

Verifying System Dynamics Simulation Results by Analytical Phase Plane Tools

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Abstract

System dynamic models are typically impossible to solve analytically and even hard to correctly simulate. This complexity is also true in equilibrium and stability analysis. This research deals with analytical and numerical phase plane analysis of such models. Example models chosen have no analytical time trajectory solutions; they are nonlinear and computationally stiff. In the study, two example models are analyzed; a prey-predator model and a competition model. Numerical phase plane constructions of these systems are first carried out. Then, analytical phase trajectories are obtained by using LambertW functions, implemented by Maple software. Numerical work is carried out by Stella simulation software, and by numerical algorithms in MAPLE. Comparisons between analytical and numerical phase trajectories are made and it is observed that significant discrepancies may exist. The study shows that, although analytical time trajectories are typically impossible to obtain, analytical phase trajectories offer important potential for system dynamics.

Introduction

The aim of this study is to plot and compare the analytical & numerical solution trajectories in phase plane, for two the selected models. Phase plots are important in doing stability analysis of complex nonlinear systems (Toro *et.al* 1988, Swart 1990) Phase planes facilitate more detailed analysis of the model dynamics, also providing insight about the correctness of the numerical solutions.

We carry out our research on two example system dynamics models. The first one is a basic predator-prey model, a second order non-linear stiff system with one stable and one unstable equilibrium point. The second model that we analyze is a basic model of competition between two species, which is a second order nonlinear model that has two unstable equilibrium points. The research procedure consists of first plotting the phase trajectories numerically, by simulation, then obtaining and plotting the analytical phase trajectories and finally comparing these two results. Numerical solutions are obtained by Stella and Vensim simulation software. Analytical phase trajectories are obtained by making use of LambertW functions, as suggested and implemented by Maple software (Corless *et.al* 1993). In the appendix we describe the LambertW function which is a transcendental function used in the analytical phase solutions of nonlinear models. A side benefit of the research is to make the system dynamics researchers more familiar with LambertW function and its potential in the analysis of nonlinear models.

Phase trajectories will be plotted with different representative initial conditions so as to uncover any significant errors that may occur in numerical solutions. Simulations will also be repeated with different time step (dt) values in order to see if and when the resulting phase trajectories start deviating qualitatively from the analytical ones.

Prey-Predator Model

The model is:

$$\frac{dx}{dt} = \alpha x - \beta xy$$

$$\frac{dy}{dt} = -\gamma y + \delta xy$$

where x is prey population, y predator population) and $\gamma, \alpha, \delta, \beta$ are (+) constants representing birth, death and interaction rates respectively.

The equilibrium points of this model can be easily obtained to be: (0,0) and $(\gamma/\delta, \alpha/\beta)$

When the stability of these points are analyzed by linearization, it is found that (0,0) is unstable and $(\gamma/\delta, \alpha/\beta)$ is borderline stable. The interesting point to be analyzed is the non-zero equilibrium $(\gamma/\delta, \alpha/\beta)$ since this is where both species co-exist. Linearized analysis also indicates that in the very close neighborhood of $(\gamma/\delta, \alpha/\beta)$, closed phase orbits exist. (Toro et.al 1988, Swart 1990, Camara *et.al* 1994 and Barlas 2005). This means that the species exhibit constant-amplitude oscillations in the close vicinity of $(\gamma/\delta, \alpha/\beta)$. One interesting question then becomes: Do these closed orbits (constant oscillations) also exist elsewhere in the phase plane, starting with any initial condition not necessarily close to $(\gamma/\delta, \alpha/\beta)$ The conjecture is ‘yes’ and often supported by simulation, but not proven. (See Figure 3, below).

