Analytical methods for structural dominance analysis in system dynamics: An assessment of the current state of affairs.

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Abstract

We provide a review of different approaches to linking model structure to observed behavior with a particular view towards using models for theory building. We identify four such approaches, namely the "classical" approach, the "pathway participation" approach, the "eigenvalue elasticity" approach, and the "eigenvector" approach, respectively. We outline our assessment of the strengths and weaknesses of each approach and point to some main challenges and tasks ahead. We find that the eigenvalue and eigenvector approaches carry the largest potential but that a more solid theoretical foundation of the method is required. Once such a foundation is developed, it will be important to develop intuitive analytical tools that can be of use to a wider system dynamics audience. Since a "grand unified theory" will never be possible, all tools will be based on approximations and it is only in their practical use that we can discover their real value.

Introduction

In recent years, there has been a growing interest among system dynamics researchers in developing methods for formal quantitative tools to help modelers understand the relationship between observed model behavior and the elements of the model structure influencing this behavior in large models. This process is obviously of great importance to both practitioners and theorists. For the former group, an understanding of the link between system structure and observed behavior is the key to finding leverage points for policy initiatives. For the theorist, the system dynamics paradigm builds on the notion that structure causes behavior, i.e., a system dynamics theory of a particular phenomenon is an account of how certain feedback loops cause certain dynamic patterns of behavior to appear. This qualitative understanding is often at least as important as the particular numerical predictions obtained, even in applied studies. Yet the rigor of such an account depends directly on the rigor with which structure-behavior link can be made in a given model.

The purpose of this paper is to give an overview of the different approaches taken to the structure-behavior problem, to outline the apparent strengths and weaknesses of each approach with a view towards using system dynamics for theory building, and to point to some main challenges and tasks ahead in order to make the methods useful for a wider audience of system dynamics practitioners and researchers.

Understanding model behavior is closely related to the process of model testing and validation, for which there is a well established tradition and an extensive literature in the field (e.g., Barlas 1989; Ford 1999; Forrester and Senge 1980; Morecroft 1985; Richardson 1984/1995; Richardson 1986). Indeed there is no sharp line between testing, validation, and analysis, and in practice, the analyst undertakes all these processes simultaneously (Forrester and Senge 1980). The present paper is focused exclusively on the tools used to linking structure to behavior, specifically how certain behavior patterns can be attributed to certain feedback loops (or external driving forces), and how the relative importance of different feedback loops may shift over time (shifting "loop dominance").

Traditionally, system dynamics analysts have relied on trial-and-error simulation to discover important system elements, by changing parameter values or switching individual links and feedback loops on and off. The tradition is well developed and includes a set of principles for partial model formulation and testing based the organizational theory of bounded rationality (Homer 1983; Morecroft 1985). The intuition guiding this effort relies on simple feedback systems with one or a few state variables, where the behavior is fully documented and understood.

Part of the tradition involves the use of well-understood "generic structures" that seem to appear again and again in system dynamics models, such as "shifting the burden to the intervener", "overshoot and carrying capacity collapse", "inventory-workforce dynamics", "drifting goal structure", etc. (See Sterman, 2000 for an account of these structures.) Clearly such structures can be a useful aid to understanding if the model is sufficiently simple to allow such a simple structure to be identified.[2]

In large-scale models with perhaps hundreds of state variables, however, the traditional approach shows significant limitations. In practice, model building and analysis is often done using a "nested" partial model testing approach where one goes from the level of small pieces of structure to entire subsystems of the model, with frequent re-use of known formulations and partial models. Although this approach does carry a long way, it can be very difficult to discover feedback mechanisms that transcend model substructures in ways not anticipated by the modeler in the original dynamic hypothesis. Thus, there is a danger that observed behavior is falsely attributed to certain feedback mechanisms when in fact another set of feedbacks is driving the outcome. Likewise, one may make false inferences about how a particular feedback mechanism modifies the behavior, e.g. whether it attenuates or amplifies a particular oscillation.[3]

Modern software packages can run extensive tests for sensitivity and "reality checks" where a large number of parameters are varied simultaneously (Peterson and Eberlein 1994). This is clearly a significant improvement over "manual" trial and error methods, particularly when these methods are combined with statistical inference methods such as Kalman Filtering or Monte Carlo maximum likelihood estimation (Eberlein 1986; Eberlein and Wang 1985; Oliva 2003; Peterson 1980; Radzicki 2004; Schweppe 1973). A variant of this approach involves using statistical experimental design and correlation methods to screen for significant model structure (parameters), as suggested by Ford and Flynn (2005). Indeed, the prospects of marrying such methods with modern search and optimization methods like classifier systems (Holland 1992) or

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[2] A devil's advocate may argue that there is also a tendency to build the models around such structure, making it less surprising that one "rediscovers" them when analyzing the model.

[3] For instance, in Sterman's simple long wave model (Sterman 1985), the self-reinforcing "hoarding mechanism" where increased delivery delays lead to further orders, was thought to play a significant role in amplifying the oscillations. However, using loop elasticity analysis, Kampmann (1996) showed how the effect of the hoarding mechanism was in fact negligible.

genetic algorithms (Goldberg 1989) seem very promising. However, these methods are more addressing issues in estimation, validation and testing than inferences about structure causing behavior, cf. the discussion above.

Clearly, a more rigorous theory for the link between feedback structure and behavior in general large-scale systems would be of great value. By way of introduction, the following section of the paper discusses what constitutes an "explanation" of the link between structure and behavior and the fundamental analytical limits to this ideal, given that we are dealing nonlinear systems. It is important to recognize that these limits exist, i.e., we that we cannot hope to completely "automate" the model analysis. That isn't necessarily a problem, though, given that the system dynamics modeling philosophy calls for a modeling process of critical inquiry and interaction with the model. We then proceed to discuss four main approaches used in system dynamics to explain the link between structure and behavior, which we have termed the *classic* approach, the *pathway participation* approach, *eigenvalue elasticity analysis,* and *eigenvector analysis*, respectively. The paper concludes with a discussion of the most important tasks ahead and speculates on the prospects for widespread use of analytical tools in the future.

Overview and fundamental limits of analytical methods

A system dynamics model can be represented mathematically as a set of ordinary differential equations

$$\frac{d\mathbf{x}(t)}{dt} \equiv \dot{\mathbf{x}}(t) = \mathbf{f}\big(\mathbf{x}(t),\mathbf{u}(t)\big), \tag{1}$$

where $\mathbf{x}(t)$ is a (column) vector of n state variables $\big(x_1(t),\ldots,x_n(t)\big)$, $\mathbf{u}(t)$ is a column vector of p exogenous variables or control variables $\big(u_1(t),\ldots,u_p(t)\big)$, $\mathbf{f}(\ )$ is a corresponding vector function, and t is simulated time.[4] In general, **f** is a nonlinear function of its arguments, and we speak of a *nonlinear* system. Conversely, if **f** is a linear function, we speak of a linear system.

Given the structure of the model (1), knowledge of the initial conditions, and the path of the input variables **u,** the behavior of the model is completely determined. In this sense, the model structure (1) constitutes a "theory" of the time behavior of the variables **x**. Although this point of view has certain merits, not least the fact that it lifts the discussion from outcomes to causes of these outcomes and from events to underlying structure (Forrester 1961; Sterman 2000), we are concerned here with a more compact "explanation" of what is going on. In fact, most system dynamics modeling projects report their results in terms of "simpler" explanations of the observed results, typically in terms of "dominant" feedback loops that produce the salient features of the behavior.

[4] In the paper, we restrict our attention to the state variables (levels) of the model for notational convenience, ignoring the auxiliary variables. Mathematically, a model can always be brought to the *reduced* form (1), but in practice, the auxiliary variables give a more intuitive account of the analysis. Likewise, we do not consider time-varying systems (where time *t* enters as an explicit argument in the function **f(**), though these can always be accommodated by an appropriate definition of the exogenous variables **u**.

