A Parameter Estimation Method to Minimize Instabilities in System Dynamic Models

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Abstract

This paper introduces a new method that facilitates the stability analysis of system dynamics models. The method is based on the concepts of asymptotic stability and Accumulated Deviations from Equilibrium (ADE) convergence. We prove several theorems that show that ADE convergence of a state variable will make its trajectory approach asymptotic stability. Achieving ADE convergence requires the solution of a policy optimization problem. We use an approach called Behavior Decomposition Weights (BDW) to reduce the search space associated with that optimization problem. We also demonstrate this method on two examples: a linear "inventory-workforce" model and a non-linear "mass business cycle model". These examples illustrate the features of this method and the potential for the development of efficient tools to improve the quality of the optimization policies.

Keywords: model analysis, stability, equilibrium point, behavior decomposition weights, optimization

Introduction

Since Forrester's pioneering work in 1961, the use of Systems Dynamic (SD) to evaluate the influence of policies on complex systems has increased dramatically. Applications of SD models can be found in numerous domains including manufacturing, energy, healthcare, management, economics, and sociology. In those models, decisions are represented by a set of parameters, referred to as "policy parameters" (Grossmann 2002). Policy optimization finds policies that optimize a given objective function to modify the system behavior by changing the parameter values. SD models coupled with policy optimization techniques have proven to be a very powerful means for improving the behavior of such systems (Mohapatra and Sharma 1985). In general, once a SD model is validated, behavior can be predicted, and current system policies revised or changed until the desired system behavior is achieved. In the case of stability analysis the goal is the minimization of the ripple effects that have a huge, negative impact on the behavior of the system.

The policy design process consists of systematic evaluation of behavior while (1) changing policy parameters at different values, (2) changing connections within causal loops, and/or (3) inserting new elements into a model (Starr 1980). The evaluation is performed using the validated SD model. Changes are based on two approaches: analytic and synthetic (Porter 1969). In the analytic approach, changes are based on an analyst's prior experience. In the synthetic approach, changes are based on either modal control theory or optimization theory (policy optimization).

Modal control methods build the desired policy by using the eigenvalues of the motion equations (Macedo 1989). Some excellent articles within this area are those of: Talavage (1980); Mohapatra and Sharma (1985); and Ozveren and Sterman (1989); among others. While these methods are very powerful, the complexity of the associated mathematics makes them difficult to use for managers and practitioners.

Optimization theory methods include mathematical programming, genetic algorithms, neural networks, response surface methodology, and algorithmic search. Some excellent articles include Grossman (2002); Bailey *et al*. (1998); Chen and Jeng (2004); Higuchi (1996); Macedo (1989); Keloharju and Wolstenholme (1989); and Burns and Malone (1974). Chen and Jeng's (2004) work is of particular interest because they combine several of the aforementioned methods. First, they transform the SD model into a recurrent neural network; then, they use a genetic algorithm to generate policies by fitting the desired system behavior to patterns established in the neural network. Chen and Jeng claim their approach is flexible in the sense that it can find policies for a variety of behavior patterns including stable trajectories. However, the transformation stage might become difficult when SD models reach real-world sizes.

In addition to the previous methods, in the literature is possible to find works related to the structural analysis of the model. They can be used to identify relevant parameters of the model that affect certain behavior modes. Very good articles are those of Saleh *et al*. (2007); Gonçalves (2006); and Guneralp (2005). These methods require the linearization of the model.

In this paper, we present a promising method for policy optimization based on the concept of Accumulated Deviations from Equilibrium (ADE). Our method relies upon a theorem that states ADE convergence of a particular state variable implies asymptotic stability for that variable. Asymptotic stability for all state variables means asymptotic stability for the entire system. The ADE method does not need the linearization of the model and can be implemented easily in any SD modeling language. Its simplicity makes it an effective tool for practitioners in the analysis of highly nonlinear dynamic systems, especially those with oscillatory behavior.

These systems are represented by models which are described by their structure and parameters. Although in a practical environment managers have not control over all parameters of the model, for large-scale complex models the number of parameters they manipulate can still affect the performance of the searching algorithm used for policy optimization. This is not the case for the small and mid-size examples provided in this paper and due to the quick convergence of the local search algorithm used; nonetheless we still consider important to demonstrate how the ADE method can be complemented with the behavior decomposition weights (BDW) approach (Saleh *et al*. 2007) to reduce the search space in the policy optimization problem.

In the following section, we describe our approach to stability analysis and include some important definitions and theorems. In the subsequent section, we introduce the ADE method and show its application to both linear and nonlinear cases. In the appendices, we provide proofs of the theorems.

Stability Analysis

Related Definitions and Theorems

Conceptually, stability in a dynamic system means that once the system reaches an equilibrium point (EP), it will stay near that equilibrium point for all future time. Formal definitions follow.

Definition 1 The point $\mathbf{x}^{eq} \in R^n$ is said to be an equilibrium point of the differential equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ * if it has the property that once the corresponding system reaches $\mathbf{x}^{eq}$ at time t_{eq} it will remain at $\mathbf{x}^{eq}$ for all future time; in other words, $\mathbf{f}(\mathbf{x}(t)) = \mathbf{0}$ for all $t \geq t_{eq}$.

Definition 2 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f}: R^n \to R^n$; $\mathbf{x}(t) = [x_s(t)]^\dagger, s = 1,..,n$. The state variable x_s is defined to be stable (around the EP x_s^{eq}) if it is bounded, that is, there is a finite number M_s such that $\left|x_s(t) - x_s^{eq}\right| \leq M_s$ ‡. If this condition holds for all state variables then the system is said to be stable.

For our purpose, we would like to augment this notion of stability to include the reduction or minimization of oscillatory behavior around the EP. Therefore, we introduce the notion of *asymptotic stability*. Conceptually, a system is to be said asymptotically stable if the system trajectory converges to the EP as time increases indefinitely. The formal definition is

Definition 3 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f}: R^n \to R^n$; $\mathbf{x}(t) = [x_s(t)], s = 1,..,n$. The state variable x_s is defined to be *asymptotically stable* (around the EP x_s^{eq}) if it is both stable (satisfies Definition 2), and additionally, we have $\lim_{t\to\infty}\left(x_s(t) - x_s^{eq}\right) = 0$. If these two conditions hold for all state variables then the system is said to be *asymptotically stable*.

The following definition provides the concept of ADE, which is the main element of our method. Theorem 1 states that the ADE convergence of a state variable of interest will make the trajectory of this variable to converge to the EP, and therefore achieving asymptotic stability. If

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* $\dot{\mathbf{x}}(t) = \partial\mathbf{x}(t)/\partial t$

† Given a n-vector $\mathbf{x}(t)$, its components are represented by the symbol $[x_s(t)]=[x_1(t), x_2(t),..., x_n(t)]^T$. In state model representation $\mathbf{x}(t)$ is called the state vector and $x_s(t)$ is the *s*-th state variable.

‡ The symbol $|c|$ represents the absolute value of *c*.

all state variables converge to the EP then the system is asymptotically stable, as stated in Definition 3.

Definition 4 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t));\ \mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f}: R^n \to R^n$; $\mathbf{x}(t) = [x_s(t)], s = 1,..,n$. For the state variable x_s the accumulated deviations from its EP x_s^{eq} is defined as $\int_0^{\infty} |x_s(t) - x_s^{eq}|\, dt$.

Theorem 1 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t));\ \mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f}: R^n \to R^n$; $\mathbf{x}(t) = [x_s(t)], s = 1,..,n$. The state variable x_s is asymptotically stable (around the EP x_s^{eq}), if $\int_0^{\infty} |x_s(t) - x_s^{eq}|\, dt$ converges.

Parameter Estimation Method

The SD model can be described by an equation of the form $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p})$, where $\mathbf{x}(t)$ is the vector of state variables (dimension *n*) and $\mathbf{p}$ is a vector of adjustable parameters (dimension *q*) with lower and upper bounds $\mathbf{p}^L$ and $\mathbf{p}^U$ respectively.

Using the results of Theorem 1 we can formulate an optimization problem that will find the parameter vector $\mathbf{p}^*$ that causes the state variable x_s to become asymptotically stable around the equilibrium point $x_s^{eq}(\mathbf{p}^*)$. We will find this optimal parameter vector by minimizing the ADE for predetermined time horizon T and making use of Theorem 1. That is, we will find the vector that makes ADE converge[§]. The optimization problem is then stated as

$$\underset{\mathbf{p}}{\text{Minimize}}\ J(\mathbf{p}) = \sum_{s=1}^{m} \left\{ w_s \int_0^T |x_s(t) - x_s^{eq}|\, dt \right\}, \text{ where } \sum_{s=1}^{m} w_s = 1 \tag{1}$$

Subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p})$$
$$\mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{p}^L \le \mathbf{p} \le \mathbf{p}^U$$

$$\mathbf{x}(t) \in R^n, \mathbf{p} \in R^q,\ \mathbf{p}^L \in R^q, \mathbf{p}^U \in R^q, t \in [0, T]$$

The use of weights, w_s, means that J($\mathbf{p}$) will support the simultaneous stabilization of any subset of *m* state variables $(m \le n)$. Positive weights can be assigned to these variables in any way,

[§] One way to check the convergence of ADE is by adding a new state variable to the model, called "ADE" (see Figure 1), and graphically verify that its graph becomes a flat line when time goes to T. If necessary the time horizon T should be increased to obtain similar effects of convergence that when time goes to infinity.

provided the normalization constraint ($\sum_{s=1}^{m} w_s = 1$) is met. This allows higher weights to be assigned to the variables that are considered more important.

If we do not know the equilibrium point x_s^{eq} in advance, we can modify J(**p**) to include it as a variable (a_s) and change to optimization of the problem[**] as follows:

$$\underset{\mathbf{p}}{\text{Minimize}}\, J(\mathbf{p}) = \sum_{s=1}^{m} \left\{ w_s \int_0^T \left| x_s(t) - a_s \right| dt \right\}, \text{ where } \sum_{s=1}^{m} w_s = 1 \tag{2}$$

This amounts to including a_s (s=1,..,m) as part of the solution vector **p**. The following theorem guarantees that the values of a_s obtained from the optimization will, in fact, coincide with the equilibrium points x_s^{eq} (s=1,..,m).

