Fifty Years of Table Functions

By Douglas Franco
ECONOINVEST C.A.
Caracas, Venezuela
58 212 9636549
dfranco@cantv.net

Abstract

Table or lookup functions, TF, are part of System Dynamics models since the very beginning fifty years ago. TF formulation, dynamics and multidimensionality are analyzed and suggestions for their improvement are presented. TF spread nonlinear dynamics to other variables involved and simplify structure, allowing the concentration on problems. Relationships between nonlinear varying eigenvalues and TF' patterns are established, which open the door for SD' software improvements and further research. A general shape, synthesizing typical TF' patterns, plots a normal curve that the Central Limit Theorem of statistics proves as the distribution of any accumulation (level) of random variables. The random face of SD, uncovered by Agent Base Simulation, converges into this normal curve, which gives statistical grounds to lookup relationships and improves policy design by fine-tuning estimation.

Key Words: Lookup, Nonlinear Dynamics, normal cdf, Eigenvectors

Introduction

An exponential growth, a straight line and a saturation curve are typical shapes of lookup functions, extensively used by Jay Forrester in Industrial, World, and Urban Dynamics. John Sterman (2000, p 817) also uses those shapes when relating the inventory coverage with price, in his outstanding treatise "Business Dynamics." Thousands of those patterns illustrate many SD books and papers along this half century. In the last decade, Agent Base Modeling uncovers random events behind SD models; even a simple decay process is a random event, where items leave the level by an exponential distribution. The Central Limit Theorem of statistics proves that accumulations of any random variable tend to distribute normally. Therefore, the normal CDF, with growing, linear, and saturation parts, combines most of the TF shapes; therefore, a universal table function emerges from the statistical distribution of levels.

Table or Lookup functions formulation, dynamics, universality, and multidimensionality are discussed and suggestions for their improvement are presented. Partial Differential Dynamics problems are solved by the appropriate use of lookup functions. The insertion of the normal CDF makes some atoms in the Hines' molecules.

Table Functions Formulation

Time can never be a part of a table function; thus, Table or Lookup functions are relationships between variables. A variable Y(t) may come from the interactions between accumulations and rates in the system.

Y(t) = TABLE (X(t)) or Y(t) = LOOKUP (X(t))

This relationship is either graphic or algebraic. Y(t) can also be a multidimensional variable like the birth rate or the level of Quality of Life in World Dynamics, where the Y(t) depends upon the product of several factors, each one a single value Table Function:

Y(t) = *factor1*(X1(t))* *factor2*(X2(t))* *factor3*(X3(t))

The quality of life level and the birth rate are multidimensional products of factors. Each factor is an individual table function.

Birth_rate =
BRN · Population · ***f1*** (Pollution) · ***f2*** (Food) · ***f3*** *(Crowding)* · ***f4*** *(Material_Standard_of_Living)*

This assumption is a good approximation to most multivariable problems. It may be the case that X_2 variation alters $factor_2$ and X_3 simultaneously. For instance, pollution may affect also food production; therefore, the quality of life may not be the product of a single value multiplier from food with a multiplier from pollution; however, for the purpose of world dynamics, this approximation is good enough.
The probability of missing a sale, depend upon the availability of the item in the inventory and the service of the salesperson. The product may be available and the salesperson ignores it. Interactions among variables may be relevant to some problems. However, there is no mathematical substitute to model the separation of variables in connection with the real operation of the problem.

Partial Differential Dynamics and Table Functions

Partial Differential Dynamics is present in System Dynamics models, since the very early Forrester' work; but Differential Equations notation is avoided. A common technique, that transforms Partial into Ordinary Differential Equations, is the one termed "Separation of Variables"` where multivariable relationships are products of single value functions, which are similar to the Jays' Lookup Functions in World Dynamics.

F(x, y, z) = X(x)*Y(y)*Z(z)

This assumption of separation of variables is the primary approach to solve partial differential equations (Cain and Meyer 2005). System Dynamics shorten the distance

between problem and solution, bypassing the unnecessary step of the partial differential equation.
Table functions are also part of many engineering models. In general, **Y(t)** is a product of **X(t)** factors**:**

$$Y(t) = (X_1(t)^a)* (X_2(t)^b)* (X_3(t)^c)$$

Each factor **Xi(t)** has to be non-dimensional, if the parameters **a, b, c** or the table function' pattern does not change when using different units. Dimensional consistency between **X**s and **Y** relates the parameters **a, b** and **c**, grouping the variables in non-dimensional modules, as in the Reynolds' Number of fluid dynamics (Jermy M., 2005**).**

Table Functions' Dynamic

The levels in the table function are engaged in their own nonlinear dynamics.

$$dy/dt = F(y, ...)$$
$$dx/dt = G(x,...)$$

A first order linear approximation of **F** and **G** gives:

$$dy/dt = \lambda_1 . y + ...$$
$$dx/dt = \lambda_2 . x + ...$$

This approximation is robust (valid in wider range of values) if the variables involved in the dynamics are eigenvectors of the linearized system and the parameters λ_1, λ_2 their eigenvalues, which are constant on local neighborhoods.