Thus, one would be interested in methods that allows for a more compact explanation or inference about the results, short of having to simulate the entire model structure. It turns out that in its ultimate form, this dream is beyond reach: Since the days of Henri Poincaré, mathematicians have known that it is impossible to find general analytical solutions to nonlinear systems. On the contrary, the development of nonlinear dynamics and chaos theory has proven that such systems, even when they have very few state variables, can produce highly complex and intricate behavior that beyond general analytic methods (e.g., Ott 1993; Richardson 1988). Thus, we shall never find a final general theory where we can infer the behavior of the system directly from its structure and shall always have to rely on simulation to discover the dynamics implied by the structure.[5]

The best we can hope for, therefore, is a set of tools that will guide intuition and help identify *dominant structure* in the model. By dominant structure we mean particular feedback loops that are in some sense "important" in shaping the "behavior" of interest. To the extent that we can identify such dominant structures, we may say that we have found a "theory" of the observed behavior.

Although the term "behavior" may appear rather loose, experience and reflection tells us that there is a limited number, perhaps a dozen or so, of relevant behavior patterns that dynamical systems can exhibit. Some of these behaviors, like exponential growth, exponential adjustment, and damped or expanding oscillations, are typical of *linear* systems. Others, like limit cycles, quasi-periodic motion, mode-locking, and chaos, can only be exhibited by nonlinear systems.

Common to the four approaches considered in this paper is that they are based on tools from linear systems theory, i.e., they approximate the nonlinear model (1) with a linearized version, using as a first-order Taylor expansion around some operating point $\mathbf{x}_0, \mathbf{u}_0$, i.e.,

$$\dot{\mathbf{x}}(t) \approx \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x} - \mathbf{x}_0) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u} - \mathbf{u}_0), \tag{2}$$

or, by an appropriate redefinition[6] of the variables $\mathbf{x}, \mathbf{u}$,

$$\dot{\mathbf{x}}(t) \approx \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \tag{3}$$

where **A** is an $n \times n$ matrix of partial derivatives $\partial f_i / \partial x_j$ and **B** is an $n \times p$ matrix of partial derivatives $\partial f_i / \partial u_j$.

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[5] This is not to say that no general analytical results exist in nonlinear systems. The field of chaos theory has uncovered a number of universal features, e.g., relating to the transition from periodic or quasi-periodic behavior to chaos, where the transitions show both qualitative and quantitative similarities that are independent of the specific forms of the model equations (see, e.g., Ott 1993). However, these universal features relate to specific situations such as period-doubling or intermittency routes to chaos.

[6] $\mathbf{x} \rightarrow \mathbf{x} - \mathbf{x_0} - \mathbf{f}(\mathbf{x_0}, \mathbf{x_0})(t - t_0)$ and $\mathbf{u} \rightarrow \mathbf{u} - \mathbf{u_0}$.

For the linear system (3), there is a well-developed and extensive theory of the system behavior as a function of its structure, expressed in the matrices **A** and **B.** One may broadly distinguish two parts of the theory, named *classical control theory* (e.g., Ogata 1990) and *modern linear systems theory* (e.g., Chen 1970; Luenberg 1979).

In classical control theory, one is concerned with system output (one or more system variables) as a function of different inputs **u**, expressed as the *transfer function G(s)* of the system. There is no space in the present paper for further details of the theory behind this method. We return to its concrete application in system dynamics in the next section.

Modern control theory or linear systems theory (LST) is concerned with the dynamical properties of the system as a direct function of the system matrices **A** and **B.** A key element in this theory is the notion of the system *eigenvalues,* i.e., the eigenvalues of the matrix **A**. If, for simplicity, we restrict ourselves to the endogenous dynamics of the system (set $\mathbf{u} = \mathbf{0}$), we can write the solution to (3) as

$$x_i(t) = c_{i,1}\exp(\lambda_1 t) + c_{i,2}\exp(\lambda_2 t) + \cdots + c_{i,n}\exp(\lambda_n t), i = 1,\ldots,n, \tag{4}$$

where $\lambda_1,\ldots,\lambda_n$ are the *n* eigenvalues of the matrix **A** and $c_{i.j}$ are constants that depend upon the eigenvectors and the initial condition of the system.[7] In other words, the resulting behavior is a weighted sum of distinct *behavior modes*, $\exp(\lambda t)$. If an eigenvalue is real, the corresponding behavior mode is exponential growth (if $\lambda > 0$) or exponential adjustment (if $\lambda < 0$). Complex-valued eigenvalues come in complex conjugate pairs $\lambda = \tau \pm i\omega$ which give rise to oscillations of frequency ω that are either expanding (if $\tau > 0$) or damped (if $\tau < 0$). In this manner, the eigenvalues serve as a compact and rigorous characterization of the behavior (of linear systems).

The concept of *dominant structure*, on the other hand, is less clear. To even claim that certain parts of the model are more important than others is perhaps to go too far. Indeed, one may characterize the different approaches recounted in this paper by the way in which they define the concept of dominant structure.

Richardson (1986) provides a taxonomy of approaches to the notion of dominant structure, where he distinguishes along three dimensions, namely *linear vs. nonlinear systems*, *model reduction vs. loop contribution*, and the *characterization of behavior* in terms of time graphs, eigenvalues or frequency response, respectively.

The *model reduction approach* seeks to replace the original model with a simplified version that is capable of reproducing the essential behavior of interest (expressed in the system eigenvalues). Eberlein (1984; 1989) represents the most extensive and systematic attempt at this. He found, however, that this is a very difficult problem and that no simple procedures exist even in the case when the system is linear, particularly when one requires a simplified model to have variables that can be related directly to the variables in the original model. It is fair to say, therefore, that

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[7] The eigenvalues λ and associated (right) eigenvectors $\mathbf{r}$ of a matrix **A** are defined as the solutions to the equation $\mathbf{Ar} = \lambda\mathbf{r}$.

this path of inquiry has largely been abandoned. Extracting the "essence" of a model therefore remains a process based upon the intuition and skill of the modeler.

The *loop contribution* or, more generally, the *structure contribution* approach, reflects the intuitive idea that if one removes the element under consideration, e.g. by breaking a link or switching off a feedback loop, and the behavior then "disappears", one would say that the element in some sense "causes" the observed behavior. This notion underlies the traditional trial-and-error approach.

If instead one considers *marginal* (infinitesimal) changes in structure, e.g. in the strength of a particular link, it is possible to derive rigorous analytical results for the resulting change in behavior, expressed as the eigenvalues of the linearized model. This is what underlies the *eigenvalue elasticity (EEA)* and *eigenvector (EVA)* approaches discussed later in the paper. One would then say that if a change in a system element has a relatively large effect upon the behavior pattern of interest, this element is "significant" in "causing" the behavior. The marginal and experimental approaches may supplement each other well, where a marginal analysis may identify elements that can then be tested experimentally for their significance.

The *classical* approach likewise considers how structural elements modify the behavior of the system, viewed in terms of the frequency response method used in classical control engineering. Unlike the other approaches, however, the method works "backwards" by starting with simple feedback systems of single loops and then considers the marginal effect of adding links and loops.

The *pathway participation* method (PPM) relates to the time path of particular system variables and is more concerned with the qualitative nature of the time path, expressed in terms of signs of the slope (whether growing or declining) and curvature (whether convex or concave). It traces the causal links in the variables influencing the system variable in question and then identifies the most important chain of links.

Table 1 summarizes the different approaches in terms of Richardson's taxonomy. With the exception of PPM, the methods rely on concepts from linear systems, yet as indicated in the table, they may nonetheless be useful in nonlinear systems as well as tools for analysis rather than as general theories, as will be discussed below.

--- Insert Table 1 about here ---

The classical approach

The first set of methods, which we call the *classical* approach, has been part of our tradition for decades and is part of the standard curriculum in system dynamics teaching at the graduate level (see, e.g. Sterman 2000, Ch. 8). It involves using the concepts from classical control theory (Ogata 1990) to very simple systems with only a few state variables.

The starting point of the classical approach is the simple first- and second-order positive and negative feedback loops found in any introductory treatment of system dynamics. The advantage of the approach is its simplicity. Although it serves at a guide to intuition, however, the obvious shortage is that it applies rigorously only to simple systems. There have been some attempts to treat higher-order systems by adding a few feedback loops (Graham 1977), but the step to large-scale models is beyond this method given its inherent limitations.