Theorem 2 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t));\ \mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f} : R^n \to R^n; \mathbf{x}(t) = \left[x_s(t)\right], s = 1,..,n$. If $\int_0^\infty \left| x_s(t) - a_s \right| dt$ converges then $a_s = x_s^{eq}$.

The objective function defined in (2) can be incorporated very easily into any SD formulation by adding a "stock and flow" piece to the model that is linked to the state variables of interest as illustrated in Figure 1. Then we define the variables DE and ADE as

DE=w1*ABS(State Var. 1-a1)+w2*ABS(State Var. 2-a2)+....+wm*ABS(State Var. m-am)

ADE=INTEG(DE,0)

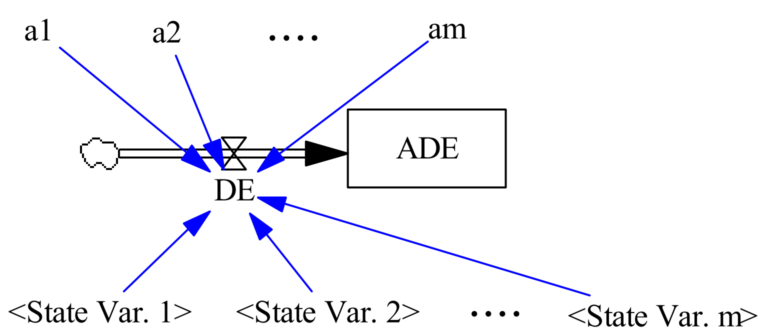

Figure 1 Stock and flow diagram for the objective function

We now illustrate the application of the proposed method using two examples[††] taken from Saleh *et al*. (2007). We will show that their results can be used to reduce the search space in our optimization problem.

[**] For example, for an inventory variable, the interval of variation of its EP in the optimization problem would be based on the minimum and maximum levels of inventory determined by the production plan.

Example 1: The Inventory-Workforce model

Our first example is a manufacturing supply chain that includes labor as an explicit factor of production. Saleh *et al*. (2007) developed a linear SD model for this supply chain by modifying Sterman's original model (2000). The interactions between inventory management policies and the labor adjustment policies are the main cause for the oscillatory behavior of the supply chain. To capture the impact of these policies, Saleh *et al*. (2007) created four state variables: *inventory*, *work in process (WIP) inventory* (see Figure 2[‡‡]), *vacancies*, and *labor* (see Figure 3).

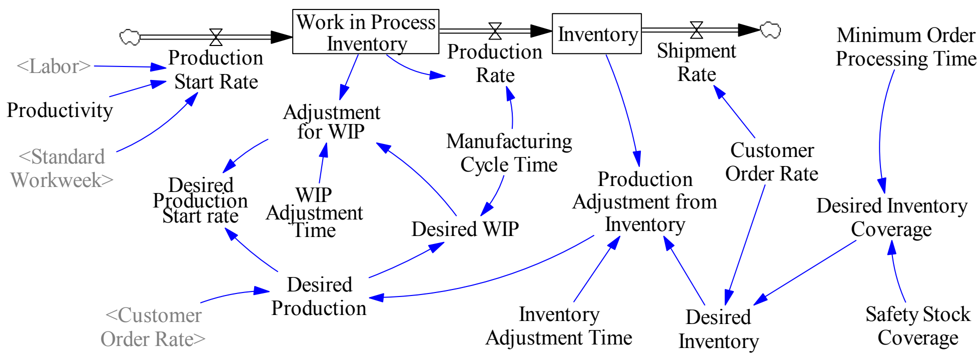

Figure 2 Structure of inventory management sector

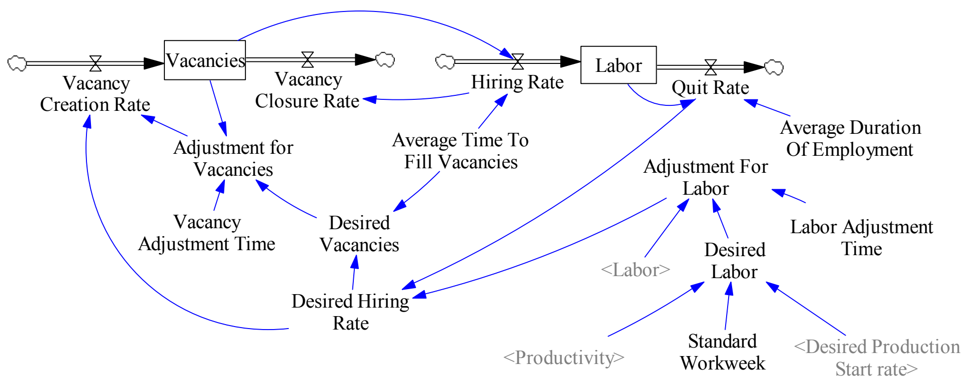

Figure 3 Structure of labor sector

The set of parameters in Table 1 define the base policy for this supply chain, called "policy I-0".

Parameter	Value	Unit
Manufacturing Cycle Time	8	Weeks
Inventory Adjustment Time	12	Weeks
Average Duration of Employment	100	Weeks
Average Time to Fill Vacancies	8	Weeks
Labor Adjustment Time	19	Weeks

[††] The files with the models are provided as supporting material. They are a copy of the models used by Saleh *et al*. (2007) but with slight modifications basically in formatting.

[‡‡] Customer Order Rate is considered an exogenous variable. Productivity and Standard Workweek are assumed to be constants with values 0.25 widgets/person-hour and 40 hours/week respectively.

Vacancy Adjustment Time	4	Weeks
WIP Adjustment Time	6	Weeks
Minimum Order Processing Time	2	Weeks
Safety Stock Coverage	2	Widgets

Table 1 Parameter values for the base policy

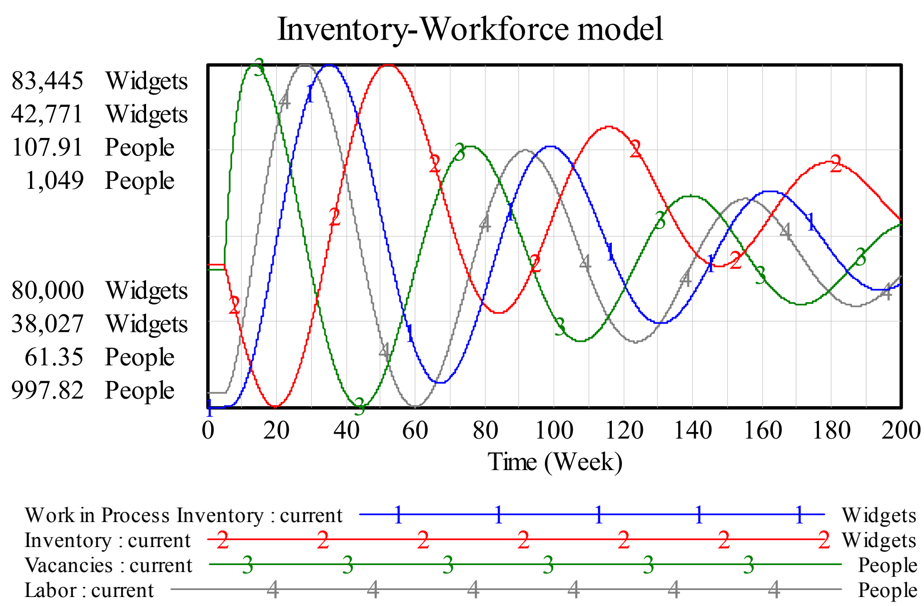

Figure 4 Behavior of state variables for the base policy I-0

At time 0, the system starts at the equilibrium points: 80000, 40000, 80, and 1000 for the variables WIP Inventory, Inventory, Vacancies and Labor respectively. Figure 4 shows the system's response to an unanticipated 2% increase in customer orders in week five. The figure shows sharp increases in all variables follow by several oscillatory fluctuations. These fluctuations are caused by delays in production. Can we determine new parameter settings that can damp these fluctuations quickly?

Policy optimization to improve the behavior of the system

To answer this question, we will apply our proposed formulation. In our initial investigation, we assign equal weights (w_s=0.25, s=1,..,4) to previously mentioned four state variables. So our optimization problem becomes

Let x_1=Work in Process Inventory, x_2=Inventory, x_3=Vacancies, x_4=Labor

$$\text{Minimize} \sum_{s=1}^{4} \left\{ 0.25 \int_0^{200} \left| x_s(t) - a_s \right| dt \right\}$$

Subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p})^{\S\S}$$

$$\mathbf{x}_0^{\mathrm{T}} = \begin{bmatrix} 80000 & 40000 & 80 & 1000 \end{bmatrix}$$

§§ See the supporting material for the model equations.

$1 \leq$ Manufacturing Cycle Time ≤ 50
$1 \leq$ Inventory Adjustment Time ≤ 50
$50 \leq$ Average Duration of Employment ≤ 150
$1 \leq$ Average Time to Fill Vacancies ≤ 50
$1 \leq$ Labor Adjustment Time ≤ 50
$1 \leq$ Vacancy Adjustment Time ≤ 50
$1 \leq$ WIP Adjustment Time ≤ 50
$1 \leq$ Minimum Order Processing Time ≤ 50
$1 \leq$ Safety Stock Coverage ≤ 50
$10000 \leq a_1 \leq 150000$
$10000 \leq a_2 \leq 150000$
$10 \leq a_3 \leq 1000$
$100 \leq a_4 \leq 10000$

To solve this optimization problem, we used the implementation of the Powell hill-climbing algorithm[***] included in our SD modeling program. The program yielded the parameter results, which we call "policy I-1", shown in Table 2. Table 2 also includes parameters a_1, a_3 and a_4 which are the new equilibrium points for state variables. The time to modify the 13 parameters of "policy I-1" (after 1757 iterations of the algorithm) was 17 seconds.