Dividing both equations,

$$dy/dx = (\lambda_1 / \lambda_2) * (y/x) = E*(y/x), \text{ where } E = \lambda_1 / \lambda_2$$

or equivalently,

$$dy/y = E* (dx/x)$$

$$Y = Yo* (X/Xo)^E = Yo* \text{ TABLE FUNCTION}$$

Each factor, **(X/Xo)^E,** is an analytical Table Function, where the relationship is algebraic rather than graphic. **E** is assumed constant elasticity in most Economic models; but, in general, table functions' patterns portray **E** as a function of **X** with a graphic, non-algebraic representation. If there are many variables, Economy uses a product of the TFs, namely,

$$Y= Yo* (X/Xo)^{E_1} * (Z/Zo)^{E_2}$$

The dynamics of many variables, encapsulated in Table Functions, originates a simpler model. Each table function spreads the dynamics to other variables involved, similar to linear models where the dynamics of the eigenvectors relates to the rest of the variables by auxiliary linear functions or weighted averages of the behavior modes(Saleh M, Oliva R, Kampmann C, Davidsen P, 2006) . The Table Functions spread the dynamic to many other dependent variables, because they map nonlinear relationships between variables given non-constant eigenvalues. There are many SD models, like World or Urban Dynamics, where experience-modeling practice achieves simple structures without using sophisticated mathematical formulas. Jay Forrester(1969), pack the world' problems in clever table functions, avoiding an endless debate about the dynamics of the Quality of Life in the World, promoting recycling and population control worldwide; the torch, passed to Urban Dynamics, still flame discussions around the dynamics of the quality of life in a city.

The elasticity **E** is a constant in most economic models; but, in general, the SD table functions portray **E** as function of **X**. For instance, the demand of a product decreases as price increases. The good is elastic when **E** is higher than one and inelastic when **E** is less than one. At very low price, customers do not care much about price; therefore, the good behaves inelastic: demand remains unchanged for small price' changes. If you keep rising the price, there is a turning point where the good become elastic because customers start to care for the price, because fewer customers are willing to pay those higher prices for that many items. The elasticity rises with price, because the probability of selling a product decreases as the price increases. There is a high enough price where there are, few customers left. Something similar can be said about the relationship between an economic factor and production. If you have few employees, specialization is not possible; but, as labor increases then the organization of work, namely production' lines, jump the output to higher levels. Finally, the law of diminishing returns saturates output from labor.

This relationship between **E** and **X** covers most SD' table functions patterns and look like a normal CDF: a sort of universal shape. The SD software is evolving toward formal mathematical tools, in the edge between build and analysis, to help model construction, simplification, analysis and testing.

Universal Table Function UTF

Most lookup patterns fit into segments of the normal distribution curve and the normal CDF links accumulations and the probability of occurrence of related events. The central limit theorem of statistics proves the normal distribution for those accumulated events, regardless of their individual distribution. Perhaps, we did not realize the random nature of SD models, until Agent Base Modeling uncover the random face of SD (Borshchev A, Filipov, 2005), even the output from one level with fixed time constant distributes exponentially, because the items do not leave the level at the same time. Besides, if we are designing policies, which covers many periods or many decay processes, then the exponential distributions add into a normal CDF.

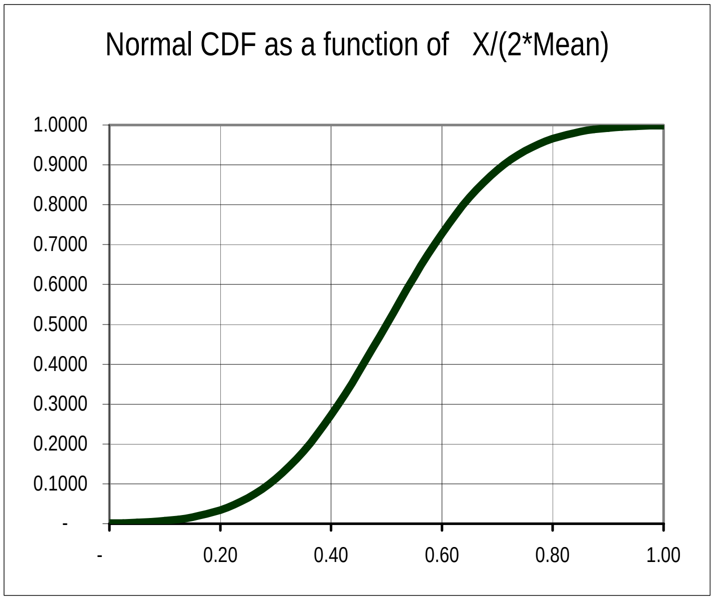

Figure 1. Normal Table Function Probability (Variable/Mean)