In a way, Graham's objective was the opposite of what seems to be the ideal pursued by the other methods: he sought to derive a set of "principles" based on simple feedback structures, as an intuitive guide or metaphor for understanding behavior. In his own words,

> "A large gap exists in the body of knowledge about the relationship of system structure and system behavior. The gap exists between formal mathematical methods and the conceptual or experiential tools that interact to give a 'gut feel' for the behavior of a system." (Graham 1977, p. 3)

Thus, the goal is to create an intuitive understanding of the mathematical results of classical control theory rather than to develop a new formal theory that can explain behavior.

Graham distills a number of principles that are based on the metaphor of a "disturbance" traveling along the chain of causal links in a feedback loop and getting amplified, damped, and possibly delayed in the process. For major negative feedback loops, which are known to tend to produce oscillation, adding minor negative loops and cross-links, or shortening the delay times increases the damping. Conversely, adding positive loops in to the oscillatory system tends to lengthen the period of oscillation whereas the effect on the damping depends upon the delays in the positive loop. Using the metaphor of pushing a child on a swing, it becomes clear that the timing of the propagation of a disturbance has as much importance for its effect on the damping as its strength.

For systems driven by a positive loop, Graham provides an intuitive account of its behavior using the *open-loop steady-state gain (OLSSG)*, which determines whether the resulting behavior is exponential growth, adjustment, or collapse. If a feedback loop consists of a chain of causal links, one may "break" the loop at any point in the chain, say after x_n, and the consider the change

$$OLSSG = \frac{\Delta x_n^*}{\Delta x_1} = \frac{\Delta x_n^*}{\Delta x_{n-1}} \cdot \frac{\Delta x_{n-1}^*}{\Delta x_{n-2}} \cdot \ldots \cdot \frac{\Delta x_2^*}{\Delta x_1}, \qquad (5)$$

where the asterix (*) denotes the steady state change. The loop may involve one or more delays and other state variables along the causal chain, hence the need to specify that the gain as the steady state value after all delays have adjusted. If the OLSSG is greater than unity, the loop will produce exponential growth or collapse. If the OLSSG is less than unity, e.g. in the classical "consumption multiplier" in macroeconomics, the resulting behavior will be exponential adjustment. This can be intuitively understood by considering how the "additional disturbance gets smaller and smaller each time it goes around the loop".

In the context of oscillation, the "classic" approach has also employed concepts closer to classical control theory by considering the phase and gain shifts that a particular structural element or subsystem (the so-called frequency response method). The frequency response is determined from the *transfer function* of the system, $G(j\omega)$, which is a complex-valued function that specifies how an input signal with frequency ω results in an output signal that may be phase shifted (delayed), and either amplified or damped. For linear systems, G can be calculated directly from the system matrices in (3). For nonlinear systems, G may be found through simulation experiments. Usually, G is represented by a "phase and gain" diagram. For instance, Figure 1 shows a typical phase-and-gain diagram where the amplitude A the phase shift ϕ of the output relative to the input varies with the frequency of the input.

--- Insert Figure 1 about here ---

Often, as is the case with the system in Figure 1 (a simple damped harmonic oscillator), the system will tend to amplify certain frequencies while attenuating other frequencies. This may be used to explain or understand the role of particular structures in the model in generating oscillation at certain frequencies, even when there are no oscillations coming in from the outside world. In this manner, the approach nicely demonstrates the "endogenous viewpoint" that behavior (oscillations) is generated internally by the system. As an analytic tool for large scale systems, however, the method does not seem to produce any additional insights.

Thus, we may conclude that the classical approaches serve mostly as intuitive metaphors to guide the analyst rather than as full analytical tools. In particular, the validity, and indeed the usefulness, of these tools in the context of large-scale models is questionable. With these tools, the effective analysis of large-scale models is still very much a function of the experience and skill of the analyst.

Pathway participation metrics

The second approach has been termed the *pathway participation* method (Mojtahedzadeh 1996; Mojtahedzadeh, Andersen, and Richardson 2004; Oliva and Mojtahedzadeh 2004). The idea is a further development of an original suggestion by Richardson (1984/1995) to provide a rigorous definition of loop polarity and loop dominance. Richardson motivated this with the common confusion associated with positive feedback loops, which may exhibit a wide range of behaviors[8], as Barry Richmond noted with wonderful humor:

> "Positive loops are ... er, well, they give rise to exponential growth ... or collapse ... but only under certain conditions ... Under other conditions they behave like negative feedback loops ..." (Richmond 1980).

Richardson proposed that the polarity of a loop be defined as the sign of the expression

$$\frac{\partial \dot{x}_i}{\partial x_i} = \frac{\partial f_i(\mathbf{x},\mathbf{u})}{\partial x_i}, \tag{5}$$

in the model (1), with a positive sign indicating a positive loop and vice versa. When several loops operate simultaneously, the sign of the expression indicates whether the positive or negative loops dominate. Note, however, that the definition only applies to *minor* loops (i.e. loops involving a single level). Put differently, it only considers the diagonal elements of the

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[8] Even sustained oscillation (cf. Graham 1977, p.76).

matrix **A** in the linearized system (3). Richardson (1984/1995) demonstrates how even with this limitation, analyzing the system with this metric can (sometimes) yield insights into behavior of higher-order systems.

The expression (5) hints that it is relevant to consider the *curvature*, i.e., the second time derivative, $\ddot{x}$, of a variable when looking for dominant structure. Although he does not say so explicitly, this is effectively the focus of Mojtahedzadeh's pathway method. Figure 2 shows how one may classify behavior by comparing the first and second time derivatives of a variable. As seen in the figure, the sign of the expression $\ddot{x}/\dot{x}$, which Mojtahedzadeh denotes the *total pathway participation metric* or *PPM*, indicates whether the behavior appears dominated by positive or negative loops, much in line with Richardson's definition of dominant polarity. A zero curvature indicates a shift in loop dominance (cf. the middle column in the figure). Note, however, that the interpretation of the middle row in the figure where the slope $\dot{x}$ is zero has no clear interpretation in terms of loop dominance. Indeed this hints at one of the weaknesses of the approach that we will return to below.

--- Insert Figure 2 about here ---

Mojtahedzadeh's method proceeds by decomposing the PPM in its constituent terms. From (1) we have

$$PPM_i = \frac{\ddot{x}_i}{\dot{x}_i} = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\dot{x}_j}{\dot{x}_i}, \tag{6}$$

where, for brevity, we have chosen to ignore the exogenous variables *u*.[9] One might say that each of the terms in the sum in (6) represents the separate influence of each of the state variables in the system on the behavior x_i. Mojtahedzadeh in fact uses a normalized measure for the terms,

$$\frac{\left(\partial f_i / \partial x_j\right)\dot{x}_j}{\sum_{k=1}^{n} \left|\left(\partial f_i / \partial x_k\right)\dot{x}_k\right|}, \tag{7}$$

which can vary between -1 and +1, to measure the relative importance of the pathway from variable *j*. By explicitly considering auxiliary variables *y* in the model, one may further decompose each term $\partial f_i / \partial x_j$ into a sum of terms

$$\frac{\partial f_i^k}{\partial x_j} = \frac{\partial f_i}{\partial y_1} \cdot \frac{\partial y_1}{\partial y_2} \cdot \ldots \cdot \frac{\partial y_{m-1}}{\partial y_m} \cdot \frac{\partial y_m}{\partial x_j}, \tag{8}$$

corresponding to a causal chain or pathway $\pi_k = \left\{x_j \to y_m \to \cdots y_2 \to y_1 \to \dot{x}_i\right\}$. Mojtahedzadeh now considers each possible pathway (7) and defines the "dominant" pathway as the one with

[9] The treatment of exogenous variables in the method is straight-forward.

the largest numerical value and the same sign as PPM_i. Having selected this dominant pathway, $\pi_{ij}^* = \{x_j \to y_m \to \cdots y_2 \to y_1 \to \dot{x}_i\}$, which originates in the state variable x_j the procedure is repeated for that state variable x_j, and so forth, until one either reaches one of the already "visited" state variables (in which case a loop has been found) or an exogenous variable (in which case an external driving force has been found). Thus, the procedure may result in three alternative forms of dominant structure illustrated in Figure 3, namely a "pure" minor or major feedback loop, a pathway from a feedback loop elsewhere in the system, or a pathway from an exogenous variable.