Figure 5 shows the behavior of the state variables when this revised policy is applied at the fifth week. While there are, indeed, changes to these variables, their fluctuations have all but disappeared. These results point out some interesting tradeoffs. Because production and labor are directly proportional, decreasing the time to adjust labor and vacancies will help production to track to the desired rates more closely. Increasing the inventory adjustment time, moreover, means fewer inventory corrections will be needed in response to the demand change. On the other hand, decreasing the time to adjust WIP inventory reduces the likelihood that the actual inventory will fall to unacceptable levels. This, in turn, means that increasing the production rate to ensure inventory levels sufficient to meet the increased demand will not be necessary. Changes in other parameters like manufacturing cycle time and average duration of employment were not very significant.

Parameter	Value	Unit
Manufacturing Cycle Time	7.71	Weeks
Inventory Adjustment Time	50	Weeks
Average Duration of Employment	109.66	Weeks
Average Time to Fill Vacancies	5.25	Weeks
Labor Adjustment Time	7.83	Weeks
Vacancy Adjustment Time	1	Weeks
WIP Adjustment Time	3.75	Weeks
Minimum Order Processing Time	1.93	Weeks
Safety Stock Coverage	1.99	Weeks
a_1 (EP for WIP)	78646.9	Widgets

[***] All the runs of the algorithm were executed on a 1.86 GHz Pentium PC with 1GB of memory.

a_2 (EP for Inventory)	40000	Widgets
a_3 (EP for Vacancies)	48.78	People
a_4 (EP for Labor)	1020	People

Table 2 Parameters values for policy I-1

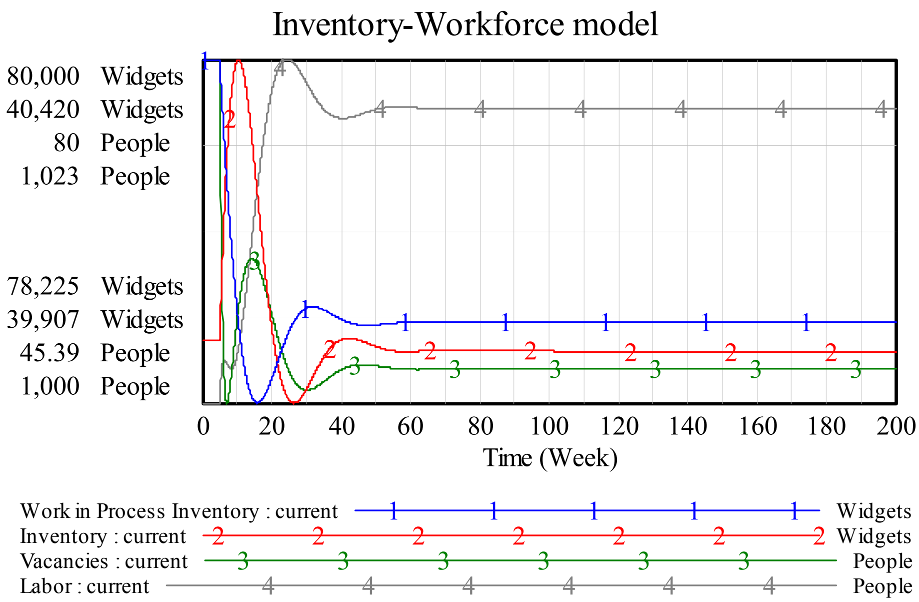

Figure 5 Behavior of state variables for policy I-1

Reducing the search space using BDW

This example has 9 parameters, not including the four equilibrium points, that can be included in the stabilization policy. We now show how to reduce the search space by using the results from the BDW analysis performed by Saleh *et al*. (2007). Using the common concept of elasticity, the authors found three parameters with high impact on the oscillatory behavior of the inventory variable and three variables with low impact. The high-impact variables were *manufacturing cycle time*, *inventory adjustment time*, and *labor adjustment time*. The low-impact variables were *average duration of employment, safety stock coverage,* and *minimum processing time.* We decided to eliminate these low-impact variables and resolve the optimization problem. Table 3 shows the results and, called "policy I-2", and Figure 6 shows the behavior of the state variables when this policy is applied at the fifth week. The time to modify the 10 parameters of "policy I-2" (after 1404 iterations of the algorithm) was 14 seconds.

Parameter[†††]	Value	Unit
Manufacturing Cycle Time	7.93	Weeks
Inventory Adjustment Time	22.25	Weeks
[Average Duration of Employment]	100	Weeks
Average Time to Fill Vacancies	1	Weeks
Labor Adjustment Time	3.31	Weeks

[†††] The parameters in brackets were considered constants in this optimization problem, keeping their values from the base policy.

Vacancy Adjustment Time	50	Weeks
WIP Adjustment Time	9.09	Weeks
[Minimum Order Processing Time]	2	Weeks
[Safety Stock Coverage]	2	Widgets
a_1 (EP for WIP)	80880.4	Widgets
a_2 (EP for Inventory)	40799.3	Widgets
a_3 (EP for Vacancies)	10.20	People
a_4 (EP for Labor)	1020.14	People

Table 3 Parameters values for policy I-2

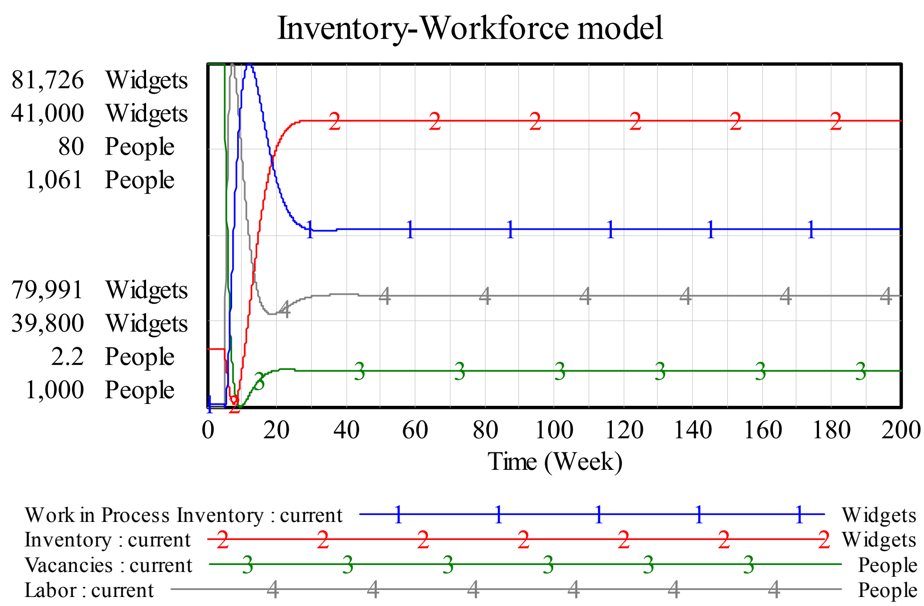

Figure 6 Behavior of state variables for policy I-2

The qualitative behavior of the state variables under this policy is same as under I-1. There are, however, some notable quantitative differences. Two high-impact leverage parameters, *inventory adjustment time* and *labor adjustment time*, show more than 50% change under this new policy. And, two equilibrium points, *WIP inventory* and *inventory*, are a little bit higher under this policy.

Example 2: The Mass business cycle model

Mass (Mass 1975) developed a non-linear SD model to explore the economic processes underlying business-cycle behavior. Business cycles are recurring fluctuations in the macro-economy that affect total production, prices, employment, inventories and capital investment. In this example, we are using the simplified version of the Mass' model designed by Kampmann and Oliva (2006). This version contains a production sector that includes the inventory sector (Figure 7[‡‡‡]) plus two factors of production: labor sector (Figure 8) and capital sector (Figure 9).

[‡‡‡] Man-Hours per year Normal is assumed to be constant with value 2080 hours/man-year, reflecting a normal forty-hour work week for fifty-two weeks per year.

We will focus our analysis on the three main state variables of the model: Capital, Inventory and Labor.

