COMPUTATIONAL AND DYNAMIC COMPLEXITY IN ECONOMICS

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Abstract:
This paper examines the rising competition between computational and dynamic conceptualizations of complexity in economics. While computable economics views the complexity as something rigorously defined based on concepts from probability, information, and computability criteria, dynamic complexity is based on whether a system endogenously and deterministically generates erratically dynamic behavior of certain kinds. On such behavior is the phenomenon of emergence, the appearance of new forms or structures at higher levels of a system from processes occurring at lower levels. While the two concepts can overlap, they represent substantially different perspectives. A competition of sorts between them may become more important as new, computerized market systems emerge and evolve to higher levels of complexity of both kinds.

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I. INTRODUCTION

As reported by Horgan (1997, p. 305), Seth Lloyd has gathered at least 45 definitions of *complexity*. Rosser (1999) argued for the usefulness in studying economics of a definition he called *dynamic complexity* that was originated by Day (1994). This is that a dynamical economic system fails to generate convergence to a point, a limit cycle or an explosion (or implosion) endogenously from its deterministic parts. It was argued that nonlinearity was a necessary but not sufficient condition for this form of complexity,[1] and that this definition constituted a suitably broad "big tent" to encompass the "four C's" of *cybernetics, catastrophe, chaos, and "small tent" (or heterogeneous agents) complexity*. Other approaches used in economics have included *structural* (Pryor, 1995; Stodder, 1995),[2] *hierarchical* (Simon, 1962), and *computational* (Lewis, 1985; Albin with Foley, 1998; Velupillai, 2000).

In recent years (Veluppilai, 2005a,b,c; Markose, 2005) there has been a tendency to argue that the latter concept is superior because of its foundation on more well-defined ideas, such as *algorithmic complexity* (Chaitin, 1987) and *stochastic complexity* (Rissanen, 1989, 2005). These are seen as founded more deeply on work of Shannon (1948) and Kolmogorov (1983). Mirowski (2006) argues that markets themselves should be seen as algorithms that are evolving to higher levels in a Chomskyian (1959) hierarchy

[1] This is incorrect. Goodwin (1947) showed such endogenous dynamic patterns in coupled linear systems with lags. Similar systems were analyzed by Turing (1952) in his paper that has been viewed as the foundation of the theory of morphogenesis, a complexity phenomenon par excellence. However, the overwhelming majority of such dynamically complex systems involve some nonlinearity, and the uncoupled normalized equivalent of the coupled linear system is nonlinear..

[2] Structural complexity appears in the end to amount to "complicatedness," which Israel (2005) argues is merely an epistemological concept rather than an ontological one, with "complexity" and "complicatedness" coming from different Latin roots (*complecti*, "grasp, comprehend, or embrace" and *complicare*, "fold, envelop"), even if many would confuse the concepts (including even von Neumann, 1966). Rosser (2004) argues that complicatedness as such poses essentially trivial epistemological problems, how to figure out a lot of different parts and their linkages.

of computational systems, especially as they increasingly are carried over computers and become resolved through programmed double-auction systems and the like. McCauley (2004, 2005) and Israel (2005) argue that such dynamic complexity ideas as *emergence* are essentially empty and should be abandoned for either more computational-based or more physics-based, the latter especially relying on *invariance* concepts.

In contrast, this paper will argue that while these ideas serve useful purposes, dynamic complexity and such concepts as emergence are useful for understanding economic phenomena and are not as incoherent and undefined as has been argued. A sub-theme of some of this literature, although not all of it, has been that biologically based models or arguments are fundamentally unsound mathematically and should be avoided in more analytical economics. Instead, this paper will argue that such approaches can be used in conjunction with the dynamic complexity approach to explain emergence mathematically and that such approaches can explain certain economic phenomena that may not be easily explained otherwise.

II. COMPUTATIONALLY BASED ARGUMENTS

Velupillai (2000, pp. 199-200) summarizes the foundations of what he has labeled *computable economics*[3] in the following.

> "Computability and randomness are the two basic epistemological notions I have used as building blocks to define computable economics. Both of these notions can be put to work to formalize economic theory in effective ways. However, they can be made to only on the basis of two theses: the Church-Turing thesis, and the Kolmogorov-Chaitin-Solomonoff thesis."

