aligned} x(t) \;= e^{tA}x(0) &= (tI + tA + \frac{(tA)^2}{2!} + \frac{(tA)^3}{3!} + ... + \frac{(tA)^n}{n!} + ...)x(0) \\ &= (tRL^H + \frac{t^2(R\Lambda L^H)^2}{2!} + \frac{t^3(R\Lambda L^H)^3}{3!} + ... + \frac{t^n(R\Lambda L^H)^n}{n!} + ...)x(0) \\ &= R(tI + t\Lambda + \frac{(t\Lambda)^2}{2!} + \frac{(t\Lambda)^3}{3!} + ... + \frac{(t\Lambda)^n}{n!} + ...)L^H\, x(0) \\ &= Re^{t\Lambda}L^H\, x(0) \end{aligned}$$

This compact form has to be fully expanded to gain more insights of the system behaviour:

$$\begin{aligned} x(t) &= \begin{bmatrix} r_1 & r_2 & .. & r_n \end{bmatrix} \begin{bmatrix} e^{t\lambda_1} & & & 0 \\ & e^{t\lambda_2} & & \\ & & . & \\ 0 & & & e^{t\lambda_n} \end{bmatrix} \begin{bmatrix} \ell_1^H \\ \ell_2^H \\ . \\ \ell_n^H \end{bmatrix} x(0) \\ &= \begin{bmatrix} e^{t\lambda_1}r_1 & e^{t\lambda_2}r_2 & .. & e^{t\lambda_n}r_n \end{bmatrix} \begin{bmatrix} \ell_1^H x(0) \\ \ell_2^H x(0) \\ . \\ \ell_n^H x(0) \end{bmatrix} \\ &= e^{t\lambda_1}r_1\,\ell_1^H x(0) + e^{t\lambda_2}r_2\,\ell_2^H x(0) + ... + e^{t\lambda_n}r_n\,\ell_n^H x(0) \end{aligned}$$

Since $\ell_j^H x(0)$ produces a number, it is easy for us to present the details of the eigensolution of a particular state variable x_i:

$$\begin{aligned} x_i(t) &= e^{t\lambda_1}r_{1i}\,\ell_1^H x(0) + e^{t\lambda_2}r_{2i}\,\ell_2^H x(0) + ... + e^{t\lambda_n}r_{ni}\,\ell_n^H x(0) \\ &= \sum_{j=1}^{n} e^{t\lambda_j}r_{ji}\,\ell_j^H\, x(0) \end{aligned}$$

B Computation of eigenvalue sensitivity

For any square matrix A (n×n) has a distinct eigenvalue λ_i, the sensitivity of the eigenvalue with respect to any entry of the matrix A, a_{pq}, is equal to the product of the *pth* component in the left eigenvector and the *qth* component in the right eigenvector (both are associated with λ_i):

$$S_{pq}^{\lambda_i} = \ell_{ip}^{H} \times r_{iq}$$

Proof :
For the eigenvalue λ_i, we have:

$$Ar_i = \lambda_i r_i$$

Differentiating this equation with respect to the entry a_{pq} gives:

$$\begin{aligned} \frac{\partial A}{\partial a_{pq}} r_i + A \frac{\partial r_i}{\partial a_{pq}} - \frac{\partial \lambda_i}{\partial a_{pq}} r_i - \lambda_i \frac{\partial r_i}{\partial a_{pq}} &= 0 \\ (\frac{\partial A}{\partial a_{pq}} - \frac{\partial \lambda_i}{\partial a_{pq}} I) r_i + (A - \lambda_i I) \frac{\partial r_i}{\partial a_{pq}} &= 0 \end{aligned} \tag{37}$$

Eq. (37) is now pre-multiplied by $\ell_i{}^H$. The second term becomes zero and it can be rewritten as:

$$\begin{aligned} \ell_i^H \frac{\partial \lambda_i}{\partial a_{pq}} r_i &= \ell_i^H \frac{\partial A}{\partial a_{pq}} r_i \\ \frac{\partial \lambda_i}{\partial a_{pq}} \ell_i^H r_i &= \ell_i^H \frac{\partial A}{\partial a_{pq}} r_i \\ S_{pq}^{\lambda_i} &= \ell_{ip}^H r_{iq} \end{aligned}$$

C Computation of eigenvector-related sensitivity

In order to compute eigenpair sensitivity rigorously, we will bring in some mathematic concepts to facilitate our calculation. On the other hand, we change our notations a bit for convenience: the eigenvector sensitivity notation $S^{r_{ij}}_{a_{pq}}$ is substituted with r'_{ij}, $S^{\ell^H_{ij}}_{a_{pq}}$ with $\ell^{H\prime}_{ij}$ as well. Let us start from Eq. (21) and we show it again:

$$C_{ij} = r'_{ij}\,\ell^H_i + r_{ij}\,\ell^{H\prime}_i \tag{38}$$

In a broader sense, for all n state variables in a system, we have n such equations. Therefore we can rewrite them into a compact matrix form:

$$C_i = \begin{bmatrix} r'_{i1}\,\ell^H_i + r_{i1}\,\ell^{H\prime}_i \\ \dots \\ r'_{in}\,\ell^H_i + r_{in}\,\ell^{H\prime}_i \end{bmatrix} = r'_i\ell^H_i + r_i\ell^{H\prime}_i \tag{39}$$

where C_i is an $n \times n$ matrix associated with *ith* behaviour mode and each row corresponds to the eigenvector sensitivities in relation with different state variable. So far, we have adjusted our goal a little bit and we are going to prove:

For a matrix A whose entries are all real values, C_i associated with a distinct eigenvalue (λ_i) restricted to satisfy a constraint of the eigenpair in the form of $r^H_i\,\ell_i = 1$ yields a constant matrix, and $C_i = -(A - \lambda_i I)^{\#}A'\,r_i\ell^H_i - r_i\ell^H_i A'(A - \lambda_i I)^{\#}$, where A' is $\partial A/\partial a_{pq}$ for shorthand.