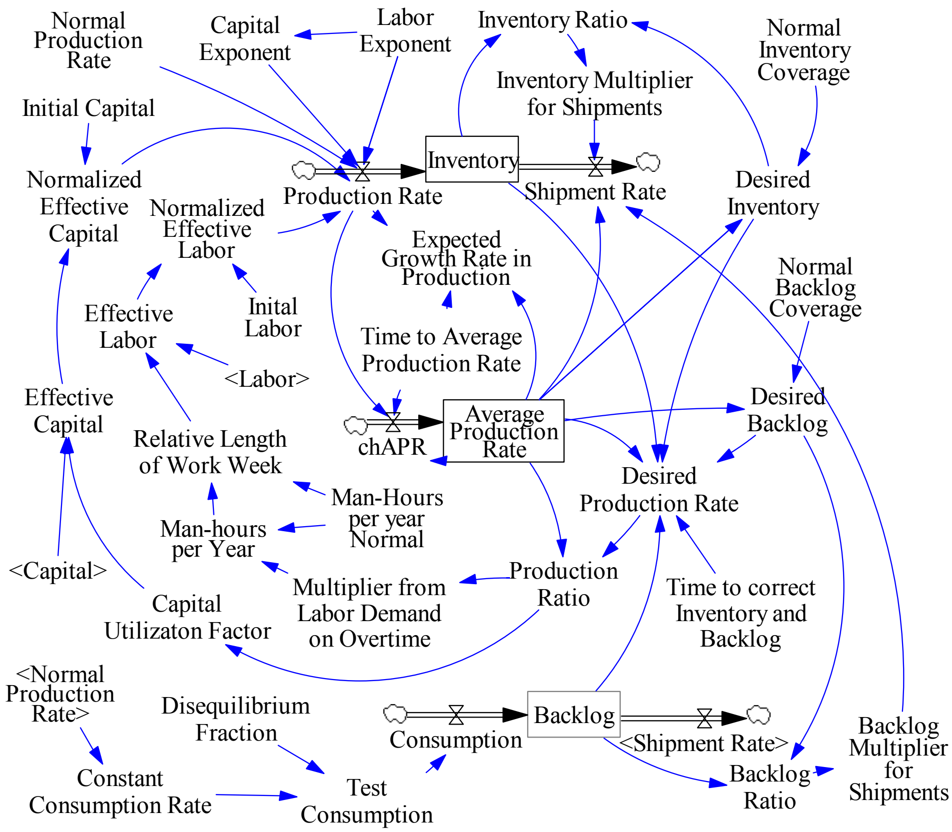

Figure 7 Structure of inventory management sector

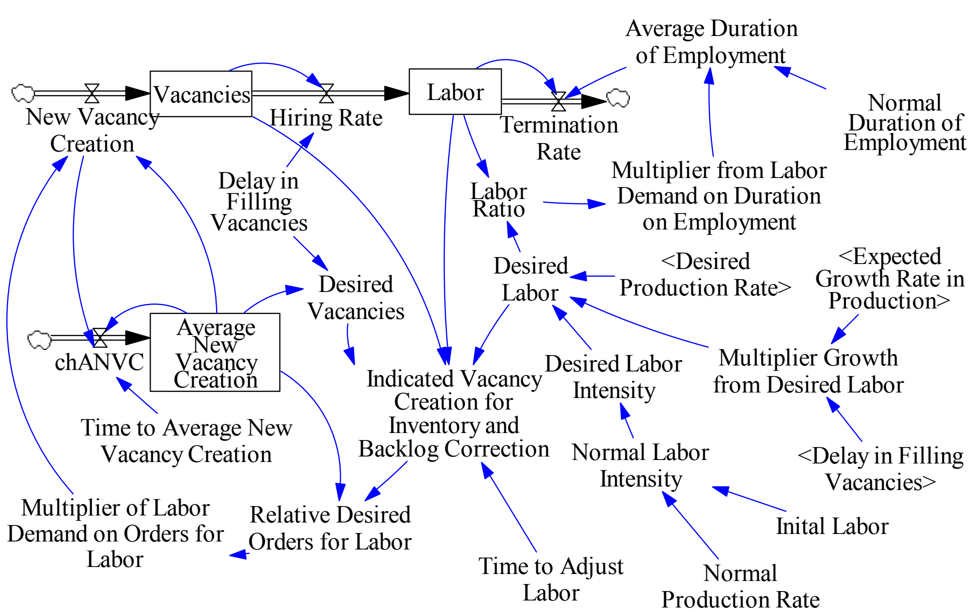

Figure 8 Structure of labor sector

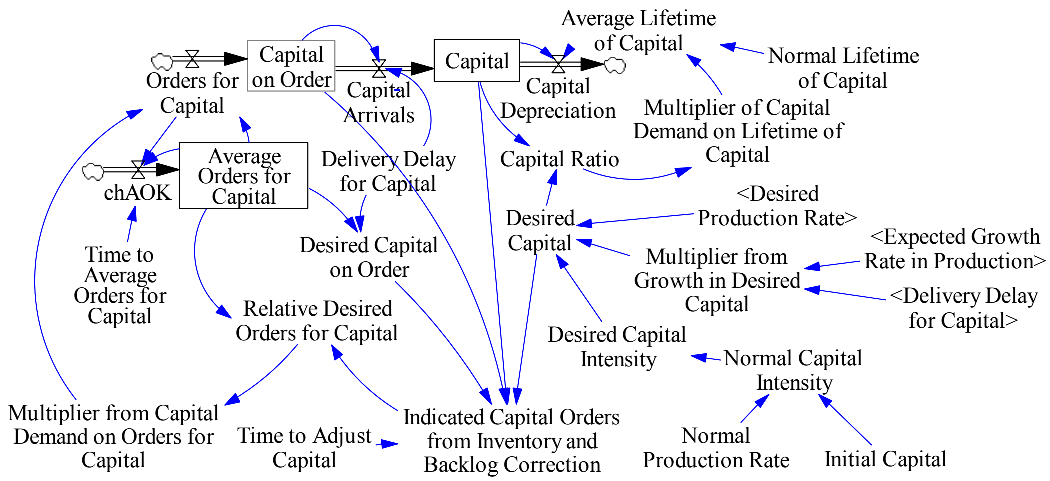

Figure 9 Structure of capital sector

The parameters of the base policy, called "policy M-0", are shown in the next table.

Parameter	Value	Unit
Normal Production Rate (NProd)	3E06	Units/year
Initial Capital (IK)	7.5E06	Capital Units
Initial Labor (IL)	1500	People
Time to Average Production Rate (tAPR)	1	Years
Normal Inventory Coverage (NIC)	0.5	Years
Time to Correct Inventory and Backlog (tCIB)	0.8	Years
Normal Backlog Coverage (NBC)	0.2	Years
Delay in Filling Vacancies (dFV)	0.25	Years
Time to Average New Vacancy Creation (tANVC)	0.5	Years
Time to Adjust Labor (TAL)	0.5	Years
Normal Duration of Employment (NDE)	2	Years
Time to Average Orders for Capital (tAOK)	4	Years
Delivery Delay for Capital (dDK)	2	Years
Time to Adjust Capital (tAK)	4	Years
Normal Life of Capital (NLK)	15	Years

Table 4 Parameter values for the base policy M-0

Figure 10 shows the behavior of the system that was started slightly out of equilibrium. Since there are no additional perturbations, the model, after some initial fluctuations, settles to equilibrium within approximately 26 to 30 years.

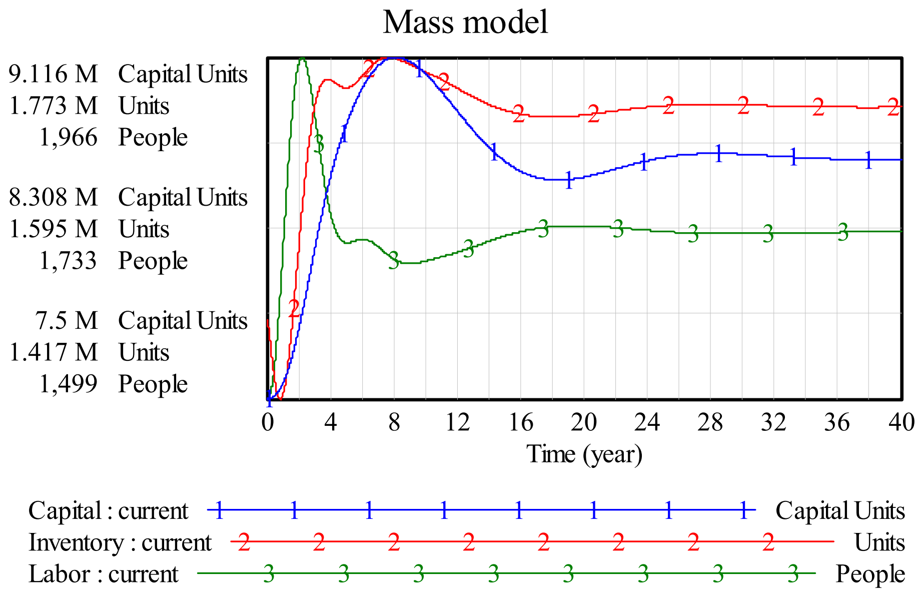

Figure 10 Behavior of state variables for the base policy M-0

Policy optimization to improve the behavior of the system

Similarly to what we did in Example 1, we will apply the formulation proposed in the parameter estimation problem to obtain a stabilization policy for the state variables. To do this, we again use equal weights for those variables. With this assumption, we get the following formulation

Let x_1=Capital, x_2=Inventory, x_3=Labor

$$\text{Minimize} \sum_{s=1}^{3} \left\{ 0.33 \int_{0}^{40} \left| x_s(t) - a_s \right| dt \right\}$$

Subject to

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{p})^{\S\S\S}$$

$$\mathbf{x}_0^{\ T} = [7.5E06 \quad 1.5E06 \quad 1500]$$

$1E06 \le$ Normal Production Rate $\le 1E07$

$1E06 \le$ Initial Capital $\le 1E07$

$1E02 \le$ Initial Labor $\le 1E04$

$0.1 \le$ Time to Average Product Rate ≤ 5

$0.1 \le$ Normal Inventory Coverage ≤ 5

$0.1 \le$ Time to Correct Inventory and Backlog ≤ 5

$0.1 \le$ Normal Backlog Coverage ≤ 5

$0.1 \le$ Delay in Filling Vacancies ≤ 5

$0.1 \le$ Time to Average New Vacancy Creation ≤ 5

$0.1 \le$ Time to Adjust Labor ≤ 5

$0.1 \le$ Normal Duration of Employment ≤ 5

$1 \le$ Time to Average Order for Capital ≤ 10

$1 \le$ Delivery Delay Capital ≤ 10

§§§ See the supporting material for the model equations.

$1 \leq$ Time to Adjust Capital ≤ 10
$1 \leq$ Normal Lifetime of Capital ≤ 20
$1E06 \leq a_1 \leq 1E07$
$5E05 \leq a_2 \leq 5E06$
$1E02 \leq a_3 \leq 1E04$

The optimal parameters yield the improved policy, called "policy M-1", shown in Table 5. Figure 11 shows the behavior of the main state variables for this revised policy. The time to modify the 18 parameters of "policy M-1" (after 1643 iterations of the algorithm) was 25 seconds.