Prey-predator model in phase representation becomes:

$$\frac{d}{dx} y = \frac{(-\gamma + \delta x)y}{(\alpha - \beta y)x}$$

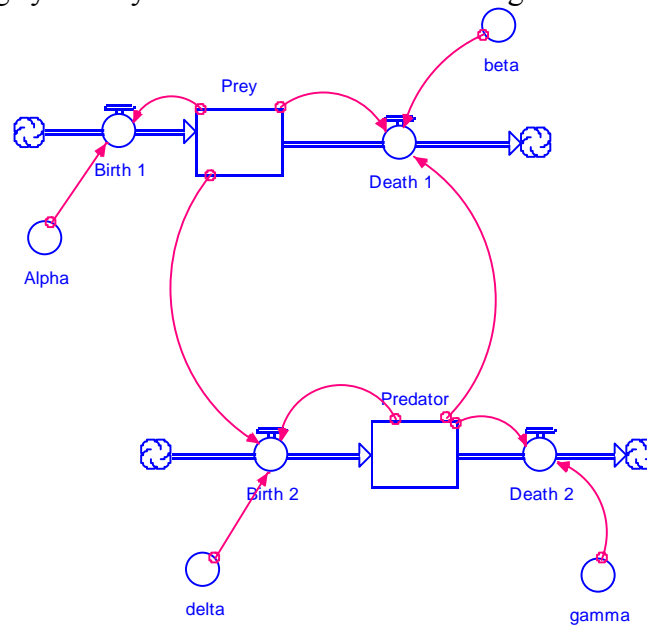
General solution of phase trajectories obtained using Lambert;s W function and Maple software is presented below:

$$y(x) = e^{\left(\frac{-\text{LambertW}\left(-\frac{\beta x \left(-\frac{\gamma}{\alpha} \right) e^{\left(\frac{\delta x}{\alpha} + \frac{C}{\alpha} \right)}}{\alpha} \right)}{\alpha} \right) \alpha + \delta x - \gamma \ln(x) + C}$$

where C is some constant, depending on initial conditions.

LambertW term seen in the general solution of the Prey-Predator model is a transcendental function recently re-discovered in literature.(Hayes 2005, Coreless *et.al* 2000) Lambert's W functions, $W(x)$, are defined to be solutions of the equation $W(x) / \exp(W(x)) = x$. This implicit function has multiple branches for $x < 0$; however, it has only two real-valued branches. We define $W_0(x)$ to be the principal branch, where $W > -1$ for $x < 0$, and $W_{-1}(x)$ to be the other real branch, where $W < -1$ for $x < 0$. (For further information see the appendix).

The corresponding system dynamics stock-flow model is given below:



$$\text{Predator}(t) = \text{Predator}(t - dt) + (\text{Birth}_2 - \text{Death}_2) * dt$$

$$\text{INIT Predator} = 15$$

INFLOWS:

$$\text{Birth}_2 = \text{Predator} * \text{Prey} * \text{delta}$$

OUTFLOWS:

$$\text{Death}_2 = \text{Predator} * \text{gamma}$$

$$\text{Prey}(t) = \text{Prey}(t - dt) + (\text{Birth}_1 - \text{Death}_1) * dt$$

$$\text{INIT Prey} = 40$$

INFLOWS:

$$\text{Birth}_1 = \text{Prey} * \text{Alpha}$$

OUTFLOWS:

$$\text{Death}_1 = \text{Predator} * \text{Prey} * \text{beta}$$

$$\text{Alpha} = 0.2$$

$$\text{beta} = 0.01$$

$$\text{delta} = 0.016$$

$$\text{gamma} = 0.4$$

Figure 1 Stock-flow Diagram of the Prey-Predator Model and related equations

The model, with proper parameter values, is equivalent to the following equation:

$$\frac{d}{dx} y = \frac{(-0.4 + 0.016x)y}{(0.2 - 0.01y)x}$$

Analytical trajectory solution for the above set of parameters is obtained as:

$$y(x) = \frac{e^{\left(\text{LambertW} \left(-\frac{e^{\left(\frac{2}{25}x + \frac{2}{25}C \right)}}{20x^2} \right) + \frac{2}{25}x + \frac{2}{25}C \right)}}{x^2}$$

In the following sections, we plot the above analytical trajectories and the simulated numerical trajectories with different initial conditions and compare them.