[3] "Computable economics" was neologized by Velupillai in 1990 and is distinguished from "computational economics," symbolized by the work one finds at conferences of the Association for Computational Economics and its journal, *Computational Economics.* The former focuses more on the logical foundations of the use of computers in economics while the latter tends to focus more on specific applications and methods.

Church (1936) and Turing (1937) independently realized that several broad classes of functions could be described as "recursive" and were "calculable" (programmable computers had not yet been invented). Turing (1936-37) was the first to realize that Gödel's (1931) Incompleteness Theorem provided a foundation for understanding when problems were not "calculable," called "effectively computable" since Tarski (1949). Turing's analysis introducing the generalized concept of the *Turing machine*, now viewed as the model for a rational economic agent within computable economics (Velupillai, 2005c, p. 181). While the original Gödel theorem relied upon a Cantor diagonal proof arising from self-referencing,[4] the classic manifestation of non-computability in programming is the *halting problem*, that a program will simply run forever without ever reaching a solution (Blum, Cucker, Shub, and Smale, 1998).

Much of recent computable economics has involved showing that when one tries to put important parts of standard economic theory into forms that might be computable, it is found that they are not effectively computable in any general sense. These include Walrasian equilibria (Lewis, 1991), Nash equilibria (Prasad, 1991; Tsuji, da Costa, and Doria, 1998), more general aspects of macroeconomics (Leijonhufvud, 1993), and whether a dynamical system will be chaotic or not (da Costa and Doria, 2005).[5]

Indeed, what are viewed as dynamic complexities can arise from computability problems that arise in jumping from a classical and continuous, real number framework

[4] Implications of self-referencing as related to Gödel's Incompleteness Theorem have been studied by Binmore (1987) and Koppl and Rosser (2002). One issue that can arise is multiple levels of conscious playing, one player thinking about the other player thinking about the first player thinking about the other player and so on up to n-levels (Stahl, 2000).

[5] Another main theme of computable economics involves considering which parts of economic theory can be proved when such classical logical axioms are relaxed as the Axiom of Choice and the exclusion of the middle. Under such *constructive* mathematics problems can arise for proving Walrasian equilibria (Pour-El and Richards, 1979; Richter and Wong, 1999; Velupillai, 2002, 2006) and Nash equilibria (Prasad, 2004), however, we shall not pursue such themes here further.

to a digitized, rational numbers-only framework. An example is the curious "finance function" of Clower and Howitt (1978) in which solution variables jump back and forth over large intervals discontinuously as the input variables go from integers, to non-integer rationals to irrational numbers and back. Velupillai (2005c, p. 186) notes the case of a Patriot missile missing its target by 700 meters and killing 28 soldiers as "friendly fire" in Dhahran, Saudi Arabia in 1991 due to a computer's non-terminating cycling through a binary expansion on a decimal fraction. Finally, the discovery of chaotic sensitive dependence on initial conditions by Lorenz (1963) because of computer roundoff error is famous, a case that is computable but undecidable.

There are actually several computability based definitions of complexity, although Velupillai (2000, 2005b,c) argues that they can be linked as part of the broader foundation of computable economics. The first is the Shannon (1948) measure of information content, which can be interpreted as attempting observe structure in a stochastic system. It is thus derived from a measure of entropy in the system, or its state of disorder. Thus, if p(x) is the probability density function of a set of K states denoted by values of x, then the *Shannon entropy* is given by

$$H(X) = -\boldsymbol{\Sigma}_{x=1}^{K} \ln (p(x)). \qquad (1)$$

From this is it is trivial to obtain the *Shannon information content* of $X=x$ as

$$SI(x) = \ln(1/p(x)). \qquad (2)$$

It came to be understood that this equals the number of bits in an algorithm that it takes to compute this code. This would lead Kolmogorov (1965) to define what is now known as *Kolmogorov complexity* as the minimum number of bits in any algorithm that does not prefix any other algorithm a(x) that a Universal Turing Machine (UTM) would require to compute a binary string of information, x, or,

$$K(x) = \min \left| a(x) \right|, \tag{3}$$

where $| \;\; |$ denotes length of the algorithm in bits.[6] Chaitin (1987) would independently discover and extend this *minimum description length* (MDL) concept and link it back to Gödel incompleteness issues, his version being known as *algorithmic* complexity, which would get taken up later by Albin (1982) and Lewis (1985) in economic contexts.[7]