Parameter	Value	Unit
Normal Production Rate (NProd)	3.012E06	Units/year
Initial Capital (IK)	7.5E06	Capital Units
Initial Labor (IL)	1240.76	People
Time to Average Production Rate (tAPR)	0.89	Years
Normal Inventory Coverage (NIC)	0.5	Years
Time to Correct Inventory and Backlog (tCIB)	0.1	Years
Normal Backlog Coverage (NBC)	0.2	Years
Delay in Filling Vacancies (dFV)	0.36	Years
Time to Average New Vacancy Creation (tANVC)	3.5	Years
Time to Adjust Labor (TAL)	1.80	Years
Normal Duration of Employment (NDE)	1.89	Years
Time to Average Orders for Capital (tAOK)	8.62	Years
Delivery Delay for Capital (dDK)	1	Years
Time to Adjust Capital (tAK)	1	Years
Normal Life of Capital (NLK)	15.04	Years
a_1 (EP for Capital)	8.611E06	Capital Units
a_2 (EP for Inventory)	1.727E06	Units
a_3 (EP for Labor)	1434.37	People

Table 5 Parameters values for policy M-1

Figure 11 shows that the system has reached equilibrium points and remains stable after only 6 years. Moreover, the system has no significant fluctuations in capital and inventory, although their equilibrium points have increased. This was achieved by increasing several parameter values including *time to average orders for capital*, *time to average new vacancy creation* and *time to adjust labor*, and decreasing several other parameter values including *delivery delay for capital*, *time to correct inventory and backlog*, *time to adjust capital,* and *initial labor*.

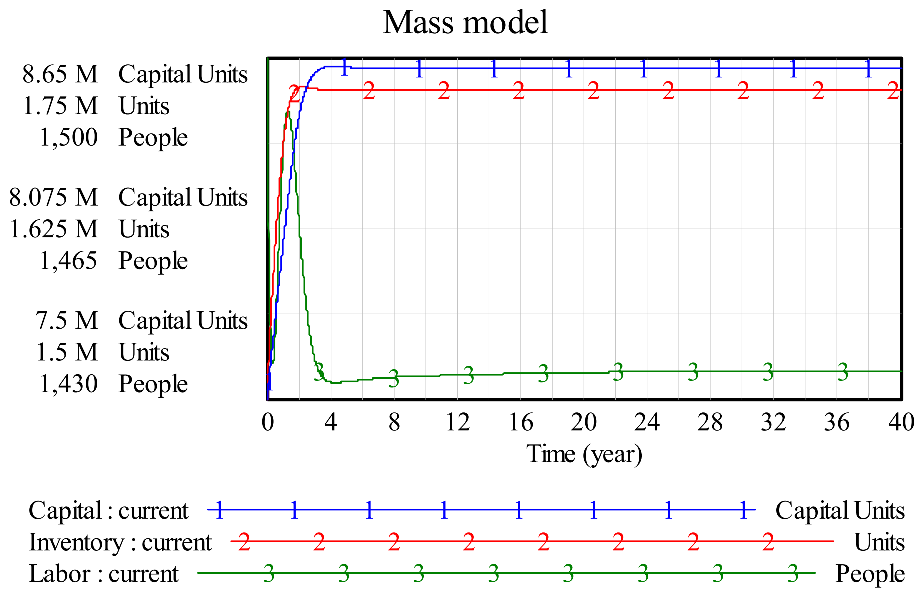

Figure 11 Behavior of state variables for policy M-1

Reducing the search space using BDW

The stabilization policy comprises 15 parameters, not including the equilibrium points. Saleh *et al*. (2007) used weight elasticities to identify four key parameters that impact the behavior of Capital significantly: *time to average orders for capital*, *delivery delay for capital*, *time to adjust capital*, and *normal life of capital*. They also identified five parameters that impact the interactions between Capital and Inventory significantly: *normal inventory coverage*, *initial labor*, *normal production rate*, *time to adjust capital*, and *time to average orders for capital*. The total number of distinct parameters in the optimization can be reduced to seven; a significant reduction in the size of the search space.

Table 6 shows the resulting policy, "policy M-2", and Figure 12 shows the behavior of the state variables when this policy is applied. The time to modify the 10 parameters of "policy M-2" (after 617 iterations of the algorithm) was 9 seconds.

Parameter[****]	Value	Unit
Normal Production Rate (NProd)	3E06	Units/year
[Initial Capital (IK)]	7.5E06	Capital Units
Initial Labor (IL)	1459.69	People
[Time to Average Production Rate (tAPR)]	1	Years
Normal Inventory Coverage (NIC)	0.50	Years
[Time to Correct Inventory and Backlog (tCIB)]	0.8	Years
[Normal Backlog Coverage (NBC)]	0.2	Years
[Delay in Filling Vacancies (dFV)]	0.25	Years

[****] The parameters in brackets were considered constants in this optimization problem, keeping their values from the base policy.

[Time to Average New Vacancy Creation (tANVC)]	0.5	Years
[Time to Adjust Labor (TAL)]	0.5	Years
[Normal Duration of Employment (NDE)]	2	Years
Time to Average Orders for Capital (tAOK)	10	Years
Delivery Delay for Capital (dDK)	1	Years
Time to Adjust Capital (tAK)	2.72	Years
Normal Life of Capital (NLK)	14.89	Years
a_1 (EP for Capital)	8.628E06	Capital Units
a_2 (EP for Inventory)	1.735E06	Units
a_3 (EP for Labor)	1679.26	People

Table 6 Parameters values for policy M-2

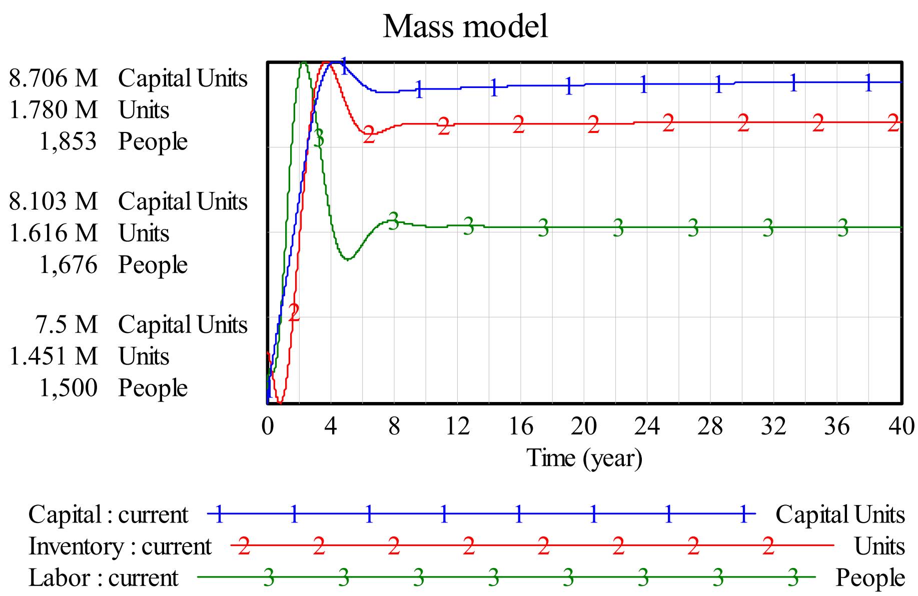

Figure 12 Behavior of state variables for policy M-2

This policy improves stability mainly by increasing the parameter *time to average orders for capital* and decreasing the parameters *delivery delay for capital*, and *time to adjust capital*. This policy shows some amplification before reaching the steady state for the variables capital, inventory and labor. Moreover, the equilibrium points are higher for these three variables than the ones with the base policy.

Conclusions

We proposed a new approach to modeling and solving policy optimization problems in system dynamics models. This approach provides a direct connection between the parameters of the model and the underlying mechanisms that govern its behavioral changes. Moreover, the solution approach, which uses ADE, does not require direct knowledge of the internal structure of the model. It also does not require linearization of the system or eigenvalue calculations.

We noted that other authors have shown how to use weight elasticities to identify a subset of the parameters that have the most impact on system stability. We showed how to use those parameters to reduce the search space for the optimization. Moreover, our method can be used with other approaches like LEEA (Kampmann and Oliva 2006) that also identify important parameters of the model.

We demonstrated the potential benefits of this approach on two example models. We argued that the simplicity of our approach makes it an effective tool for practitioners – especially when dealing with systems that exhibit highly non-linear, oscillatory behavior.

The Powell hill-climbing algorithm performed well for the examples presented in this paper, taken just few seconds to generate the different policies. We see from the results of the experiments that performance is related to the number of parameters, linearity/nonlinearity of the model, and number of iterations required to solve the problem.

Future Work

The Powell hill-climbing algorithm belongs to a family of local search techniques that work well when the initial solution is close to the equilibrium point. However, stabilization policies obtained using this algorithm, are constrained to equilibrium states near the initial solution, which may not be the global optimum. Although, it is not required to find the global optimum to obtain a satisfactory reduction in instability, more efficient searching algorithms that escape local convergence may produce better solutions that may lead to fewer oscillations and faster stability. Therefore, we will experiment with other types of techniques that avoid premature convergence to a local optimum such as Particle Swarm Optimization (Kennedy and Eberhart 1995).

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APPENDIX A: ADDITIONAL DEFINITIONS AND THEOREMS

Definition A.1 Linearization around an Operating Point
The linearization of the nonlinear system equations at an operating point is done by using the Taylor series expansion, as it is shown next.

Consider the nonlinear system defined by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{x}(t) = [x_s(t)], s = 1,..,n$; $\mathbf{f} : R^n \to R^n$; $\mathbf{f}(\mathbf{x}(t)) = [f_s(\mathbf{x}(t))], s = 1,..,n$

The linear approximation $\dot{z}_s(t)$ for the *s*-th component of vector $\dot{\mathbf{x}}(t)$ around the operating point $\mathbf{x}_\theta = [x_{\theta 1}, x_{\theta 2}, \cdots, x_{\theta n}]$ is given by

$$\dot{z}_s(t) = f_s(\mathbf{x}_\theta) + \sum_{i=1}^{n} \left\{ \left. \frac{\partial f_s}{\partial x_i} \right|_{\mathbf{x}=\mathbf{x}_\theta} (z_i - x_{\theta i}) \right\}$$

Definition A.2 Linearized Model of a Nonlinear System
Consider the nonlinear system defined by
$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$; $\mathbf{f} : R^n \to R^n$

The linearized model $\dot{\mathbf{z}}(t)$ of system $\dot{\mathbf{x}}(t)$ around *m* operating points $\{\mathbf{z}(t_{p-1})$, p=1,..,m; $t_0<t_1<...<t_m\}$ is represented by the following equations

$$\dot{\mathbf{z}}(t) = \begin{cases} \mathbf{A}_1\mathbf{z}(t) + \mathbf{b}_1, t_0 \le t < t_1; & \text{Initial condition} : \mathbf{z}(t_0) = \mathbf{x}(t_0) \\ \mathbf{A}_2\mathbf{z}(t) + \mathbf{b}_2, t_1 \le t < t_2; & \text{Initial condition} : \mathbf{z}(t_1) \\ \vdots & \vdots \\ \mathbf{A}_m\mathbf{z}(t) + \mathbf{b}_m, t_{m-1} \le t < t_m; & \text{Initial condition} : \mathbf{z}(t_m) \end{cases}$$

where $\mathbf{z}(t_p) = \lim_{t \to t_p^-} \mathbf{z}(t_p), p = 1,..,m-1$ (3)

This definition implies that we are approximating trajectory $\mathbf{x}(t)$ by trajectories $\mathbf{z}(t)$ of *p* linear systems. Note that $\mathbf{z}(t)$ is a continuous piecewise function. This is because $\mathbf{z}(t)$ is differentiable and therefore continuous in $[t_{p-1}, t_p), p = 1,..,m$, and condition (3).

