The Basics of System Dynamics: Discrete vs. Continuous Modelling of Time[1]

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Abstract
System Dynamics deals with modelling of processes over time. In this paper we discuss two ways to model changes over time: finite vs. infinitesimal. This leads to two different concepts of time: discrete time as a succession of time points and time intervals vs. continuous time. Although the System Dynamics concept of distinguishing between stocks and flows suggests a discrete modelling of time, System Dynamics is considered mostly a modelling technique based on continuous time. In the paper we argue to see System Dynamics modelling compatible with both the continuous and the discrete concept of time. We will show that this "hybrid" potential makes System Dynamics a superior technique for modelling time, which combines the advantages of continuous and discrete time concepts.

Keywords
System Dynamics, Time, Discrete Time, Continuous Time

INTRODUCTION

System Dynamics (SD) models deal with time. In every SD model time is *the* central independent variable and the main organizing principle. Despite of its central role, both theory and practice of SD does not much bother about how to model time. For SD learners the usage of *dt* as the symbol for a time-step might seem somewhat confusing. In SD theory *dt* represents an infinitesimal time-step and in practical modelling *dt* stands for the duration of the finite time-step of the simulation. For experts in SD this ambiguity seems to be no problem at all.

In this paper we will look "behind the curtains" of how time is used. We will show that there are two different modes of dealing with time, which are both related to SD: (a) the continuous concept of time and (b) the discrete concept of time. The discrete concept of time is based upon the distinction between time-points and time intervals. The continuous concept of time is closely related to infinitesimal mathematics and does not need any time intervals for explaining change, as change happens as momentarily change at each point of time.

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MODELLING OF TIME

In this section we will discuss two different modes of how to model time. The main difference between these modes is how they deal with changes over time, which may be finite or infinitesimal.

Finite Changes over Time

This mode of dealing with changes over time resembles our everyday experience. It is based on a very simple principle: *In order to notify some change, some time has to pass by*.[4] As a consequence changes are related to time intervals, whereas the state of a variable is specified for specific points in time. A simple example for this approach is the relation between a population and changes of population. The number of people is specified for some point in time. The population changes over time due to births, deaths, immigration and emigration. All these figures are actually counted as statistical data for finite time-intervals (days, months, years).

The distinction between time-points and time-intervals yields to two distinct types of data: data related to time-points and data related to time-intervals. For simplicity let us call the data related to time-points "stocks" and the data related to time-intervals "flows". We will discuss later whether these connotations are identical or just similar to the stock-flow-concept in SD. Flows are associated to stocks as stocks are exclusively changed over time via flows. Often it is possible to distinguish two structurally different subtypes of flows: inflows and outflows. For a population births and immigration are inflows, whereas emigration and deaths are outflows. To distinguish between births (inflow) and deaths (outflow) is crucial for practical understanding of populations. It would not make practical sense to define e.g. deaths as "negative births".

If we see stocks as state variables for time-points and flows as changes of the stocks for certain time-intervals, the relation between stocks and flows is trivial arithmetic. For a given time interval (t_0,t_1) and given flows for that time interval we can calculate the "new" value of the stock at the end of the time interval according to the following equation:

$$\text{stock}(t_1) = \text{stock}(t_0) + \text{inflows}(t_0,t_1) - \text{outflows}(t_0,t_1) \qquad [1]$$

This simple procedure of calculating the new value of a stock needs just elementary arithmetic, without any infinitesimal differentiation or integration process. In some cases the value of the stock at t_0 and t_1 are given and we can calculate the (in)flow for the time interval (t_0,t_1). For example, a mileage counter in a car shows at the start of a trip 10382 miles, and at the end of the trip two hours later a value of 10502 miles. The distance travelled in two hours was 10502 miles - 10382 miles = 120 miles, which implies the car drove at an average speed of 60 miles per hour.

This example gives us also a clue that the flows as changes of stocks over time intervals are not precisely the same as flows in SD. In SD the flow would be 60 miles/h, which is a speed. In the example we have a "flow" of 120 miles distance travelled in 2 hours. Absolute changes

[4] This principle holds no matter whether the changes are continuous (like water filling a bathtub via a faucet) or instantly (like changes of a bank account via a transferral) Even for defining the infinitesimal change at some point in time we need to know the development over a short time-span around the time-point of interest.

over time intervals and relative changes per time unit are not the same, although the relation between them is very easy to calculate:

$$\text{absolute change}(t_0,t_1) = \text{average change per time unit} \cdot (t_1 - t_0) \qquad [2]$$

Here $(t_1 - t_0)$ is the length of the time interval. Often [2] is used for calculating the average change per time unit with the absolute change and the length of the time interval given.