Proof

If $P_{n\times n-1}$ is a matrix whose columns form an orthonormal basis for $R(A - \lambda_i\,I)$ ($R(\cdot)$ denotes range and $N(\cdot)$ denotes the nullspace). The orthmnormal basis can be formed by performing the Gram-Schmidt orthogormalization. Then $W = (r_i|P)$[1] is nonsingular and it is easy to verify that:

$$W^{-1} = \left(\frac{\ell^H_i}{P^H(I - r_i\ell^H_i)}\right)^2 \tag{40}$$

Matrix $W^{-1}(A - \lambda_i I)W$ has the form

$$\begin{aligned} W^{-1}(A - \lambda_i I)W &= \begin{pmatrix} 0 & 0 \\ 0 & P^H(I - r_i\,\ell^H_i)(A - \lambda_i I)P \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & P^H(A - \lambda_i I)P \end{pmatrix} \end{aligned}$$

since λ_i is simple, $P^H(A - \lambda_i I)P$ is nonsingular and the group inverse of $(A - \lambda_i I)$

$$(A - \lambda_i I)^{\#} = W\begin{pmatrix} 0 & 0 \\ 0 & \{P^H(A - \lambda_i I)P\}^{-1} \end{pmatrix}W^{-1}$$

is well defined. We can further verify that the non-uniqueness of r_i and ℓ_i will not affect the computation of $(A - \lambda_i I)^{\#}$. Additional material on group inverse can be found in Campbell and

[1]Vector $r_{i(n\times 1)}$ forms the first column of matrix W while $P_{n\times n-1}$ fill in the rest column space.
[2]Vector ℓ^H_i forms the first row of matrix W while $P^H(I - r_i\ell^H_i)$ fill in the rest row space.

Meyer (1979). Since we will make use of the group inverse's properties in our analysis, so we introduce some of them now.

1. The nullspace of the original matrix and its group inverse matrix is the same, i.e., $N(A-\lambda I) = N(A-\lambda I)^{\#}$
2. If A is a group matrix and $b \in R(A)$, then the set of all solutions for p in $Ap = b$ is given by $p = A^{\#}b + N(A)$
3. For an eigenvalue λ,
$$\lambda^{\#} = \begin{cases} 1/\lambda, & \lambda \neq 0 \\ 0, & \lambda = 0 \end{cases}$$
A vector r is an eigenvector for A corresponding to the eigenvalue λ if and only if r is an eigenvector for $A^{\#}$ corresponding to $\lambda^{\#}$, i.e., $Ar = \lambda r$ if and only if $A^{\#}r = \lambda^{\#}r$.

From Eq. (9), we know that
$$(A' - \lambda_i' I)r_i + (A - \lambda_i I)r_i' = 0$$
with addition to property 1 and 2, there must exist a scalar β such that

$$\begin{aligned} r_i' &= \beta r_i - (A - \lambda_i I)^{\#}(A - \lambda_i I)' r_i \\ &= \beta r_i - (A - \lambda_i I)^{\#} A' \, r_i - (A - \lambda_i I)^{\#} \lambda_i' \, r_i \\ &= \beta r_i - (A - \lambda_i I)^{\#} A' \, r_i \end{aligned} \tag{41}$$

Recall the normalization condition we have established in Eq. (13), we rewrite it to be:

$$\ell_i^H r_i' + r_i^H \ell_i' = 0 \tag{42}$$

Utilizing Eq. (41) and (42) derives the scalar $\beta = -r_i^H \ell_i'$, which is substituted into Eq. (41) to produce the following expression:

$$r_i' = -r_i^H \ell_i' \, r_i - (A - \lambda_i I)^{\#} A' \, r_i \tag{43}$$

Analogously, we can obtain the general solution to the left eigenvector sensitivity by Eq. (11):

$$\ell_i^{H\prime} = \beta \ell_i^H - \ell_i^H A'(A - \lambda_i I)^{\#}$$

Plug it into Eq. (42) to generate $\beta = -\ell_i^H r_i'$. As a result, the left eigenvector sensitivity can be sorted out to be:

$$\ell_i^{H\prime} = -\ell_i^H r_i' \ell_i^H - \ell_i^H A'(A - \lambda_i I)^{\#} \tag{44}$$

In light of Eq. (43), (44) and (42), our objective Eq. (39):

$$\begin{aligned} C_i &= r_i' \ell_i^H + r_i \ell_i^{H\prime} \\ &= -(r_i^H \ell_i') \, r_i \ell_i^H - r_i(\ell_i^H r_i')\ell_i^H - (A - \lambda_i I)^{\#} A' \, r_i \ell_i^H - r_i \ell_i^H A'(A - \lambda_i I)^{\#} \\ &= -(r_i^H \ell_i' + \ell_i^H r_i') r_i \ell_i^H - (A - \lambda_i I)^{\#} A' \, r_i \ell_i^H - r_i \ell_i^H A'(A - \lambda_i I)^{\#} \\ &= -(A - \lambda_i I)^{\#} A' \, r_i \ell_i^H - r_i \ell_i^H A'(A - \lambda_i I)^{\#} \end{aligned} \tag{45}$$