Lemma A.1[††††] (Convergence of Infinite Series)
If the series $\sum_{i=1}^{\infty} \Psi_i$ converges, then $\lim_{m \to \infty} \Psi_m = 0$

[††††] Refer to Spivak (1967) to see the proof of this lemma.

Theorem A.1 Consider the system defined by $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$; $\mathbf{x}(0) = \mathbf{x}_0$; where $\mathbf{x}(t) \in R^n$ $\mathbf{A} \in R^{n \times n}, \mathbf{b} \in R^{n \times 1}$. If matrix **A** has distinct nonzero eigenvalues, then the solution to this system can be expressed as

$$\mathbf{x}(t) = [x_s(t)],\ s = 1,..,n$$

$$x_s(t) = x_s^{eq} + \sum_{j \in H_1} \left\{ \mathrm{Re}\left(\alpha_j r_{sj} + \frac{\beta_j}{\lambda_j} r_{sj} \right) e^{\mathrm{Re}(\lambda_j)t} \right\} + \sum_{k \in H_2} \left\{ 2v_{sk}\, e^{\mathrm{Re}(\lambda_k)t} \sin\left(\mathrm{Im}(\lambda_k)t + \theta_{sk} \right) \right\}$$

where

$x_s(t)$ denotes the state variable *s* with equilibrium point x_s^{eq}, s=1,..,n

λ_q and $\mathbf{r}_q$ are the corresponding eigenvalues and eigenvectors of **A**, q=1,..,n

Re(z) and Im(z) mean the real and imaginary parts of $z \in C$

H_1 is a set of indexes j such that $\mathrm{Im}(\lambda_j) = 0$

H_2 is a set of indexes k such that $\mathrm{Im}(\lambda_k) \neq 0$, where k denotes the conjugate pair of eigenvalues λ_k and λ_{k+1}, i.e. one index k represents two eigenvalues. Therefore, $\mathrm{Re}(\lambda_k) = \mathrm{Re}(\lambda_{k+1})$ and $\mathrm{Im}(\lambda_k) = \mathrm{Im}(\lambda_{k+1})$

The constants $\alpha_j, \beta_j, v_{sk}$, and θ_{sk} are defined as follows

$$[\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]^T = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \cdots \quad \mathbf{r}_n]^{-1} \mathbf{x}_0$$

$$[\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n]^T = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \cdots \quad \mathbf{r}_n]^{-1} \mathbf{b}$$

$v_{sk} = \left\| \alpha_k r_{sk} + \frac{\beta_k}{\lambda_k} r_{sk} \right\|$, where $\|z\|$ is the modulus[‡‡‡‡] of $z \in C$

$\theta_{sk} = \arctan\left(\mathrm{Re}\left(\alpha_k r_{sk} + \frac{\beta_k}{\lambda_k} r_{sk} \right) \Big/ -\mathrm{Im}\left(\alpha_k r_{sk} + \frac{\beta_k}{\lambda_k} r_{sk} \right) \right)$, expressed in radians

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[‡‡‡‡] Given z=a+bi then $\|z\| = \sqrt{a^2 + b^2}$

APPENDIX B: PROOF OF THEOREMS

B.1 Proof of Theorem 1

Firstly, we will prove by contradiction that if $\int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt$ converges then the state variable x_s is stable around the EP x_s^{eq} (4)

So let's assume to contrary that state variable x_s is not stable, which by Definition 2 means that $\left| x_s(t) - x_s^{eq} \right|$ is not bounded, i.e.

$\forall M_s, \exists t_M$ such that $\left| x_s(t) - x_s^{eq} \right| > M_s$

Let's make $y_s(t) = x_s(t) - x_s^{eq}$, thus

$$\forall M_s, \exists t_M \text{ such that } \left| y_s(t_M) \right| > M_s \tag{5}$$

Expressing $\int_0^\infty \left| y_s(t) \right| dt$ as Riemann sums (Yuen and Yuan 2000)

$$\int_0^\infty \left| y_s(t) \right| dt = \sum_{i=1}^{\infty} \left| y_s(c_i) \Delta t_i \right| \tag{6}$$

where $\Delta t_i = t_i - t_{i-1}$, and $c_i \in [t_i, t_{i-1}]$

By hypothesis we know that the integral $\int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt$ converges (i.e. it is bounded), and thus there is a number W_s such that $\int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt \le W_s, \forall t \ge 0$

Expressing the above statement in terms of $y_s(t)$:

$$\text{there is a number } W_s \text{ such that } \int_0^\infty \left| y_s(t) \right| dt \le W_s, \forall t \ge 0 \tag{7}$$

From (6) and (7)

$$\sum_{i=1}^{\infty} \left| y_s(c_i) \Delta t_i \right| \le W_s \Rightarrow \left| y_s(c_i) \Delta t_i \right| \le W_s, \forall i \Rightarrow \left| y_s(c_i) \right| \le \frac{W_s}{\left| \Delta t_i \right|}, \forall i \tag{8}$$

Because "t" is a continuous variable from 0 to infinity, then there is an index i=b such that $c_b = t_M$. Moreover, condition (8) holds for every c_i and particularly for $c_b = t_M$, therefore

$$\left| y_s(t_M) \right| \le \frac{W_s}{\left| \Delta t_b \right|} \tag{9}$$

Condition (5) holds for every M_s and particularly for $M_s = \frac{W_s}{|\Delta t_b|}$, thus

$$|y_s(t_M)| > \frac{W_s}{|\Delta t_b|}$$

But this is a contradiction to the statement in (9). Therefore, the assumption that the equilibrium point x_s^{eq} is not stable is false.

Secondly, we will prove that if $\int_0^\infty |x_s(t) - x_s^{eq}|\,dt$ converges then $\operatorname*{Lim}_{t\to\infty}\left(x_s(t) - x_s^{eq}\right) = 0$ (10)

In order to do that, we will linearize the nonlinear system around *m* operating points. It is important to note that the equilibrium points of these linear systems do not have to coincide with the equilibrium point of the nonlinear system. However, we will show that when the system is asymptotically stable the equilibrium points of the linear systems tend to converge to the equilibrium point of the nonlinear system when t goes to infinity.

Let's make the transformation $y_s(t) = x_s(t) - x_s^{eq}, \forall s$. The equilibrium point for the new nonlinear system will be the origin, i.e. $y_s^{eq} = 0, \forall s$ and therefore

$\int_0^\infty |x_s(t) - x_s^{eq}|\,dt = \int_0^\infty |y_s(t)|\,dt$, $\forall s$ implying that

if $\int_0^\infty |x_s(t) - x_s^{eq}|\,dt$ converges then $\int_0^\infty |y_s(t)|\,dt$ converges (11)

Applying Definition A.2, we can approximate $\dot{\mathbf{y}}(t)$ by $\dot{\mathbf{z}}(t)$ after linearizing the system $\dot{\mathbf{y}}(t)$ around *m* operating points $\{\mathbf{z}(t_{p-1}), p=1,..,m; t_0<t_1<...<t_m\}$ as follows:

$$\dot{\mathbf{z}}(t) = \begin{cases} \mathbf{A}_1\mathbf{z}(t) + \mathbf{b}_1, t_0 \le t < t_1 \\ \mathbf{A}_2\mathbf{z}(t) + \mathbf{b}_2, t_1 \le t < t_2 \\ \vdots \\ \mathbf{A}_m\mathbf{z}(t) + \mathbf{b}_m, t_{m-1} \le t < t_m \end{cases}$$

Let's consider $\Delta t_p = t_p - t_{p-1} = \text{constant} = h > 0,\;\; p = 1,..,m$. Thus, the interval of validity for each linear system is $[t_{p-1}, t_{p-1} + h)$

Now the integral $\int_0^\infty |y_s(t)|\,dt$ can be calculated as the sum of the integrals of *m* linear systems when *m* goes to infinity as follows:

$$\int_0^{\infty} |y_s(t)|\, dt = \lim_{m\to\infty} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_{i-1}+h} |z_s(t)|\, dt \tag{12}$$

Let's make $\Psi_i = \int_{t_{i-1}}^{t_{i-1}+h} |z_s(t)|\, dt$ and $S_m = \sum_{i=1}^{m} \Psi_i$ (13)

Replacing S_m in (12) we have

$$\int_0^{\infty} |y_s(t)|\, dt = \lim_{m\to\infty} S_m = \sum_{i=1}^{\infty} \Psi_i \tag{14}$$

From (11) we know that $\int_0^{\infty} |y_s(t)|\, dt$ converges and therefore from (14) we obtain that

$\sum_{i=1}^{\infty} \Psi_i$ converges also. Therefore, from Lemma A.1 we obtain that $\lim_{m\to\infty} \Psi_m = 0$ (15)

From (15) and (13) we obtain:

$$\lim_{m\to\infty} \int_{t_{m-1}}^{t_{m-1}+h} |z_s(t)|\, dt = 0 \tag{16}$$

From Theorem A.1 we know that solution of the system $\dot{\mathbf{z}} = \mathbf{A}_m \mathbf{z}(t) + \mathbf{b}_m,\ t_{m-1} \le t < t_m$, is given by:

$$z_s(t) = z_{sm}^{eq} + \sum_{j\in H_1} \left\{ \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm} \right) e^{\mathrm{Re}(\lambda_{jm})t} \right\} + \sum_{k\in H_2} \left\{ 2 v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \sin\left(\mathrm{Im}(\lambda_{km})t + \theta_{skm} \right) \right\} \tag{17}$$

Note that all the parameters on the right-hand side of the equality have also a subindex *m*, denoting that they are dependant of the *m*-th linear model. In other words, each linear model *p* (p=1,..,m) has its own parameters (constants, eigenvalues and eigenvectors).