--- Insert Figure 3 about here ---

By dividing the observed model behavior into different phases according to the taxonomy in Figure 2 and then applying the method just described at different points in during these phases, one identifies how the dominant structure changes over time.

The PPM method is still mostly used at an early explorative stage on rather simple models, where it does appear to aid insight into the dynamics (e.g., Oliva and Mojtahedzadeh 2004). The method has been implemented in a software package, *Digest,* (Mojtahedzadeh *et al.* 2004), yet its use by practitioners still seems limited. (Something that can be said of all the methods except the classical approach.)

From the studies performed so far, it is clear that the main strength of this method is its relative computational simplicity (it does not require computing eigenvalues, which is a numerically demanding task), and the intuitive and direct connection it makes between the observed behavior and the influencing structural elements. Unlike the other approaches which operate in the "frequency domain", the method considers the time path of a specific variable directly.

There are, however, some important outstanding issues that remain to be clarified. First, the method is not suitable for oscillatory systems. The problem is easy to recognize when one considers how the PPM measure will vary over the course of a sinusoidal outcome: The sign of the PPM will shift twice during each cycle, indicating that the behavior is alternately dominated by positive and negative loops, even though the system structure, and hence the loop dominance, may remain unchanged all the time. Richardson (1984/1995) already alluded to this problem by noting that the measure only considers the diagonal elements in the system matrix in (4). This is a significant limitation, given the prevalence and importance of oscillation in system dynamics analysis.

A second limitation of the current implementation of PPM is that it uses a depth-first search for the single most influential pathway for a variable. This strategy does not capture the situation where more than one structure may contribute significantly to the model behavior and, through the depth-first algorithm, may miss alternative paths that could prove to yield a larger total value of the metric. This problem could likely be addressed by modifying the search algorithm and is most likely of minor importance.

Another issue is how to treat the case when $\dot{x} = 0$ since it appears in the denominator of the terms in (6). However, it is not clear that it is necessary to do this division, given that it is easy to identify the nine cases in the figure by simply examining its sign. Thus, the issue is probably not of much significance.

The fourth issue, on the other hand, is more significant, namely the emphasis on identifying a single "dominant" structure. In reality, of course, the behavior of a variable is influenced by many loops and pathways at once. Reducing the consideration to a single one of these may miss important features of the structure-behavior relationships. For instance, a variable may be influenced by two negative loops and one positive, with the "sum" of the two negative loops dominating the influence of the positive loop, even though that loop by itself has the strongest influence on the behavior. It is more appropriate to consider the relative importance of alternative pathways, yet the method does not address how one would partition the behavior among pathways (the three structures in Figure 3) – only among individual links.

Thus, while the notion of pathways seems an interesting and useful idea, it may be that it will ultimately be more effective to use list, ranked in order of magnitude, of the pathways that influence a variable.

Finally, the method shares a weakness with the traditional method in that it considers primarily partial system structures rather than global system properties. In contrast, the two eigenvalue methods to which we now turn are based on a rigorous characterization of the entire system (at a given point in time).

Eigenvalue elasticity analysis

The third method may be termed *eigenvalue elasticity analysis* (or EEA for short) and builds upon the tools from "modern" linear systems theory (LST), applied to the linearized model (3). The method is concerned with the structural elements that significantly affect the system eigenvalues or behavior modes — the values λ in (4). Specifically, it measures influence by the *elasticity* of an eigenvalue λ with respect to some parameter g in the model, defined as $\varepsilon = (\partial\lambda / \partial g)(g / \lambda)$, i.e. the fractional change in the eigenvalue relative to the fractional change in the parameter. The advantage of this fractional measure is that it is dimensionless, i.e., independent upon the choice of units, include the time scale unit. Sometimes, the *influence measure* is use in stead, defined as $\mu = (\partial\lambda / \partial g)g$. This measure has dimension [1/time] and so depends upon the choice of is time unit, but it is generally easier to interpret for complex-valued eigenvalues and avoids numerical problems with very small or zero eigenvalues (see Kampmann 1996; Saleh *et al.* 2007).

The idea behind EEA was first introduced in system dynamics by Forrester (1982) in the context of economic stabilization policy. For purposes of policy analysis in oscillating systems, one may define a number of criteria from engineering control theory, all of which relate to the eigenvalues of the system, as summarized in Table 2. Figure 4 provides a graphical characterization of the eigenvalues and policy criteria in the complex plane. Though these measures are not new, the EEA method is unique in its attempt to use them to gain qualitative intuitive understanding of the system. A significant step in this direction was first suggested by Forrester (1983) with the notion that the elasticities of any links in the model (corresponding to elements of the matrix **A** in the linearized system (3)), can be interpreted as the sum of elasticities of all feedback loops containing that link. We have chosen to name this approach *loop eigenvalue elasticity analysis (LEEA)*.

--- Insert Table 2 about here ---

--- Insert Figure 4 about here ---

Kampmann (1996) provided a rigorous definition of LEEA and also pointed to the fact that feedback loops are not independent. In other words, given the possibly very large number of loops in a given model, it only makes sense to speak of individual contributions of a limited set of *independent* loops.[10] He proved that a fully connected system (where there is a feedback loop between any pair of variables – the typical case in system dynamics models) with N links and n variables has a total of N-n+1 independent loops and provided a procedure for constructing this set and calculating the loop elasticities.

Kampmann's analysis points to a fundamental issue relating to the notion of feedback loops as a way to explain behavior: the significance assigned to a particular loop depends upon the context (the chosen independent loop set). In other words, feedback loops are derived and relative concepts rather than fundamental independent building blocks of systems.[11] Oliva (2004) further refined the definition of independent loop sets by introducing the *Shortest independent loop set (SILS)* along with a procedure for constructing the set. Although a SILS is not generally unique, experience seems to suggest that it is easier to interpret (Oliva and Mojtahedzadeh 2004). Yet the issue remains that feedback loops are relative concepts.

The EEA/LEEA method has been applied in a number of contexts (e.g., Abdel-Gawad *et al.* 2005; Gonçalves 2003; Gonçalves, Lerpattarapong, and Hines 2000; Güneralp 2006; Kampmann and Oliva 2006; Saleh and Davidsen 2001a, b; Saleh *et al.* 2007), but remains a tool employed by specialists and fundamental research, not least because it has not been incorporated into standard software packages. Thus, the potential of the method for widespread practice remains unexplored.

One might be skeptical that a method derived from linear systems theory may have any use for the nonlinear models found in system dynamics. Kampmann and Oliva (2006) considered the question what types of models the method would be particularly suited for. They defined three categories of models, based upon the behavior they are designed to exhibit: 1) *linear and quasi-linear models*, 2) *nonlinear single-transient models*, and 3) *nonlinear periodic models*. The first category encompasses models of oscillations, possibly combined with growth trends, with relatively stable equilibrium points, (e.g., the classical industrial dynamics models Forrester

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[10] Kampmann demonstrated how the theoretical maximum number of loops grows combinatorically to with the number of variables, reaching astronomical values even for relatively small models.

[11] There is one way in which every possible feedback loop of the model may be assigned a unique elasticity, namely by using Mason's rule on the characteristic polynomial of the system – see Kampmann (1996), Hines (2005). Yet, as Kampmann (1996) points out, the measure is not analytically valid in that it is not possible to change the strength of one loop without affecting the others. How this elasticity measure relates to the independent loop set elasticities is one of the tasks that remain to be solved.

1961). Nonlinearities may modify behavior (particularly responses to extreme shocks) but the instabilities and growth trends can be analyzed in terms of linear relationships. Kampmann and Oliva concluded that LEEA showed the most promise and potential for this class of models because the analytical foundations are solid and valid, and because the method has the ability to find high-elasticity loops even in large models very quickly without much intervention on the part of the analyst.