Infinitesimal Changes over Time

The second way to model change over time is to make time intervals infinitesimally short using a mathematical limes-process. Mathematically is this the first derivative of some state variable with time as the independent variable. We can do so only if the state function over time is differentiable. This implies necessarily that there are no discontinuities in the state function. The derivative function usually is also defined for a continuum of time points that makes it possible to repeat the differentiation process (provided that the derivative is differentiable, too), yielding the second derivative function. We see that the concept of derivatives allows formulating higher order changes for single time points. For the concept of finite changes over time this is not so easy, since first order changes are defined not for time-points, but for time intervals. When comparing two changes of adjacent time interval we get some figure which is related to the overall length of the adjacent intervals.

The concept of continuous changes is typically modelled by differential equations or infinitesimal accumulation processes (i.e. mathematical integration) over time. It has a strong foundation in the mathematical field of Calculus and is applied for modelling time-related processes in many sciences.

What role do changes over time intervals play in this approach? Since all changes are related to time points via momentarily rates of change, there is no more need for dealing with time-intervals. This elimination of time-intervals has the advantage of mathematical elegance, allowing higher derivatives, but has also the consequence that there is no need for discerning between data related to points in time vs. time-interval-related data, as in the finite approach of change.

In the finite concept of time, we can discern between stocks and flows in a trivial manner: stocks are data related to time-points and flows are changes of stocks, related to time-intervals. Under the infinitesimal concept of time both stocks and flows are related to time-points, so they cannot be any longer discerned in that simple way as in the finite concept of time. Actually in the continuous model of time it is often a matter of the standpoint whether a certain variable is considered to be a stock or a flow. For example considering a linear motion over time the velocity can be considered as a stock, being changed by acceleration as the associated flow. Velocity can also be considered as a flow changing the position(place) as a stock over time. So the stock-flow-concept becomes somewhat arbitrary when we deal with continuous changes over time.

Two Models of Time: Discrete vs. Continuous

The two modes of modelling changes over time can be formalized to two distinct concepts of seeing and modelling time: discrete time and continuous time. The concept of discrete time is based upon a distinction between time-points and time intervals. Typically the time axis is divided into a number of adjacent time-segments (which usually are of fixed length). Both the number of time intervals and time points that are specified are finite. Actually the number of time points = Number of time-intervals +1.

The concept of continuous time models time as a continuum of subsequent time-points. This implies that data given for some time-span are specified as a continuous function over time. Typically this function is also assumed to be differentiable, so that changes of the state variable over time can be modelled as the first derivative of the state function. Time intervals do not play a specific role as an organizing principle. Of course one can look at any time intervals, but there are no pre-defined time-steps of finite length, which structure the whole model.

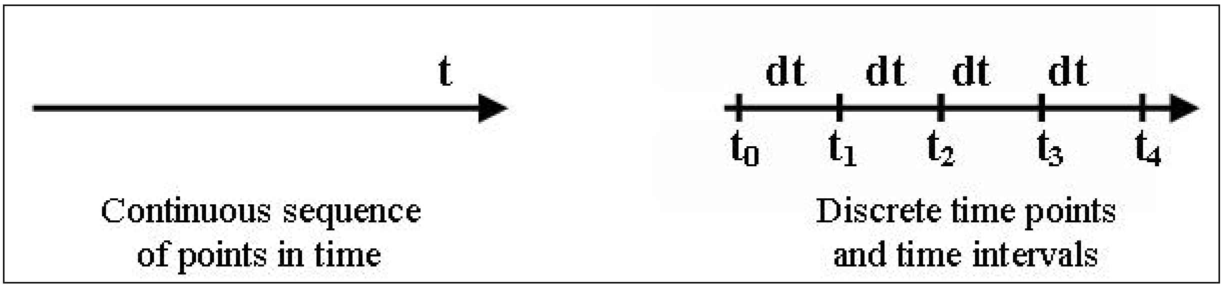

Figure 1: Continuous Time vs. Discrete Time

Figure 1 sketches the concepts of continuous time on the left and the discrete time model on the right. The distinction between a continuous and discrete concept of time can be explained due to different types of time-related data. If the number of data given is finite, the discrete model of time applies quite naturally, no matter whether the data given belong to time-points or represent changes over time-intervals. For continuous data, like continuous waves in signal theory, the continuous concept of time fits naturally.