Taking absolute value in both sides and rearranging terms

$$\left| \sum_{j\in H_1} \left\{ \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm} \right) e^{\mathrm{Re}(\lambda_{jm})t} \right\} + \sum_{k\in H_2} \left\{ 2 v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \sin(\mathrm{Im}(\lambda_{km})t + \theta_{skm}) \right\} \right| = \left| z_s(t) - z_{sm}^{eq} \right|$$

By the property of absolute value: $\big| |a| - |b| \big| \le |a+b|$

$$\left| \left| \sum_{j\in H_1} \left\{ \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm} \right) e^{\mathrm{Re}(\lambda_{jm})t} \right\} \right| - \left| \sum_{k\in H_2} \left\{ 2 v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \sin(\mathrm{Im}(\lambda_{km})t + \theta_{smk}) \right\} \right| \right| \le \left| z_s(t) - z_{sm}^{eq} \right|$$

By the property of absolute value: $|a-b| \le |a| + |b|$ and knowing that $|\sin(t)| \le 1, \forall t \in R$ and $2v_{skm}e^{\mathrm{Re}(\lambda_{km})t} \ge 0$ then the following inequality holds

$$\left\| \sum_{j\in H_1} \left\{ \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm} \right) e^{\mathrm{Re}(\lambda_{jm})t} \right\} \right| - \sum_{k\in H_2} \left\{ 2v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \right\} \right| - \left| z_{sm}^{eq} \right| \le \left| z_s(t) \right| \tag{18}$$

Integrating both terms of the inequality (18) from t_{m-1} to $t_{m-1}+h$

$$\int_{t_{m-1}}^{t_{m-1}+h} \left\| \sum_{j\in H_1} \left\{ \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm} \right) e^{\mathrm{Re}(\lambda_{jm})t} \right\} \right| - \sum_{k\in H_2} \left\{ 2v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \right\} \right| dt - \int_{t_{m-1}}^{t_{m-1}+h} \left| z_{sm}^{eq} \right| dt \le \int_{t_{m-1}}^{t_{m-1}+h} \left| z_s(t) \right| dt$$

Applying different properties of absolute value and the integral we obtain

$$\left\| \sum_{j\in H_1} \left\{ \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm} \right) \int_{t_{m-1}}^{t_{m-1}+h} e^{\mathrm{Re}(\lambda_{jm})t}\, dt \right\} \right| - \sum_{k\in H_2} \left\{ 2v_{skm} \int_{t_{m-1}}^{t_{m-1}+h} e^{\mathrm{Re}(\lambda_{km})t}\, dt \right\} \right| - \int_{t_{m-1}}^{t_{m-1}+h} \left| z_{sm}^{eq} \right| dt$$

$$\le \int_{t_{m-1}}^{t_{m-1}+h} \left| z_s(t) \right| dt \tag{19}$$

Evaluating the integrals $\int_{t_{m-1}}^{t_{m-1}+h} e^{\mathrm{Re}(\lambda_{jm})t}\, dt$ and $\int_{t_{m-1}}^{t_{m-1}+h} e^{\mathrm{Re}(\lambda_{km})t}\, dt$:

$$\int_{t_{m-1}}^{t_{m-1}+h} e^{\mathrm{Re}(\lambda_{jm})t}\, dt = \left. \frac{e^{\mathrm{Re}(\lambda_{jm})t}}{\mathrm{Re}(\lambda_{jm})} \right|_{t_{m-1}}^{t_{m-1}+h} = e^{\mathrm{Re}(\lambda_{jm})t_{m-1}} \left[\frac{e^{\mathrm{Re}(\lambda_{jm})h} - 1}{\mathrm{Re}(\lambda_{jm})} \right] \tag{20}$$

In a similar fashion

$$\int_{t_{m-1}}^{t_{m-1}+h} e^{\mathrm{Re}(\lambda_{km})t}\, dt = e^{\mathrm{Re}(\lambda_{km})t_{m-1}} \left[\frac{e^{\mathrm{Re}(\lambda_{km})h} - 1}{\mathrm{Re}(\lambda_{km})} \right] \tag{21}$$

Evaluating the integral $\int_{t_{m-1}}^{t_{m-1}+h} \left| z_{sm}^{eq} \right| dt$

$$\int_{t_{m-1}}^{t_{m-1}+h} \left| z_{sm}^{eq} \right| dt = \left| z_{sm}^{eq} \right| h \tag{22}$$

Replacing (20-22) in (19)

$$\left\| \sum_{j\in H_1} \left\{ \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm} \right) e^{\mathrm{Re}(\lambda_{jm})t_{m-1}} \left[\frac{e^{\mathrm{Re}(\lambda_{jm})h} - 1}{\mathrm{Re}(\lambda_{jm})} \right] \right\} \right| - \sum_{k\in H_2} \left\{ 2v_{skm}\, e^{\mathrm{Re}(\lambda_{km})t_{m-1}} \left[\frac{e^{\mathrm{Re}(\lambda_{km})h} - 1}{\mathrm{Re}(\lambda_{km})} \right] \right\} \right| -$$

$$\left| z_{sm}^{eq} \right| h \le \int_{t_{m-1}}^{t_{m-1}+h} \left| z_s(t) \right| dt \tag{23}$$

Let’s define the following constants

$$E_{sjm} = \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm}\right)\left[\frac{e^{\mathrm{Re}(\lambda_{jm})h} - 1}{\mathrm{Re}(\lambda_{jm})}\right]$$

$$F_{skm} = 2v_{skm}\left[\frac{e^{\mathrm{Re}(\lambda_{km})h} - 1}{\mathrm{Re}(\lambda_{km})}\right]$$

Replacing these constants in (23)

$$\left\|\sum_{j\in H_1}\left\{E_{sjm} e^{\mathrm{Re}(\lambda_{jm})t_{m-1}}\right\} - \sum_{k\in H_2}\left\{F_{skm}\, e^{\mathrm{Re}(\lambda_{km})t_{m-1}}\right\}\right| - \left|z_{sm}^{eq}\right| h \le \int_{t_{m-1}}^{t_{m-1}+h} \left|z_s(t)\right| dt$$

Taking the limit when *m* goes to infinity, and knowing that:

$$(m \to \infty) \Rightarrow (t_{m-1} \to \infty) \Rightarrow (t \to \infty) \tag{24}$$

$$\lim_{t_{m-1}\to\infty}\left\|\sum_{j\in H_1}\left\{E_{sjm} e^{\mathrm{Re}(\lambda_{jm})t_{m-1}}\right\} - \sum_{k\in H_2}\left\{F_{skm}\, e^{\mathrm{Re}(\lambda_{km})t_{m-1}}\right\}\right| - \lim_{m\to\infty}\left|z_{sm}^{eq}\right| h \le \lim_{m\to\infty}\int_{t_{m-1}}^{t_{m-1}+h} \left|z_s(t)\right| dt \tag{25}$$

However, by (16) we know that $\lim_{m\to\infty}\int_{t_{m-1}}^{t_{m-1}+h} \left|z_s(t)\right| dt = 0$, and therefore the only way to satisfy this condition is if the terms on the left-hand side of inequality (25) are zero.

The first term, $\lim_{t_{m-1}\to\infty}\left\|\sum_{j\in H_1}\left\{E_{sjm} e^{\mathrm{Re}(\lambda_{jm})t_{m-1}}\right\} - \sum_{k\in H_2}\left\{F_{skm}\, e^{\mathrm{Re}(\lambda_{km})t_{m-1}}\right\}\right|$, can take two values when t_{m-1} goes to infinity: zero or infinity. The requirement for this term to be zero is that the real part of all the eigenvalues of $\mathbf{A}_m$ has to be negative. (26)

The second term, $\lim_{m\to\infty}\left|z_{sm}^{eq}\right| h$, will be zero only if $\left|z_{sm}^{eq}\right|$ is zero (because h>0). Therefore, $z_{sm}^{eq} = 0$ when *m* goes to infinity, which coincides with the equilibrium point of the nonlinear system $\mathbf{y}(t)$ that is also zero.