The second class is typical of scenario models like the World model (Forrester 1971; Meadows *et al.* 1972), the urban dynamics model (Forrester 1969), the energy transition model (Sterman 1981), to name a few, that show a *single transient behavior pattern*, like overshoot and collapse or a turbulent transition to a new equilibrium. In these models, nonlinearities usually play an essential role in the dynamics. Yet it is possible to divide the behavior into distinct phases where certain loops tend to dominate the behavior. In this class of models LEEA also shows promise by measuring shifts in structural dominance by the change in elasticities. Yet requires more input from the analyst (e.g. in defining the different phases of the transition) and it has no obvious advantage over other methods, like PPM.

The third class*, nonlinear periodic models*, are those that exhibit fluctuating behavior in which nonlinearities play an essential role, such as like limit cycles, quasiperiodic behavior, or chaos, (see, e.g., Richardson 1988). Here the utility of the method is much less clear and depends upon the specifics of the model in question. For example, the classic Lorenz model that exhibits limit cycles, period doubling and deterministic chaos does not lend itself to any insight using LEEA (Kampmann and Oliva 2006).[12] In other cases, the method of breaking the behavior into phases with different dominant structures may yield significant from LEEA. For instance, Sterman's simple long wave model (Sterman 1985) lends itself well to this approach (e.g., Güneralp 2006; Kampmann 1996).

In the present paper, we add a fourth category of models or behavior for which the method has not been explored yet. We name this category *nonlinear multi-modal models*. These encompass the cases where one behavior mode interacts with and, therefore, modifies another behavior mode – something that can only happen in nonlinear systems. The most common example is mode-locking or entrainment, in which oscillations become synchronized (e.g., Haxhold *et al.* 1995). Another example is mode modification, where one behavior mode (growth or oscillation) affects the character of another (typically oscillation). An example of this is the interaction of the business cycle with the economic long wave, where the former tends to get more severe during long wave downturns (Forrester 1993). Whether LEEA can contribute to this class of models remains to be seen.

Compared to the former two methods, the EEA/LEEA is mathematically more general and rigorous, though many of the mathematical issues in the method remain to be addressed, as we summarize below. This rigor is also the main strength of the method, since it provides an

[12] This is particularly the case in systems with "strong" nonlinearities such as min and max functions. In these systems, the behavior may change abruptly (eigenvalues suddenly shift) in what is called "border-collision bifurcations" (Mosekilde and Laugesen 2006; Zhusubaliyev and Mosekilde 2003).

unambiguous and complete measure of the influence of the entire feedback structure on all behavior modes.

A weakness or challenge that is starting to show up is the computational intensity in calculating eigenvalues and elasticities. This is not so much an issue of computer time and memory space as of the stability of numerical methods. Kampmann and Oliva (2006) found that the numerical method used sometimes proved unstable, yielding meaningless results. Clearly, there is a need to explore this issue further, possibly building upon the developments in control engineering.

A more serious weakness is the difficulty in interpreting the results: Eigenvalues do not directly relate to the observed behavior of a particular variable. The concepts of eigenvalues and elasticities are rather abstract and unintuitive (Ford 1999). There is a need for tools and methods that can translate them into visible, visceral, and salient measures. Here, the measures in Table 2 may provide a guide. In particular, it is possible to use (linear) filtering in the frequency domain to define a behavior of interest. For example, an analyst may be concerned with structures causing a typical business cycle (3-4-year oscillation) and, by specifying a filter that "picks out" that range of fluctuation, could obtain measures for structures that have elasticities in that range. Because filters are linear operators, all the analytical machinery of the LEEA method will also apply in this case – a significant advantage.

Using filters will also solve an issue that appears in large-scale models, namely the presence of several identical or nearly identical behavior modes. Saleh *et al.* (2007) do consider the analytical problems associated with repeated eigenvalues, where it becomes necessary to use generalized eigenvectors, and where other behavior modes appear involving power functions of time. A filter essentially constitutes a weighted average of behavior modes and in this fashion avoids the "identity problem" of non-distinct eigenvalues.

The most serious theoretical issue, in our view, is how the results are interpreted using the feedback loop concept. As mentioned, the concept is relative (to a choice of an independent loop set). Moreover, practice reveals that the number of loops to consider is rather large and that the loops elasticities often do not have an easy or intuitive explanation. A lot of care must be taken when interpreting the results. For instance, Kampmann and Oliva (2006) found that "phantom loops" – loops that cancel each other by logical necessity and are essentially artifacts of the equation formulations used in the model – could nonetheless have large elasticities and thus seriously distort the interpretation of the results. The problems may not be intractable, but their resolution will require careful mathematical analysis.

Finally, a problem with EEA and LEEA is that it only considers changes to behavior modes, not the degree to which these modes are expressed in a system variable of interest. This issue is addressed by also considering the eigen*vectors* of the system, which is the foundation for the analysis in the next section.

Eigenvector analysis

The last set of methods, which are still in early development, we have termed the *eigenvector-based* approach (EVA). EVA attempts to improve the EEA/LEEA method by considering how much an eigenvalue or behavior mode is expressed in a particular system variable. One may present the logic of the method and how EEA and EVA complement each other as shown in Figure 5. As shown by Kampmann (1996), in a sense there is a one-to-one correspondence between eigenvalues and loop gains whereas the eigenvectors arise from the remaining "degrees of freedom" in the system.[13] The observed behavior of the state variables in the model is then the combined outcome of the behavior modes (from the loop gains) and the weights for each mode (from the eigenvectors) in the respective state variable.

A number of researchers have attempted to develop EVA methods. Some emphasize the curvature (second time derivative) of the behavior, similar to the starting point of the PPM method (Güneralp 2006; Saleh 2002; Saleh and Davidsen 2001a, b). The slope or rate of change $\dot{x}(t)$ of a given variable x in the linearized system may be written by

$$\dot{x}(t - t_0) = w_1 \exp\left(\lambda_1 (t - t_0)\right) + \cdots + w_n \exp\left(\lambda_n (t - t_0)\right), \tag{9}$$

where the weights w_i are related to the eigenvectors. Then the curvature at time t_0 is

$$\ddot{x}(t_0) = w_1 \lambda_1 + \cdots + w_n \lambda_n. \tag{10}$$

One may therefore interpret (10) has the sum of contribution from individual behavior modes. Güneralp (2006) suggested using the terms on the right-hand side of (10) as weights to combine elasticities of individual behavior modes ε_i with respect to some system element (like a link gain or a loop gain) into a weighted sum

$$\bar{\varepsilon} = \frac{\sum_{i=1}^{n} w_i \lambda_i \varepsilon_i}{\sum_{i=1}^{n} \left| w_i \lambda_i \right|}, \tag{11}$$

as a measure of the overall significance of that system element. He further normalized the elasticity measure by the elasticity measure for other system elements, i.e., assuming there are K such elements (loops or links), the relative importance ρ_k of the k-th element is defined as

$$\rho_k = \frac{\bar{\varepsilon}_k}{\sum_{j=1}^{K} \left| \bar{\varepsilon}_j \right|}, \tag{12}$$

with the motivation that elasticities may vary greatly in numerical values, making comparisons at different points in time difficult, whereas ρ_k is a relative measure varying between +1 and -1.

[13] As mentioned above, a system with N links and n variables will have N-n+1 independent feedback loops. The total degrees of freedom is equal to the N link gains that can all be varied independently. The remaining n-1 degrees of freedom relate to the determination of the eigenvector direction in the n-dimensional state space. Since an eigenvector can only be determined up to a scalar multiplier, i.e., if **r** is an eigenvector, then so is c**r**, where c is any scalar, the degrees of freedom of the eigenvector space is n-1, not n.

He then proceeded to illustrate the method on two models that have already been analyzed with EEA and PPM, namely the Lotka-Volterra model and the simple long wave model. His results shed an alternative light on the behavior of these models, though in our opinion, there is no dramatic improvement in intuition. In particular, the mathematical meaning, consistency and significance of the doubly normalized measure (12) remain to be clarified.

It is still too early to tell what the most useful approach will be, but one may note that the emphasis on the curvature shares the basic weakness in the PPM approach in dealing with oscillations, as discussed above.