While these concepts usefully linked probability theory and information theory with computability theory, they all share the unfortunate aspect of being non-computable. This would be remedied by the introduction of *stochastic complexity* by Rissanen (1978, 1986, 1989, 2005). The intuition behind Rissanen's modification of the earlier concepts is to focus not on the direct measure of information but to seek a shorter description or model that will depict the "regular features" of the string. For Kolmogorov a model of a string is another string that contains the first string. Rissanen (2005, pp. 89-90) defines a likelihood function for a given structure as a class of parametric density functions that can be viewed as respective models, where θ represents a set of k parameters and x is a given data string indexed by n:

$$M_k = \{f(x^n, \theta): \theta \in \mathbf{R}^k). \tag{4}$$

For a given f, with f(y^n) a set of "normal strings," the *normalized maximum likelihood function* will be given by

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[6] It should be understood that whereas on the one hand Kolmogorov's earliest work axiomatized probability theory, his efforts to understand the problem of induction would lead him to later argue that information theory precedes probability theory (Kolmogorov, 1983). McCall (2005) provides a useful discussion of this evolution of Kolmogorov's views.

[7] Closely related would be the *universal prior* of Solomonoff (1964) that puts the MDL concept into a Bayesian framework. From this comes the rather neatly intuitive idea that the most probable state will also have the shortest length of algorithm to describe it. Solomonoff's work was also independently developed, drawing on the probability theory of Keynes (1921).

$$f^*(x^n, M_k) = f(x^n, \theta^*(x^n))/[\int_{\theta(yn)} f(y^n, \theta(y^n))dy^n], \quad (5)$$

where the denominator of the right-hand side can be defined as being $C_{n,k}$.

From this the *stochastic complexity* is given by

$$-\ln f^*(x^n, M_k) = -\ln f(x^n, \theta^*(x^n)) + \ln C_{n,k}. \quad (6)$$

This term can be interpreted as representing "the 'shortest code length' for the data x^n that can be obtained with the model class M_k." (Rissanen, 2005, p. 90). With this we have a computable measure of complexity derived from the older ideas of Kolmogorov, Solomonoff, and Chaitin. The bottom line of Kolmogorov complexity is that a system is complex if it is not computable. The supporters of these approaches to defining economic complexity (Israel, 2005; Markose, 2005; Velupillai, 2005b,c) point out the precision given by these measures in contrast to so many of the alternatives.

III. DYNAMIC COMPLEXITY AND EMERGENCE

In contrast with the computationally defined measures described above, the dynamic complexity definition stands out curiously as essentially for its negativity: dynamical systems that do *not* endogenously and deterministically generate certain "well-behaved" outcomes. The charge that it is not precise carries weight. However, the virtue of it is precisely its generality guaranteed by its vagueness. It can apply to a wide variety of systems and processes that many have described as being "complex." Of course, the computationalists argue with reason that they are able to subsume substantial portions of nonlinear dynamics with their approach, as for example with the already mentioned result on the non-computability of chaotic dynamics (da Costa and Doria, 2005).

However, most of this recent debate and discussion, especially by Israel (2005), McCauley (2005), and Velupillai (2005b,c) has focused on a particular outcome that is associated with some interacting agents models within the smaller tent complexity part of the broader big tent dynamic complexity concept. This property or phenomenon is *emergence*. It was much discussed by cyberneticists and general systems theorists (von Bertalanffy, 1962), including under the label *anagenesis* (Boulding, 1978; Jantsch, 1982), although it was initially formalized by Morgan (1923), drawing upon the idea of *heteropathic laws* due to Mill (1843, Book III).[8] Much recent discussion has focused on Crutchfield (1994) because he has associated it more clearly with processes within computerized systems of interacting heterogeneous agents and linked it to minimum length computability concepts related to Kolmogorov's idea, which it makes it easier for the computationalists to deal with. In any case, the idea is of the dynamic appearance of something new endogenously and deterministically from the system, often also labeled *self-organization*.[9] Furthermore, all of these cited here would add another important element, that it appears at a higher level within a dynamic hierarchical system as a result of processes occurring at lower levels of the system. Crutchfield (1994) allows that what is involved is symmetry breaking bifurcations, which leads McCauley (2005, pp. 77-78) to be especially dismissive, identifying it with biological models (Kaufmann, 1993) and declaring that "so far no one has produced a clear empirically relevant or even theoretically clear example." The critics complain of implied *holism* and Israel identifies it with Wigner's (1960) "mystical" alienation from the solidly grounded view of Galileo.