Fixed vs. Constant Time-steps

Discrete Event Simulations are typically based on fixed iteration. A typical example is a stochastic process on a discrete Markov Chain with a finite number of states and a transition matrix describing the probabilities for changing from one state to another during one iteration step. The overall state of the system after *n* steps can be calculated by the product of the *n* transition matrices or the n^{th} power of a fixed transition matrix. In such a system there is no way to vary the duration of a time-step, since a time-step is represented just by one iteration step.

In SD modelling the time-step is a parameter, which can be chosen freely and varied over different simulation runs. Within one simulation run it is typically kept constant, but this is just a property of SD simulation software, which traditionally does not adopt strategies of varying step widths within one simulation run.

TIME IN SYSTEM DYNAMICS

The traditional view

Sterman (2000, p.193f.) describes four equivalent representations of a stock and flow structure: Hydraulic Metaphor, Stock-Flow Diagram, Integral Equation, and Differential equation (see Figure 2). In the “Hydraulic Metaphor” the stock is represented through the water in a bathtub at any time. The amount of water in the bathtub either increases (water flowing in through tap) or decreases (water flowing out through drain), excluding outside factors such as evaporation. A "Stock-Flow Diagram" already has an unambiguous mathematical meaning as a stock accumulates its flows. The stock increases through material inflow and decreases through material outflow. The “Integral Equation” describes the same stock-flow principle, as the new *Stock(t)* is defined through the initial *Stock(t_0)* plus all *Inflow(s)* subtracted by all the *Outflow(s)* between the time t_0 and time *t*. Finally, “Differential Equation” describe the net rate change of a stock as the *Inflow(t)* subtracted by the *Outflow(t)*.

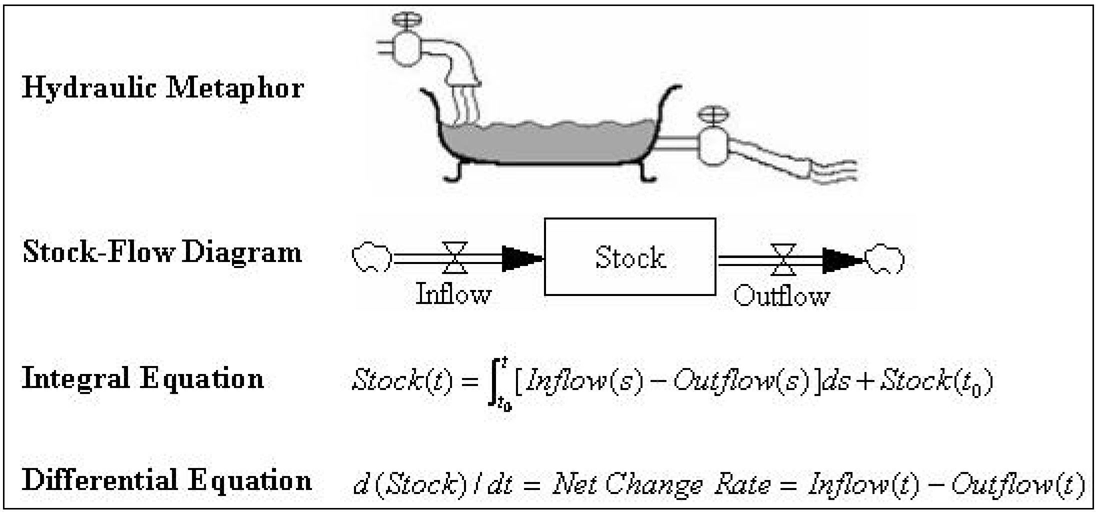

Figure 2: Four Representations of Stock-Flow Structure (Sterman, 2000, p.194)

Sterman stresses that these four modes of representing a stock-flow system are equivalent: “*Each representation contains precisely the same information.*” (Sterman 2000, p.194). We agree with Sterman under the assumption that the whole modelling is based on the concept of continuous time. However, we assume that within a discrete concept of time the four representations are not necessarily precisely equivalent. Given finite time intervals as simulation steps there might be flows changing a stock that are *not* capable with integral equations or differential equations. We will see that SD models can cope with such types of flows, too. In such a case the stock-flow-diagram (representing an actual SD model) and the representation by a differential- or integral equation is not precisely the same, so that Sterman’s statement “precisely the same information” no longer holds.