Other researchers have looked directly at the weights w_i in (9), i.e., on the relative weight of the modes for a particular variable, from a policy criterion perspective, similar to Forrester's original focus and the starting point for the EEA analysis (Gonçalves 2006; Saleh *et al.* 2006, 2007). For instance, Saleh *et al.* (2007) look at how alternative stabilization policies affect the behavior of business cycle models, using both a simple inventory-workforce model (Sterman 2000), and a more extensive model based on Mass (Mass 1975) and used in the LEEA analysis of Kampmann and Oliva (Kampmann and Oliva 2006). Using the procedure in Figure 5, they decompose the net stabilizing effect of a policy into its effect on the behavior mode itself (LEEA) and its effect on the expression of that mode in the variable of interest (EVA). It is fair to say, however, that the EVA work is still to early in its development to judge its potential.

Discussion

As mentioned in the introduction, we should not hope for a "grand unified theory" that will automatically provide modelers with "the" dominant structure. Given the analytical intractability of nonlinear high-order systems found in our field, the most we can hope for is a set of tools that will guide the analysis and aid the development of the modeler's intuition.

That said, however, we are left with an impression that the analytical foundation for these tools is in need of further development before one rushes into implementing them into software packages. We are quite satisfied with the current state of affairs in this regard, where code, models, and documentation are made freely to download (most of the cited papers provide URL were their code is available). Understanding *how* and *why* the tools work the way they do is crucial, and this will require that a number of puzzles, uncertainties, and technical problems be addressed. Only then will the time come to submit the methods for wider application to test their real-world utility. It is too early to tell, in other words, what method is "the best".

While the classical method remains a useful intuitive guide and teaching tool for graduate students, there are no signs that it may be developed further. (That said, it is possible that the classical control transfer function method may be employed in the eigensystem approaches to explore nested canonical systems, though this is purely speculative). The pathway method would benefit from a firmer mathematical foundation. In particular, it would be important to compare how its results and conclusions compare to those found in the LST. It is possible that the pathway method may eventually be merged with the LST approaches as a subset of a general analytical toolbox. We believe that there is a great deal of promise in combining the eigenvalue and eigenvector analysis in the LST approaches. This combination will yield a complete system characterization and an understanding of both how particular feedback loops are involved in generating a behavior mode, and how system elements determine the expression of that behavior mode in a particular variable. A "unified" LST approach along the lines suggested in Figure 5 thus seems within reach.

We conclude the paper with a concrete analytical “to-do list:”

Loop analysis: We need to understand better how the different loop representations (independent loop sets) relate to one another, specifically how they relate to the elasticity measures from Mason’s rule (see footnote 11).

Loop and link vector spaces: As shown by Kampmann (1996), the link eigenvalue elasticities constitute a *current* in the causal network of the model, which is why the number of independent loops is equal to N-n+1. However, the dual “voltage space” of the link elasticities, which has dimension n-1 may provide a clue to the effect on the eigenvectors. There may be an equally fundamental mathematical property here waiting to be discovered.

Non-distinct eigenvalues. We need a full account of the case where the eigenvalues are not distinct. This includes first an assessment of how common this situation might be in real-world models, how to identify and use the generalized eigenvectors needed for this approach, how to characterize the behavior (which will now include power functions of time), and what happens when two identical eigenvalues bifurcate into two distinct ones (does behavior change smoothly?).

Alternative output criteria. This involves using some of the measures outlined in Table 2, which will include both EEA and EVA analysis. As mentioned in the text, one can exploit the linearity property of most filters.

Applications to nonlinear models. We need more experiments and examples of using the EEA/EVA method to different nonlinear models, perhaps based on the taxonomy suggested in this paper (quasi-linear, nonlinear single-transient, nonlinear periodic, nonlinear multi-mode). It would be interesting to apply it to a truly large-scale model, like Sterman’s energy transition model (Sterman 1981), for instance.

Relationship between PPM and EEA/EVA methods. So far, the comparison between the two methods has been by looking at results of their application (e.g., Kampmann and Oliva 2006). It ought to be possible, however, to derive analytical results showing the relationship between EEA/EVA measures and the PPM measures.

Special cases and exceptions. It seems that certain structures pose an analytical puzzle. These are relatively simple structures which should lend themselves well to formal analysis. For instance, a “figure-8” second-order loop where two levels interact through two first-order loops via the same auxiliary variable (as happens in the Lotka-Volterra model, see Güneralp 2006, Fig. 1) will not be considered a loop unless the auxiliary variable is eliminated from the equations, yet it may conceptually be an important loop. Another special case is the “phantom loops” discovered by Kampmann and Oliva (2006). There are probably several more such cases which need to be fully understood.

Numerical issues. Finding a stable an efficient method of eigenvalue calculation will be crucial for implementing the method in mainstream software packages.

Visualization. Ultimately, the aim of the exercise will be to provide intuitive illustrative measures for the practitioner. It may be that this can best be achieved through various visual aids. For instance, one could make the links between variable in the model diagram "glow" with different color and brightness depending on their significance to a given measure. The visual aids may capitalize on the ability of the human eye to detect patterns that may ultimately prove much more efficient and effective than the measures currently under consideration.

Bibliography

Abdel-Gawad A, Abdel-Aleem B, Saleh M, Davidsen P. 2005. Identifying dominant behavior patterns, links and loops: Automated eigenvalue analysis of system dynamics models. *Proceedings of the 2005 Int. System Dynamics Conference*. Boston.

Barlas Y. 1989. Multiple Tests for Validation of System Dynamics Type of Simulation Models. *European J. of Operations Research* **42**(1): 59-87.

Chen CT. 1970. *Introduction to Linear System Theory*. Holt, Rinnehart and Winston: New York.

Eberlein R. 1984. Simplifying Models by Retaining Selected Behavior Modes. PhD Thesis, Sloan School of Management, Mass. Inst. of Technology, Cambridge, MA.

Eberlein RL. 1986. Full Feedback Parameter Estimation. *Proceedings of the 1986 Int. Conference of the Systems Dynamics Society*, pp. 69-83. Sevilla, Spain.

Eberlein RL. 1989. Simplification and understanding of models. *System Dynamics Review* **5**(1): 51-68.

Eberlein RL, Wang Q. 1985. Statistical Estimation and System Dynamics Models. *Proceedings of the 1985 Int. Conference of the Systems Dynamics Society*, pp. 206-222. Keystone, CO.

Ford A, Flynn H. 2005. Statistical screening of system dynamics models. *System Dynamics Review* **21**(4): 273-303.

Ford DN. 1999. A Behavioral Approach to Feedback Loop Dominance Analysis. *System Dynamics Review* **15**(1): 3-36.

Forrester JW. 1961. *Industrial Dynamics*. Productivity Press: Cambridge MA.

Forrester JW. 1969. *Urban Dynamics*. Productivity Press: Cambridge, MA.

Forrester JW. 1971. *World Dynamics*. Productivity Press: Boston.

Forrester JW. 1993. System Dynamics and the Lessons of 35 Years. In DeGreene KB (ed.), *Systems-Based Approach to Policymaking*, pp. 199-240. Kluwer Academic Publishers: Norwell, MA.

Forrester JW, Senge PM. 1980. Tests for Building Confidence in System Dynamics Models. *TIMS Studies in Management Science* **14**: 209-228.

Forrester N. 1982. A Dynamic Synthesis of Basic Macroeconomic Policy: Implications for Stabilization Policy Analysis. PhD Thesis, Sloan School of Management, Mass. Inst. of Technology, Cambridge, MA.

Forrester N. 1983. Eigenvalue analysis of dominant feedback loops. *Proceedings of the 1983 Int. System Dynamics Conference*. Chestnut Hill, MA.

Goldberg DE. 1989. *Genetic Algorithms in Search, Optimization and Machine Learning*. Addison-Wesley: Reading, MA.

Gonçalves P. 2003. Demand bubbles and phantom orders in supply chains. PhD Thesis, Sloan School of Management, Mass. inst. of Technology, Cambridge, MA.

Gonçalves P. 2006. Eigenvalue and Eigenvecor Analysis of Dynamic Systems. University of Miami, School of Business Administration, Working Paper. Coral Gables, FL.

Gonçalves P, Lerpattarapong C, Hines JH. 2000. Implementing formal model analysis. *Proceedings of the 2000 Int. System Dynamics Conference*. Bergen, Norway.