Initial condition set 1

For initial values prey x=40 and predator y=15 analytic trajectory is:

$$y(x) = -20 \text{ LambertW} \left(-\frac{1200 e^{\left(\frac{2}{25}x - \frac{79}{20} \right)}}{x^2} \right)$$

The solution obtained above is for the principal branch of Lambert function. The principal branch only gives the lower part of the solution. By lower part we mean the part below the isocline y=20. For the above portion, Lambert's branch parameter k=-1 provides the solution. The complete closed analytic trajectory is union of principal and the upper (-1) branch as given in Figure 2. We should inform that Lamberts function continues to give values out of ~ (13 and 42) range, but they consist of imaginary parts.

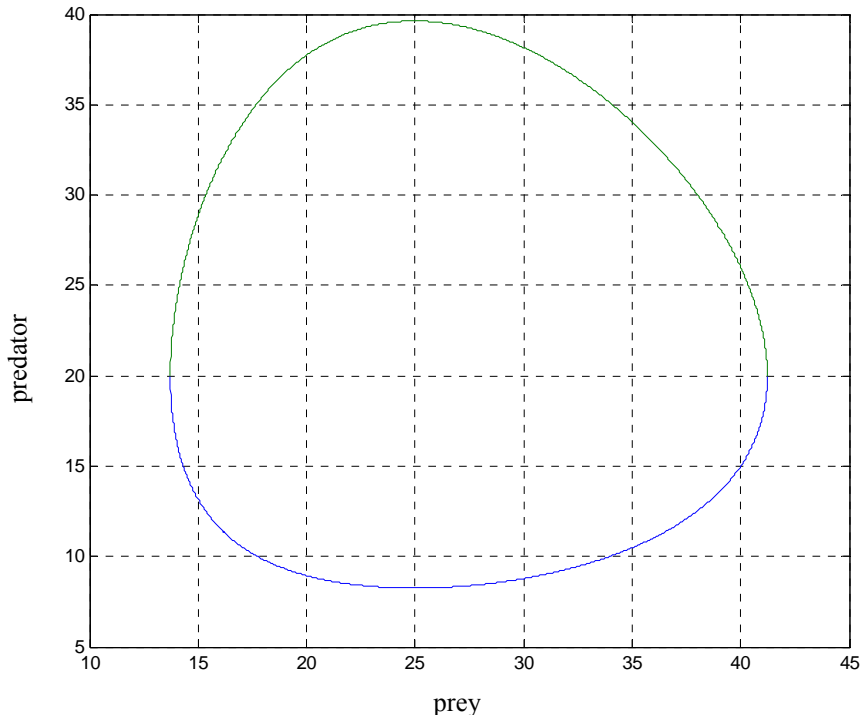


Figure 2 Plot of analytic trajectory with initial conditions prey x=40 and predator y=15 using MAPLE

The exact analytic trajectory shown in Figure 2 shows that the closed trajectories believed (by linearized analysis) to hold in close vicinity of the equilibrium point, is also valid globally, elsewhere in the phase plane.

After analytic trajectories, we next obtain numerical phase and field plots of the differential equation using Rosenbrock numerical algorithm of Maple (Figure 3). Finally we obtained Stella phase plots with different dt's ranging from 0.5 to 0.001 with Euler and R-K methods. (But we only show 0.5 and 0.1 dt valued results Figures 4-9). For dt value 0.5 only R-K4 is able to generate a an exact closed curve. Other methods such as Euler or R-K2 produce erroneous growing spirals or saw tooth patterns on a close curve. With dt 0.1 (or even smaller ones) Euler still yields growing spiral (Figure 4 and 5) but R-K2 yields a smoother closed curve (Figure 6 and 7).