Elasticity starts less than one at low probabilities, but when the probability approaches to one, it grows exponentially. Normal curve portrays a varying elasticity. For instance, the relationship between inventory and percentage of lost sales may be like Figure 3.

The average probability of having a solicited item in stock increases with inventory size. More items in the inventory decrease the probability of an out stock. However, if your stock is too low, then there is a high probability of missing one or another item, one or another day.

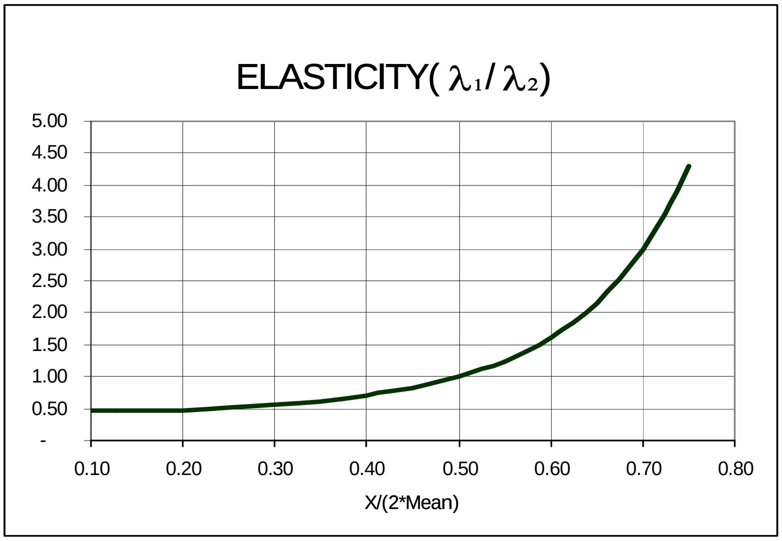

Figure 2. Elasticity (Probability)

The loss of customers related to “out of stock” explodes below a certain low level in the inventory where the unsatisfied clients run away. By contrast, if you have more and more items, sales saturate. In Urban Dynamics, the probability of funding a housing program increases as the Tax ratio increases; but there is a minimum quantity required to finance al least one housing project. Sometimes, the problem locates in a sub range of all possible values. For instance, A and B reflect half of the normal range, A the top and B the bottom.

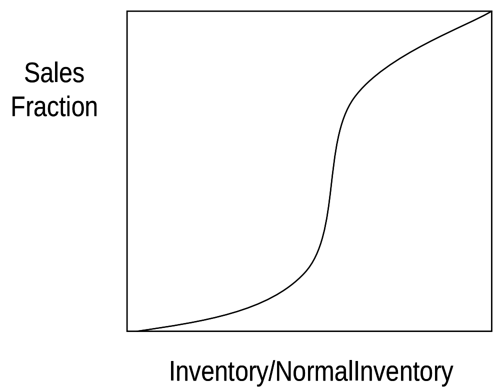

Figure 3. Sales Fraction as a function of Inventory/Normal Inventory

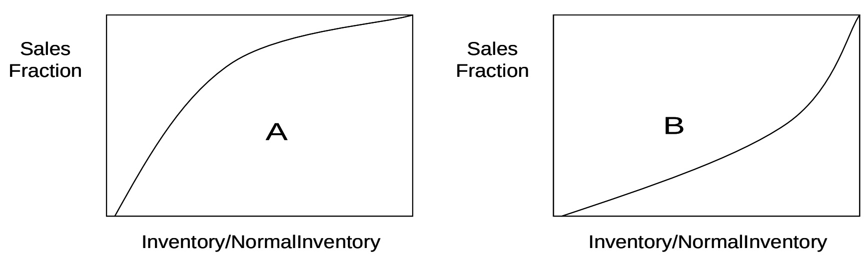

Figure 4. Half of the Normal Pattern

For instance, Forrester in World Dynamics, p63, use the following table function to represent the quality of life as a function of the available Food Ratio. The probability of having food for everyone increases as food ratio increases.