Graham AK. 1977. Principles on the Relationship Between Structure and Behavior of Dynamic Systems. PhD Thesis, Sloan School of Management, Mass. Inst. of Technology, Cambridge, MA.

Güneralp B. 2006. Towards Coherent Loop Dominance Analysis: Progress in Eigenvalue Elasticity Analysis. *System Dynamics Review* **22**(3): 263-289.

Haxhold C, Kampmann CE, Mosekilde E, Sterman JD. 1995. Mode Locking and Entrainment of Endogenous Economic Cycles. *System Dynamics Review* **11**(3): 177-198.

Hines JH. 2005. How to Visit a Great Model Like Yours. *Proceedings of the 2005 Int. System Dynamics Conference*. Boston.

Holland JH. 1992. *Adaptation in Natural and Artificial Systems*. MIT Press: Cambridge, MA.

Homer JB. 1983. Partial-Model Testing as a Validation Tool for System Dynamics. *Proceedings of the 1983 Intl. System Dynamics Conf.*, pp. 920-932. Chestnut Hill, MA.

Kampmann CE. 1996. Feedback Loop Gains and System Behavior (unpublished manuscript). *Proceedings of the 1996 Int. System Dynamics Conference*, pp. 260-263 (summarized). Cambridge, MA.

Kampmann CE, Oliva R. 2006. Loop Eigenvalue Elasticity Analysis: Three Case Studies. *System Dynamics Review* **22**(2): 146-162.

Luenberg D. 1979. *Introduction to Dynamic Systems: Theory, Models and Applications*. Wiley: New York.

Mass NJ. 1975. *Economic Cycles: An Analysis of Underlying Causes*. Productivity Press: Cambridge, MA.

Meadows DH, Meadows DL, Randers J, Behrens III WW. 1972. *The Limits to Growth: A Report for the Club of Rome's Project on the Predicament of Mankind*. Universe Books: New York.

Mojtahedzadeh MT. 1996. A path taken: Computer-assisted heuristics for understanding dynamic systems. PhD Thesis, Rockefeller College of Pubic Affairs and Policy, State University of New York at Albany, Albany, NY.

Mojtahedzadeh MT, Andersen D, Richardson GP. 2004. Using *Digest* to implement the pathway participation method for detecting influential system structure. *System Dynamics Review* **20**(1): 1-20.

Morecroft JDW. 1985. Rationality in the Analysis of Behavioral Simulation Models. *Management Science* **31**(7): 900-916.

Mosekilde E, Laugesen JL. 2006. Nonlinear Dynamic Phenomena in the BEER Model. Department of Physics, The Technical University of Denmark. Kongens Lyngby, Denmark.

Ogata K. 1990. *Modern Control Engineering*. 2nd ed. Prentice Hall: Englewood Cliffs, NJ.

Oliva R. 2003. Model Calibration as a Testing Strategy for System Dynamics Models. *European Journal of Operational Research* **151**(3): 552-568.

Oliva R. 2004. Model Structure Analysis Through Graph Theory: Partition Heuristics and Feedback Structure Decomposition. *System Dynamics Review* **20**(4): 313-336.

Oliva R, Mojtahedzadeh M. 2004. Keep it simple: Dominance assessment of short feedback loops. *Proceedings of the 2004 Int. System Dynamics Conference*. Oxford, UK.

Ott E. 1993. *Chaos in Dynamical Systems*. Cambridge University Press: New York.

Peterson DW. 1980. Statistical Tools for System Dynamics. In Randers J (ed.), *Elements of the System Dynamics Method*, pp. 224-241. Productivity Press: Cambridge MA.

Peterson DW, Eberlein RL. 1994. Reality Checks: A Bridge Between Systems Thinking and System Dynamics. *System Dynamics Review* **10**(2/3): 159-174.

Radzicki MJ. 2004. Expectation Formation and Parameter Estimaion in Uncertain Dynamical Systems: The System Dynamics Approach to Post Keynesian-Institutional Economics. *Proceedings of the 2004 Int. System Dynamics Conference*. Oxford, UK.

Richardson GP. 1984/1995. Loop Polarity, Loop Dominance, and the Concept of Dominant Polarity. *System Dynamics Review* **11**(1): 67-88.

Richardson GP. 1986. Dominant structure. *System Dynamics Review* **2**(1): 68-75.

Richardson GP (ed.). 1988. *System Dynamics Review: Special Issue on Chaos*. Vol. 4(1-2).

Richmond B. 1980. A new look at and old friend. Plexus Resource Policy Center, Thayer School of Engineering, Dartmouth College. Hanover, NH.

Saleh M. 2002. The characterization of model behavior and its causal fundation. PhD Thesis, Dept. of Information Science, University of Bergen, Bergen, Norway.

Saleh M, Davidsen P. 2001a. The origins of behavior patterns. *Proceedings of the 2001 Int. System Dynamics Conference*. Atlanta.

Saleh M, Davidsen P. 2001b. The origins of business cycles. *Proceedings of the 2001 Int. System Dynamics Conference*. Atlanta.

Saleh M, Oliva R, Davidsen P, Kampmann CE. 2006. Eigenvalue Analysis of System Dynamics Models: Another Perspective. *Proceedings of the 2006 Int. System Dynamics Conference*. Neijmegen, The Netherlands.

Saleh M, Oliva R, Davidsen P, Kampmann CE. 2007. Analysis of System Dynamics Models Using Behavioral Decomposition Weights. Mays Business School, Texas A&M University, Working Paper.

Schweppe F. 1973. *Uncertain Dynamical Systems*. Prentice-Hall: Englewood Cliffs, Nj.

Sterman JD. 1981. The Energy Transition and the Economy: A System Dynamics Approach. PhD Thesis, Sloan School of Management, Mass. Inst. of Technology, Cambridge, MA.

Sterman JD. 1985. A Behavioral Model of the Economic Long Wave. *Journal of Economic Behavior and Organization* **6**(1): 17-53.

Sterman JD. 2000. *Business dynamics: Systems thinking and modeling for a complex world*. Irwin McGraw-Hill: Boston.

Zhusubaliyev ZT, Mosekilde E. 2003. *Bifurcation and Chaos in Piecewise-Smooth Dynamical Systems*. World Scientific: Singapore.

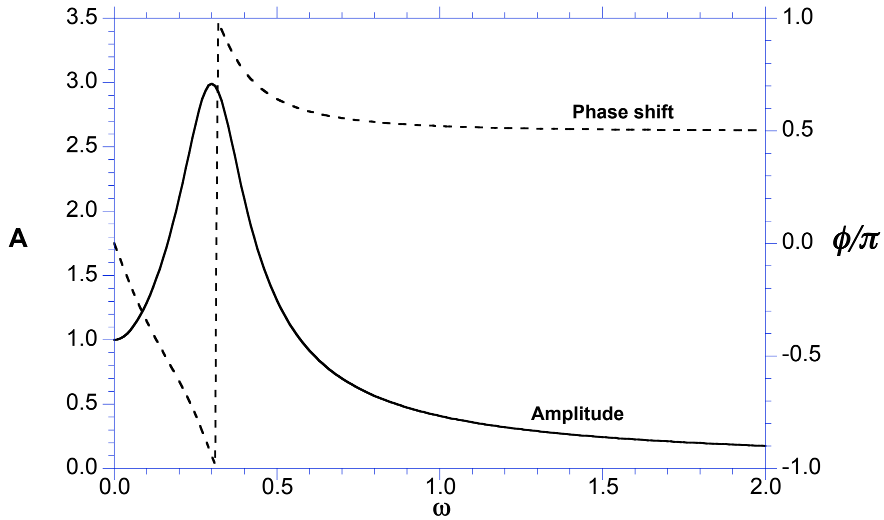

Figure 1
Phase-and-gain diagram (Bode diagram) showing the inventory A sin(ω t + ϕ) with amplitude A and phase shift ϕ, relative to a sinusoidal demand sin(ω t), for varying values of the frequency ω of the demand fluctuation.