The core Elements of System Dynamics and Time

The core idea of SD modelling is the accumulation of flows over finite time intervals of duration *dt*. Forrester (1968) argues clearly that the concept of stocks as accumulations is more appropriate for SD than the concept of flows as derivatives: “*Formulation of systems in terms of differential equations obscures for many students the direction of causality within systems […]. Representing a system in terms of integral equations gives a more immediate and evident equivalence between the model and the real world.*” (Forrester 1968, p. 6-12.).

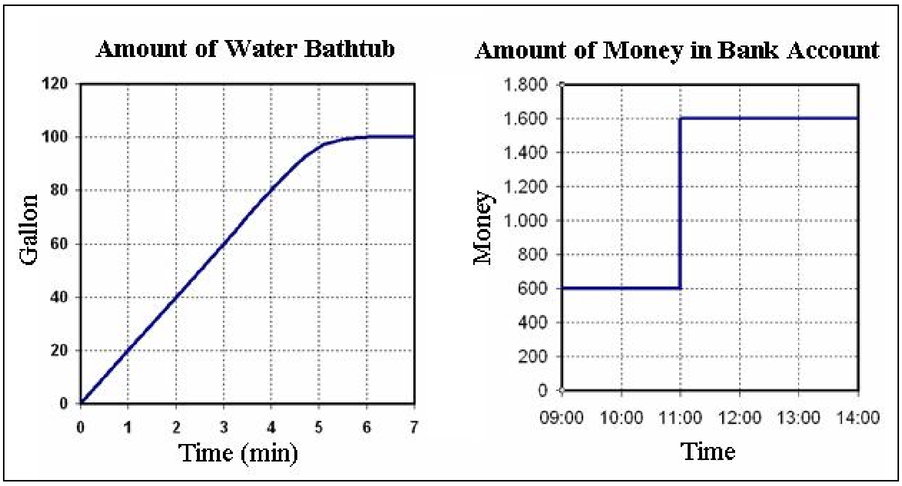

Figure 3: Accumulation of Bathtub and Bank Account (Ossimitz and Lapp 2006, p.81)

Moreover, the accumulation process in SD is more powerful than infinitesimal mathematical integration. It allows accumulating not only smooth flows as the continuous inflow of water in a bathtub, but also flows that have the character of instant pulses such as incoming payments on a bank account that instantly change the value of the accumulated money, as illustrated in Figure 3. This is possible due to the finite time structure of SD-models. In a strictly infinitesimal integration process a pulse-like instant increase of a stock could not be modelled, since at the time-points of the instant change of the state variable no derivative can be specified.

We can illustrate the necessity of accumulation of both continuous and non-continuous flows also by using Sterman`s bathtub metaphor (Sterman 2000, p.194): There might be not only a continuous inflow through the faucet, one might also add a certain amount of water in a moment by pouring a bucket of water into the bathtub. This would give the stock of water in the bathtub an instant rise which cannot be modelled precisely with infinitesimal changes over time.

Another advantage of the SD integration process in comparison to differential equations is the possibility to attach several inflows and outflows to the same state variable (see Figure 4). For a population we might discern between four distinct flows: births, deaths, immigration and emigration. The derivative of a state-function at a given point of time is just a single figure. It represents just the net-change of the corresponding change variable. Taking this into account the development of different flows over time gives a much more realistic view of why the state variable behaves in a specific manner.