[8] I thank K. Vela Velupillai for making me aware of these arguments. Morgan was a key part of what is now known as the "British emergentist" school of thought (McLaughlin, 1992).
[9] This term has been especially associated with Bak (1996) and his *self-organized criticality*, although he was not the first to discuss self-organization in these contexts.

Now the complaint of McCauley amounts to an apparent lack of *invariance*, a lack of ergodicity or steady state equilibria, with clearly identifiable symmetries whose breaking brings about these higher-level reorganizations or transformations.

> "We can understand how a cell mutates to a new form, but we do not have a model of how a fish evolves into a bird. That is not to say that it has not happened, only that we do not have a model that helps us to imagine the details, which must be grounded in complicated cellular interactions that are not understood." (McCauley, 2005, p. 77)[10]

While he is probably correct that the details of these interactions are not fully understood, a footnote on the same page points in the direction of some understanding that has appeared, not tied directly to Crutchfield or Kaufmann. McCauley notes the work of Hermann Haken (1983) and his "examples of bifurcations to pattern formation via symmetry breaking." Several possible approaches suggest themselves at this point.

The first is a direct argument by biochemists Eigen and Schuster (1979) ironically enough regarding the preservation, reproduction, and transmission of information content in genetical systems. They label the first appearance of self-reproducing biochemical structures as the *hypercycle*, "the simplest system that can allow the evolution of reproducible links," (Eigen and Schuster, 1979, p. 87). They propose a *threshold of information content*. Below this threshold the system can stabilize itself against the accumulation of errors in self-reproduction. Above it will occur an *error catastrophe* that results in the "disintegration of information due to a steady accumulation of errors," (Eigen and Schuster, 1979, p. 25). Assuming that V is the number of symbols, $\sigma > 1$ is the degree of "superiority of the master copy," and q is the quality of symbol copying, then the threshold below which the number of symbols must remain is given by

[10] McCauley's argument is based on Moore's (1990, 1991a,b) study of low dimensional, iterated maps that are Turing machines without attractors, scaling properties, or symbolic dynamics. McCauley argues that this view provides a foundation for complexity as ultimate surprise and unpredictability.

$$V < \ln \sigma/(1-q). \quad (7)$$

Hypercycle formation has been simulated by Mosekilde, Rasmussen, and Sorenson (1983) and the concept has been applied to the evolution of market structures based on differential rates of learning among firms by Silverberg, Dosi, and Orsenigo (1988).

It has also been linked with the concept of *autopoesis*, defined as the stable reproduction of a space-time structure (Varela, Maturana, and Uribe, 1974). While Turing defined *morphogenesis* as structural differentiation due to bifurcation at a single level, anagenesis within a hierarchical system of a new, self-reproducing hierarchical level that is self-reproducing has been labeled *hypercyclic morphogenesis* by Rosser (1991, 1992) or the *anagenetic moment* by Rosser, Folke, Günther, Isomäki, Perrings, and Puu (1994). Some of the early models of hypercycle formation required the absence of parasites, however with appropriate mixing, they may be stable against parasites (Boerlijst and Hogeweg, 1991).

This raises parallels within evolutionary game theoretic models of the issue of multi-level evolution (Henrich, 2004), with the Price-Hamilton equations providing sufficient conditions for this to occur, although the original version of these was due to Crow (1953). For its discrete form, if ΔC is the single generation change mean value of the trait in the whole population, B_w and B_b are the within- and between-group genetic regressions of fitness on the value of the trait, V_w and V_b the within and between-group genetic variances, and W the mean population fitness, then

$$\Delta C = (B_w V_w + B_b V_b)/W. \quad (8)$$

This allows for a statement of Hamilton's (1972) condition for an altruistic trait to increase (the equivalent of cooperation at a higher level) as

$$B_w/(B_b - B_w) < r, \quad (9)$$

Where r is the Sewall Wright coefficient of relationship (Crow and Aoki, 1984). The left-hand side can be interpreted as a cost-of-fitness to benefit-minus-cost-of-fitness ratio.