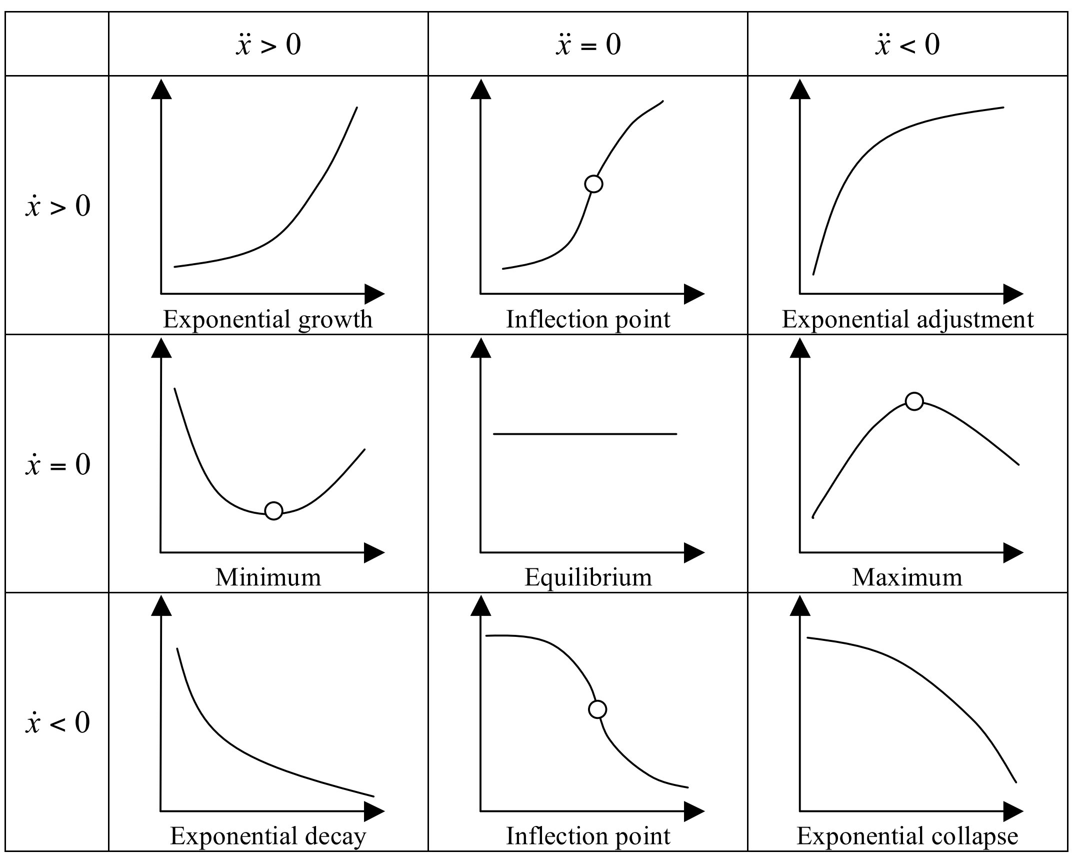

Figure 2
Characteristic behavior patterns based on the first and second time derivatives.

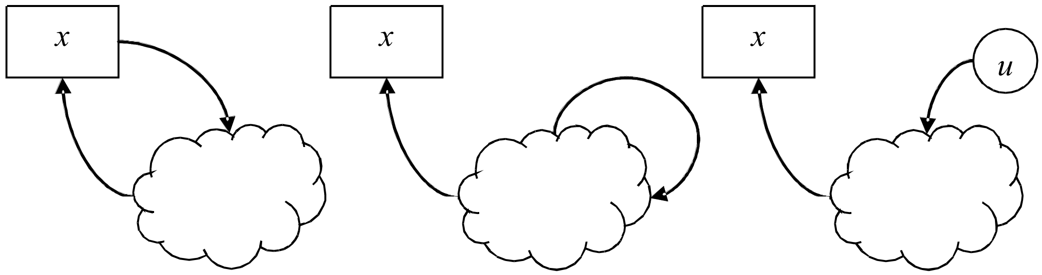

Figure 3
Three alternative forms of dominant structure in the PPM method

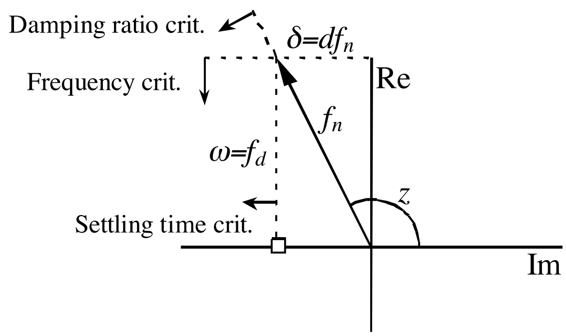

Figure 4
Characterization of eigenvalues plotted in the complex plane

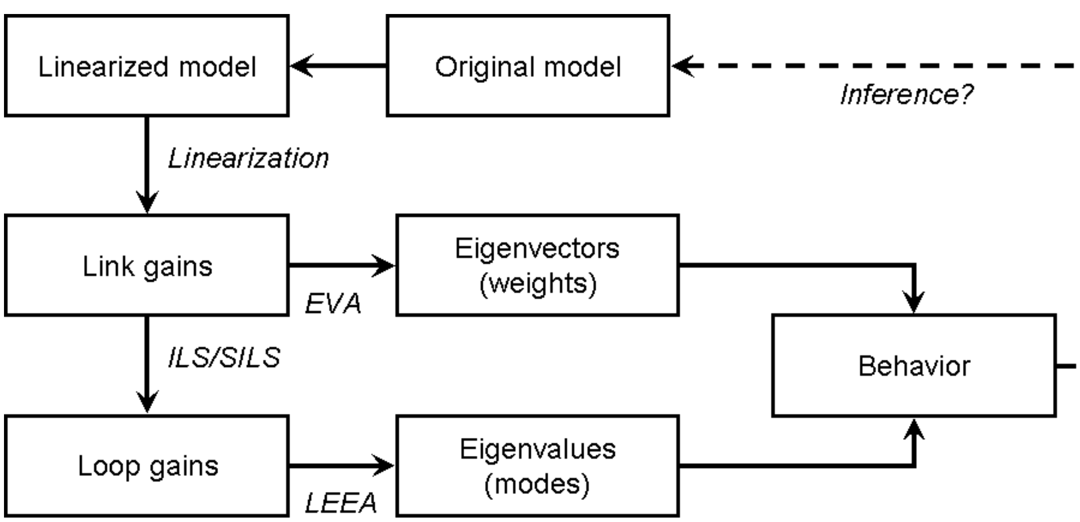

Figure 5
Schematic view of eigenvalue and eigenvector analysis approach

		Time graphs	Freq. response	Eigenvalues
Model reduction	Linear	Traditional		Eberlein ('84, '89)
	Nonlinear			
Structure contribution	Linear	Traditional, PPM	Classical	EEA/LEEA, EVA
	Nonlinear			

Table 1: Taxonomy of methods for identifying dominant structure.

PPM: Pathway participation metric. EEA/LEEA: (Loop) eigenvalue elasticity analysis. EVA: Eigenvector analysis. Adapted from Richardson (1986)

Policy Criterion	Description	Change in eigenvalue $\lambda = \delta \pm i\omega, \omega > 0$	Change in BDW w	Appropriate measure in time path
Damping	Increases the rate of decay of oscillation (or decreases the rate of expansion)	$\frac{\partial \delta}{\partial g}\frac{g}{\delta} < 0$	N/A	$\frac{x(t+T)}{x(t)}$
Frequency	Decreases the frequency of oscillation (or lengthens the period T)	$\frac{\partial \omega}{\partial g}\frac{g}{\omega} < 0$	N/A	T
Variance	Reduces the variance of a target variable (or the weighted average variances of several variables)	No simple relation	$\frac{\partial w}{\partial g}\frac{g}{w} < 0$	$\int x(t)^2\, dt$
Auto-spectrum	Reduces variance of target variable(s) within a target frequency range	No simple relation	$\frac{\partial w}{\partial g}\frac{g}{w} < 0$	Filter in frequency domain
Frequency response gain	Reduces the gain (amplification) in the target frequency range for a particular combination of disturbance exogenous and output variables.	Based upon transfer function $G(i\omega)$		

Table 2

Stabilization policy criteria and corresponding effects on eigenvalues and BDW of a policy change in a system element g.