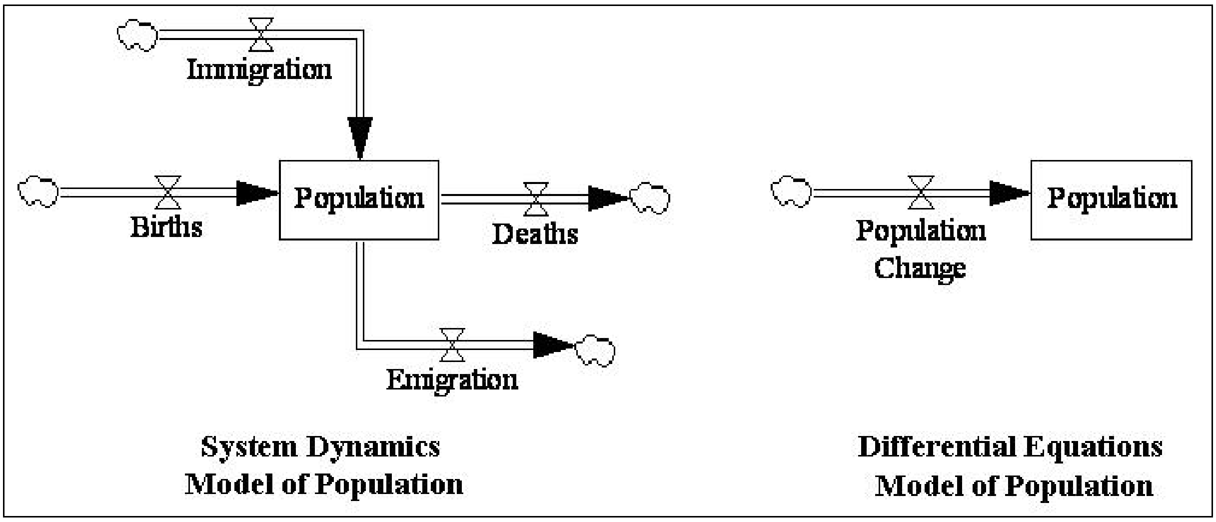

Figure 4: Using SD or Differential Equations to Model a Population

System Dynamics allows both discrete and continuous time
Altogether we can conclude that SD is compatible both with the continuous *and* the discrete concept of time. On the one hand, there is a widely accepted consensus that SD is a modelling methodology based on continuous time. The time-step can be made as small as one wishes and all SD modelling software products offer at least a Runge-Kutta-4th order simulation for good numerical results for continuous models. On the other hand, core aspects of SD modelling like the strict distinction of stocks and flows combined with the specific diagramming of stock-flow-diagrams, a finite time-step of constant length *dt*, and discontinuous modelling functions (like pulse or ramp functions), are naturally compatible with a discrete modelling of time in finite time-steps.
Concerning the way SD deals with time we can say that SD is a kind of "hybrid" methodology, being compatible with both the continuous and the discrete concept of time. A clear indication for this ambiguity is the usage and meaning of the term *dt*. In mathematical calculus *dt* means a infinitesimally short time-step, a differential. In SD *dt* means a time-step of finite length. So *dt* is both a infinitesimal and a finite time-step, thus allowing us to associate it with both a continuous and a discrete concept of time.

CONCLUSIONS

One of the biggest advantages of the System Dynamics method is its structural simplicity of modelling time, based on a clear distinction between stock variables related to time-points and flows related to time intervals. This concept fits perfectly to our everyday perception of time, where we discern naturally between time-points and time intervals.

Nevertheless SD has never adopted a clear discrete concept of time. Instead the traditional mathematical concept of infinitesimal changes at individual time-points was adapted to SD and is nowadays considered to be the theoretical framework to describe how the SD methodology deals with time. The numerical integration methods of SD fit perfectly in the mathematical tradition of numerical methods to integrate differential equations, and *dt* has always denoted an infinitesimal interval of time in calculus.

Yet the concept of continuous time has two disadvantages: (1) It needs strong mathematical assumptions of continuous state-functions and does not work if these assumptions do not apply. (2) It eliminates de facto the structural difference between points in time and time-intervals by making time-intervals infinitesimally short, so that they can be treated as time points. As a consequence the concept of continuous time makes it easier to mix up stocks and flows, because in this concept the easy structural difference between time-points and time-intervals is no more relevant.

The empirical research on “Bathtub Dynamics” (Booth-Sweeney and Sterman 2001, Kainz and Ossimitz 2002, Ossimitz 2002) shows that even highly educated persons have massive difficulties in discerning between stocks and flows. We can state it as a hypothesis for further empirical research that this might be due to a poor understanding of the different concepts of time. Maybe even the way we present SD as a methodology exclusively based on a concept of continuous time and then doing simulation runs using discrete time steps afterward keeps up confusion that leads to the poor scores in bathtub dynamics test items.

We see the conflict between a continuous and a discrete concept of time right in the core of SD methodology. By addressing it explicitly and making both concepts of time clear to SD learners might help them to understand better both System Dynamics and the world.

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