Another approach is that of the *synergetics* due to Haken (1983), alluded to above. This deals more directly with the concept of entrainment of oscillations via the *slaving principle* (Haken, 1996), which operates on the principle of *adiabatic approximation*. A complex system is divided into *order parameters* that are presumed to move slowly in time and "slave" faster moving variables or subsystems. While it may be the that the order parameters are operating at a higher hierarchical level, which would be consistent with many generalizations made about relative patterns between such levels (Allen and Hoekstra, 1992; Holling, 1992), this is not necessarily the case, and the variables may well be fully equivalent in a single, flat hierarchy, such as with the *control* and *state* variables in catastrophe theory models. Stochastic perturbations can lead to structural change near bifurcation points.

If slow dynamics are given by vector $\mathbf{F}$, fast dynamics generated by vector $\mathbf{Q}$, with $\mathbf{A}$,$\mathbf{B}$, and $\mathbf{C}$ being matrices, and $\boldsymbol{\varepsilon}$ a stochastic noise vector, then a locally linearized version is given by

$$d\mathbf{q} = \mathbf{Aq} + \mathbf{B(F)q}\ \mathbf{C(F)} + \boldsymbol{\varepsilon}. \quad (10)$$

Adiabatic approximation is given by

$$d\mathbf{q} = -(\mathbf{A} + \mathbf{B(F)})^{-1}\mathbf{C(F)}. \quad (11)$$

Fast variable dependence on the slow variables is given by $\mathbf{A} + \mathbf{B(F)}$. Order parameters are those of the least absolute value.

The symmetry breaking bifurcation occurs when the order parameters destabilize by obtaining eigenvalues with positive real parts, while the "slave variables" exhibit the opposite. Chaos is one possible outcome. However, the most dramatic situation is when the slaved variables destabilize and "revolt" (Diener and Poston, 1984), with the possibility of the roles switching within the system and former slaves replacing the former "bosses" to become the new order parameters. An example in nature of such an emerging and self-organizing entrainment might the periodic and coordinated appearance of the slime mold out of separated amoebae, which later disintegrates back into its isolated cells (Garfinkel, 1987).

Finally there is the model of Nicolis (1986), derived from the work of Nicolis and Prigogine (1977) on frequency entrainment. Rosser, Folke, Günther, Isomäki, Perrings, and Puu (1994) have argued that this can serve as a possible model for the anagenetic moment, or the emergence of a new level of hierarchy. Let there be n well-defined levels of the hierarchy, with $\mathbf{L}_1$ at the bottom and $\mathbf{L}_n$ at the top. A new level, $\mathbf{L}_{n+1}$, or *dissipative structure*, can emerge at a phase transition with a sufficient degree of entrainment of the oscillations at that level. Let there be k oscillating variables, x_j and $z_i(t)$ be an i.i.d. exogenous stochastic process with zero mean and constant variance, then dynamics are given by the coupled, nonlinear differential equations of the form

$$dx_i/dt = f_i(x_j, t) + z_i(t) + \mathbf{\Sigma}_{j=1}^{k} \int_1^k x_j(t')\mathbf{w}_{ij}(t' + \tau)dt', \qquad (12)$$

with $\mathbf{w}_{ij}$ representing a cross-correlation matrix operator. The third term is the key, either being "on" or "off," with the former showing frequency entrainment. Nicolis (1986) views this in terms of a model of neurons, with a *master hard nonlinear oscillator* being turned on by a symmetry breaking of the cross-correlation matrix operator when the probability distribution of the real parts of its eigenvalues exceeding zero.[11] Then a new variable vector will emerge at the $\mathbf{L}_{n+1}$ level that is y_j, which will damp or stimulate the oscillations at level $\mathbf{L}_n$, depending on whether the sum over them is below or above zero.