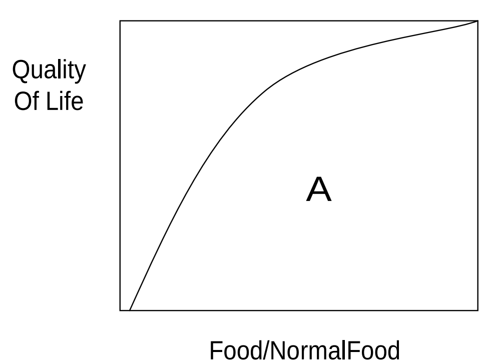

Figure 5. Quality of Life as a function of Food ratio

This function misses the fact that when people diet, they require less food intake to keep the same “quality of life,” that is why poor people can live well with less food per capita than rich people, who are not necessarily better of by overloading their refrigerators. Therefore, a more robust representation of this table function is:

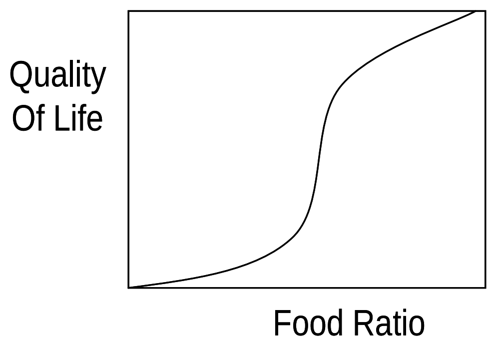

Figure 5. Enhanced Quality of life as a function of Food Ratio

For instance, countries like Bangladesh, on the verge of limits to growth from food, lessen famine by delaying the external food relief, letting people to adjust to lesser-required food per capita (Mayer 1975). A finer policy moves the operation from the top to the bottom of the curve, because the bottom region is less sensitive to inventory size. Downsizing is often an alternative when facing financial constraints. A company may choose to have fewer items in stock to cope with financial limits to growth, because increasing inventory to diminish delivery delay may not be affordable. In the upper part of the curve, limits to growth, from variables not included in the model saturates the effect.

Molecules and Table Functions

Molecules are building blocks of system dynamics models, created by Jim Hines to promote good modeling. Table functions are atoms for Hines’ molecules. In page 56, the molecule describes the effect of inventory upon sales. Using the normal CDF the molecule will look like:

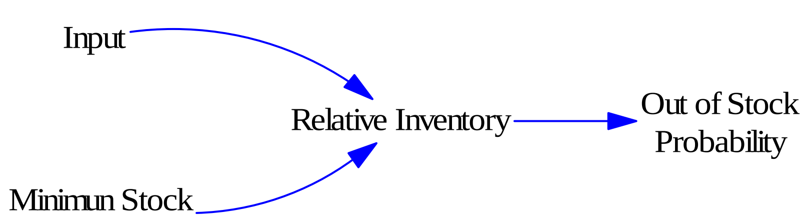

Figure 6. Atoms for Hines’ molecules

Loosing clients may add an additional effect. These data is usually available concerning the statistics of average fraction of lost sales related to average levels of inventory. The normal distribution describes the probabilities of those averages.

Hines (2005), Page 57, talks about the pressure to change value, a deviation from the normal one, a typical normal CDF case. Page 59 is about the influence of a level over draining, for instance, the population level and the average age of the population influence dying. The normal distribution with the life expectancy of the population is a good approximation of the actuarial average fraction of death every year. The probability of completing a work on time is also describable by the normal CDF. There are also, Table Functions in the molecules regarding prices and other economic issues, especially coming from The MIT National Model of the US Economy, where there are many lookup functions for production of goods, services, and consumers' satisfaction. The shapes of those Table Functions establish a major breakthrough in Economic Theory and thinking, even more significant than the now extraordinary endogenous generation of economic cycles. Policy design can benefit from those practical behavior patterns. With the use of normal curve to cast lookup relationships, the molecules get stronger covalent bonds where the atoms stick together even under high-pressure environments, because curves will reflect more data than opinions.

Conclusion

Table or lookup functions spread nonlinear dynamics to other variables involved, allowing simpler models. A Normal Table Function synthesizes most shapes used in simulation, because it is the statistical distribution for any accumulation (level) of random variables. The Normal Table Function provides statistical grounds to lookup relationships and improves policy design by fine-tuning estimation; therefore, the normal CDF strengthens lookup relationships. The normal CDF changes the kind of analytical table functions mostly used in Economic models, where constant elasticity is assumed. The normal CDF pattern fine-tunes the behavior of customers facing price increases and the relationships among economic factors, one of the major contributions of the National Model of the US Economy. Nevertheless, there are many mappings to explore between nonlinear eigenvalues and table functions' patterns, which open the doors for further research, especially to improve the available analytical tools in SD' software.

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