Now we will show that if the real part of all the eigenvalues of $\mathbf{A}_m$ is negative then $\lim_{t\to\infty}\left(z_s(t) - z_{sm}^{eq}\right) = 0$

Taking limits to both sides of equation (17) and rearranging terms we have

$$\lim_{t\to\infty}\left(z_s(t) - z_{sm}^{eq}\right) = \lim_{t\to\infty}\sum_{j\in H_1}\left\{\mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm}\right) e^{\mathrm{Re}(\lambda_{jm})t}\right\} +$$

$$\lim_{t\to\infty} \sum_{k\in H_2} \left\{ 2v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \sin\left(\mathrm{Im}(\lambda_{km})t + \theta_{skm}\right) \right\} \tag{27}$$

Because $\mathrm{Re}(\lambda_{jm}) < 0, \forall j \in H_1, \forall m$ then $\lim_{t\to\infty} \sum_{j\in H_1} \left\{ \mathrm{Re}\left(\alpha_{jm} r_{sjm} + \frac{\beta_{jm}}{\lambda_{jm}} r_{sjm} \right) e^{\mathrm{Re}(\lambda_{jm})t} \right\}$ is zero (28)

The calculation of the second limit in (27) requires the use of the sandwich theorem. The function of the second limit in can be bounded as follows:

$$-\sum_{k\in H_2} \left\{ 2v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \right\} \le \sum_{k\in H_2} \left\{ 2v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \sin(\mathrm{Im}(\lambda_{km})t + \theta_{skm}) \right\} \le \sum_{k\in H_2} \left\{ 2v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \right\}$$

Because $\mathrm{Re}(\lambda_{km}) < 0, \forall k \in H_2, \forall m$ then

$\lim_{t\to\infty}\left(-\sum_{k\in H_2} \left\{ 2v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \right\} \right) = \lim_{t\to\infty}\left(\sum_{k\in H_2} \left\{ 2v_{skm} e^{\mathrm{Re}(\lambda_{km})t} \right\} \right) = 0$, therefore by the sandwich theorem

$$\lim_{t\to\infty} \sum_{k\in H_2} \left\{ 2v_{sk} e^{\mathrm{Re}(\lambda_k)t} \sin(\mathrm{Im}(\lambda_k)t + \theta_{sk}) \right\} = 0 \tag{29}$$

Replacing the results of (28) and (29) in (27) then $\lim_{t\to\infty}\left(z_s(t) - z_{sm}^{eq} \right) = 0$

Considering that $z_{sm}^{eq} = 0$ when $m \to \infty$ and (24) the previous expression can be written as

$$\lim_{t\to\infty}\left(z_s(t) \right) = 0 \tag{30}$$

Because $\mathbf{z}(t)$ is as an approximation of $\mathbf{y}(t)$ and from (30) we conclude that

$$\lim_{t\to\infty}\left(y_s(t) \right) = 0 \tag{31}$$

But we know that $y_s(t) = x_s(t) - x_s^{eq}$. Thus, taking limits to both sides when t goes to infinity and from (31) we have

$$\lim_{t\to\infty}\left(y_s(t) \right) = \lim_{t\to\infty}\left(x_s(t) - x_s^{eq} \right) = 0 \tag{32}$$

From (11-32) we conclude that

if $\int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt$ converges then $\lim_{t\to\infty}\left(x_s(t) - x_s^{eq} \right) = 0$ (33)

From (4), (33) and Definition 3 we demonstrate that

The state variable x_s is asymptotically stable around the EP x_s^{eq} if $\int_0^\infty \left| x_s(t) - x_s^{eq} \right| dt$ converges ■

B.2 Proof of Theorem 2

We will prove by contradiction that if $\int_0^\infty |x_s(t) - a_s| \, dt$ converges then $a_s = x_s^{eq}$

So let's assume to contrary that $a_s \neq x_s^{eq}$. (34)

Let's make $y_s(t) = x_s(t) - a_s$, thus (35)

$\int_0^\infty |x_s(t) - a_s| \, dt = \int_0^\infty |y_s(t)| \, dt$, $\forall s$ implying that

if $\int_0^\infty |x_s(t) - a_s| \, dt$ converges then $\int_0^\infty |y_s(t)| \, dt$ converges (36)

From (34) and (35) we derive that $y_s^{eq} = x_s^{eq} - a_s \neq 0$ (37)

From the proof of Theorem 1 we know that if $\int_0^\infty |y_s(t)| \, dt$ converges then $\operatorname*{Lim}_{t \to \infty}(y_s(t)) = 0$, and this statement is satisfied independently of the initial conditions of the system $\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t))$ (38)

By Definition 1, if the system $\dot{\mathbf{y}}(t) = \mathbf{f}(\mathbf{y}(t))$ starts at the equilibrium point y_s^{eq} then $y_s(t) = y_s^{eq}, \forall t$, and thus

$\operatorname*{Lim}_{t \to \infty}(y_s(t)) = \operatorname*{Lim}_{t \to \infty}(y_s^{eq}) = y_s^{eq}$ (39)

From (37) and (39)

$\operatorname*{Lim}_{t \to \infty}(y_s(t)) \neq 0$

But this is a contradiction to the statement in (38). Therefore, the assumption that $a_s \neq x_s^{eq}$ is false ■

B.3 Proof of Theorem A.1

We will no provide a detailed proof of this theorem but just very general steps that lead to its demonstration.

Step 1: The solution of the linear system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}$; $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{b} \, d\tau$ (DeCarlo 1989)

Step 2: If matrix $\mathbf{A}$ has distinct eigenvalues, then it is possible to find the following transformation: $e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$ (DeCarlo 1989), where $\mathbf{T} \in C^{n \times n}$ is a matrix that has the eigenvector $\mathbf{r}_j$ as its *j*-th column, that is, $\mathbf{T} = [\mathbf{r}_1 \, \mathbf{r}_2 \cdots \mathbf{r}_n]$

D is a diagonal matrix whose diagonal entries are the eigenvalues of **A**

Step 3: If matrix **A** has distinct nonzero eigenvalues, then the solution to the system $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b};\ \mathbf{x}(0) = \mathbf{x}_0$ can be expressed as

$$\mathbf{x}(t) = \mathbf{x}^{eq} + \sum_{i=1}^{n} \left\{ \left(\alpha_i + \frac{\beta_i}{\lambda_i} \right) e^{\lambda_i t} \mathbf{r}_i \right\} \text{ (from steps 1 and 2)}$$

where λ_i and $\mathbf{r}_i$ are the corresponding eigenvalues and eigenvectors of **A**, and $\alpha_i \in C$, $\beta_i \in C$ are constants defined as

$$[\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]^T = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \cdots \quad \mathbf{r}_n]^{-1} \mathbf{x}_0$$

$$[\beta_1 \quad \beta_2 \quad \cdots \quad \beta_n]^T = [\mathbf{r}_1 \quad \mathbf{r}_2 \quad \cdots \quad \mathbf{r}_n]^{-1} \mathbf{b}$$

Step 4: We can write expression $\mathbf{x}(t) = \mathbf{x}^{eq} + \sum_{i=1}^{n} \left\{ \left(\alpha_i + \frac{\beta_i}{\lambda_i} \right) e^{\lambda_i t} \mathbf{r}_i \right\}$ as

$$\mathbf{x}(t) = \mathbf{x}^{eq} + \sum_{j \in H_1} \left\{ \left(\alpha_j + \frac{\beta_j}{\lambda_j} \right) e^{\lambda_j t} \mathbf{r}_j \right\} + \sum_{j \in \{H - H_1\}} \left\{ \left(\alpha_j + \frac{\beta_j}{\lambda_j} \right) e^{\lambda_j t} \mathbf{r}_j \right\}$$

where H_1 is a set of indexes j such that $\mathrm{Im}(\lambda_j) = 0$, and H is the set of indexes that represent all the eigenvalues of matrix **A**

Step 5: The summation $\sum_{j \in \{H - H_1\}} \left\{ \left(\alpha_j r_{sj} + \frac{\beta_j}{\lambda_j} r_{sj} \right) e^{\lambda_j t} \right\}$ can be expressed as

$$\sum_{k \in H_2} \left\{ (u_{sk} + w_{sk} i) e^{c_k t} e^{d_k i t} + (u_{sk} - w_{sk} i) e^{c_k t} e^{-d_k i t} \right\} \tag{40}$$

where

H_2 is a set of indexes k such that $\mathrm{Im}(\lambda_k) \neq 0$[§§§§]

$$\lambda_k = c_k + d_k i,\ \lambda_{k+1} = c_k - d_k i,\ \left(\alpha_k r_{sk} + \frac{\beta_k}{\lambda_k} r_{sk} \right) = u_{sk} + w_{sk} i,\ \left(\alpha_{k+1} r_{s,k+1} + \frac{\beta_{k+1}}{\lambda_{k+1}} r_{s,k+1} \right) = u_{sk} - w_{sk} i,$$

Step 6: By Euler's formula (Spivak 1967) we know that $e^{\gamma i} = \cos(\gamma) + \sin(\gamma) i$; thus replacing this formula in the expression (40) and simplifying:

$$\sum_{j \in \{H - H_1\}} \left\{ \left(\alpha_j r_{sj} + \frac{\beta_j}{\lambda_j} r_{sj} \right) e^{\lambda_j t} \right\} = \sum_{k \in H_2} \left\{ 2 e^{c_k t} \left(u_{sk} \cos(d_k t) - w_{sk} \sin(d_k t) \right) \right\}$$

Step 7: We know that $c_k = \mathrm{Re}(\lambda_k), d_k = \mathrm{Im}(\lambda_k)$ and

$$u_{sk} = \mathrm{Re}\left(\alpha_k r_{sk} + \frac{\beta_k}{\lambda_k} r_{sk} \right), w_{sk} = \mathrm{Im}\left(\alpha_k r_{sk} + \frac{\beta_k}{\lambda_k} r_{sk} \right)$$

[§§§§] Sets $\{H\text{-}H_1\}$ and H_2 point the same eigenvalues; thus, the cardinality of $\{H\text{-}H_1\}$ is twice the cardinality of H_2

If we make

$$\mathrm{v}_{\mathrm{sk}} = \left\| \alpha_{\mathrm{k}} \mathrm{r}_{\mathrm{sk}} + \frac{\beta_{\mathrm{k}}}{\lambda_{\mathrm{k}}} \mathrm{r}_{\mathrm{sk}} \right\|$$, where $\|z\|$ is the modulus of $z \in C$

$$\theta_{\mathrm{sk}} = \arctan\left(\mathrm{Re}\left(\alpha_{\mathrm{k}} \mathrm{r}_{\mathrm{sk}} + \frac{\beta_{\mathrm{k}}}{\lambda_{\mathrm{k}}} \mathrm{r}_{\mathrm{sk}} \right) \Big/ -\mathrm{Im}\left(\alpha_{\mathrm{k}} \mathrm{r}_{\mathrm{sk}} + \frac{\beta_{\mathrm{k}}}{\lambda_{\mathrm{k}}} \mathrm{r}_{\mathrm{sk}} \right) \right)$$, expressed in radians

After some simplifications the proof of the theorem is achieved.