IV. DISCUSSION AND CONCLUSIONS

Needless to say, most of the models discussed in this last section would not impress most of the adherents of the computability approach to complexity. To the extent that the models discussed have axiomatic foundations rather than being merely *ad hoc*, which many of them ultimately are, these foundations are strictly within the non-constructivist, classical mathematical mode, assuming the Axiom of Choice, the Law of the Excluded Middle, and other hobby horses of the everyday mathematicians and mathematical economists. To the extent that they provide insight into the nature of dynamic economic complexity and the special problem of emergence (or anagenesis), they do not do so by being based on axiomatic foundations[12] that would pass muster with the constructivists and intuitionists of the early and mid-20th century, much less their more recent disciples, who are following the ideal hope that "The future is a minority; the past and present are a majority," to quote Velupillai (2005b, p. 13), himself paraphrasing Shimon Peres from an interview about the prospects for Middle East peace.

In any case, even if the future belongs to computable complexity, there are a considerable array of models available for contemplating or modeling emergent

[11] In a related model, Holden and Erneux (1993) show that the systemic switch may take the form of a slow passage through a supercritical Hopf bifurcation., thus leading to the persistence for awhile of the previous state even after the bifurcation point has been passed.

[12] While this movement focuses on refining axiomatic foundations, it ultimately seeks to be less formalistic and Bourbakian. This is consistent with the history of mathematical economics, which first moved towards a greater axiomatization and formalism within the classical mathematical paradigm, only to move away from it in more recent years (Weintraub, 2002).

phenomena operating at different hierarchical levels. I would suggest at this point that a potentially interesting area to see which of the approaches might prove to be most suitable may well be in the study of the evolution of market processes as they themselves become more computerized. This is the focus of Mirowski (2006) who goes so far as to argue that fundamentally markets *are* algorithms. The simple kind of posted price, spot market most people have traditionally bought things in is at the bottom of a Chomskyian hierarchy of complexity and self-referenced control. Just as newer algorithms may contain older algrorithms within them, so the emergence of newer kinds of markets can contain and control the older kinds as they move to higher levels in this Chomskyian hierarchy. Futures markets may control spot markets, options markets may control futures markets, and the ever higher order of these markets and their increasing automation pushes the system to a higher level towards the unreachable ideal of being a full-blown Universal Turing Machine (Cotogno, 2003).

Mirowski brings to bear more recent arguments in biology regarding coevolution, noting that the space in which the agents and systems are evolving itself changes with their evolution. To the extent that the market system increasingly resembles a gigantic assembly of interacting and evolving algorithms, both biology and the problem of computability will come to bear and will come to bear and influence each other (Stadler, Stadler, Wagner, and Fontana, 2001). In the end the distinction between the two may become irrelevant.

In that regard what may become more important as the system moves to higher order levels will be the old struggle between the continuous and the discrete, with such intermediate compromises such as those posed in Blum, Cucker, Shub, and Smale (1998) looming as more important over time. While this struggle may result in system-wide crashes or failures due to the sorts of computer errors that pushed the Patriot missile 700 meters off in 1991, more likely will be an outcome in which the way that a particular algorithm or system behaves or interacts with the other parts will depend on how it looks at the system and from which level. In that regard, I conclude with a quote from Joseph Schumpeter (1939, pp. 226-227).

> "Our theory of the mechanism of change stresses discontinuity. It takes the view that, as we may put it, evolution proceeds by successive revolutions, or, that there are in the process jerks or jumps which account for many of its features. As soon, however, as we survey the history of society or of any particular sector of social life, we become aware of a fact which seems at first sight to be incompatible with that view: every change seems to consist in the accumulation of many small influences and events and comes about precisely by steps so small as to make any exact dating and any sharp distinction of epochs almost meaningless…Now it is important to note that there is no contradiction whatever between our theory and a theory of history which bases itself on those facts. What difference there is, is a difference of purpose and method only. This becomes evident if we reflect that any given industrial development, for instance the electrification of the household, may involve many discontinuities incident to the setting up of new production functions when looked at from the standpoint of individual firms and yet appear, when looked at from other standpoints as a continuous process proceeding steadily from roots centuries back. By one of the many roughnesses or even superficialities forced upon us by the nature of the task this volume is to fulfill, we may characterize this as a difference between microscopic and macroscopic points of view: there is as little contradiction between them as there is between calling the contour of a forest discontinuous for some and smooth for other purposes."

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