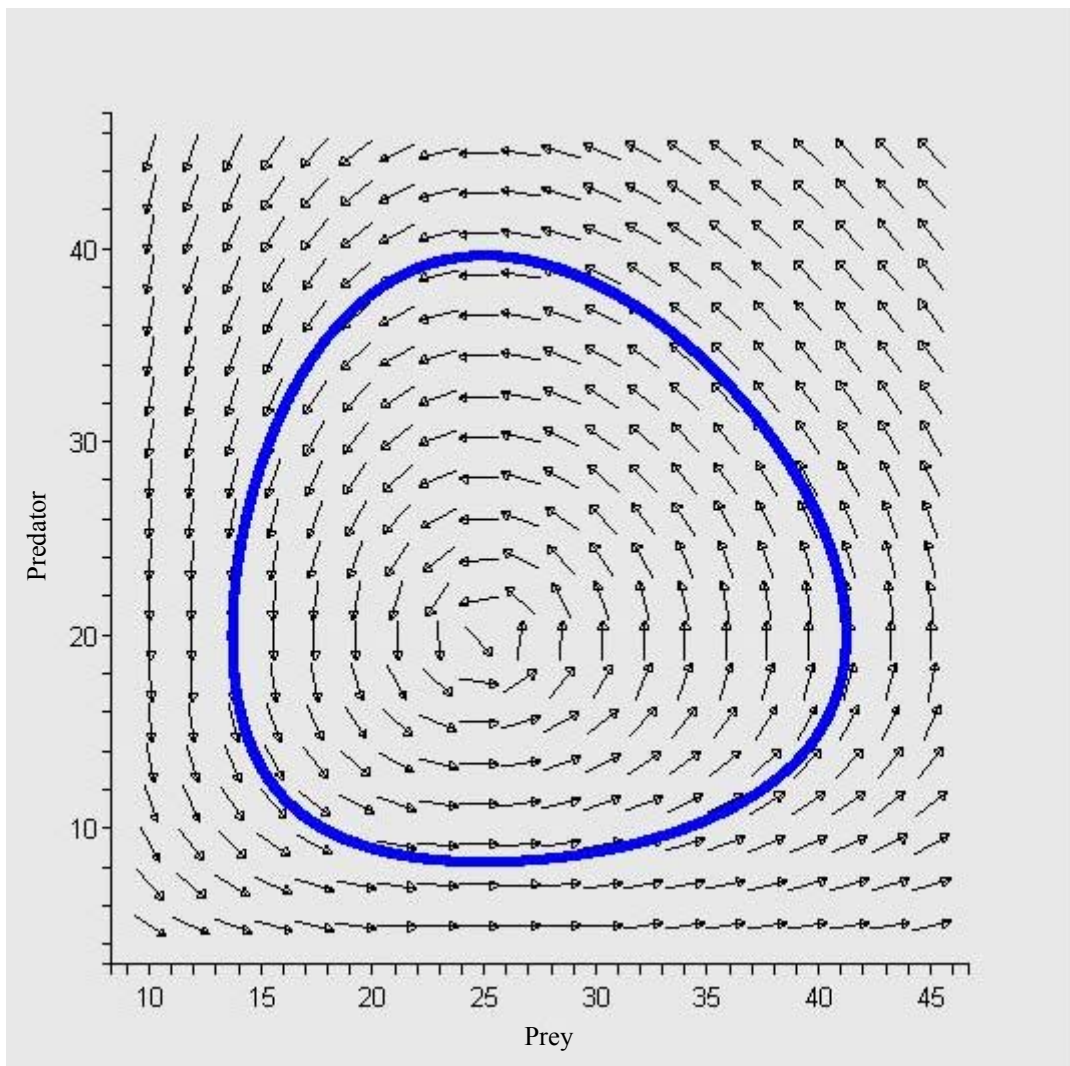
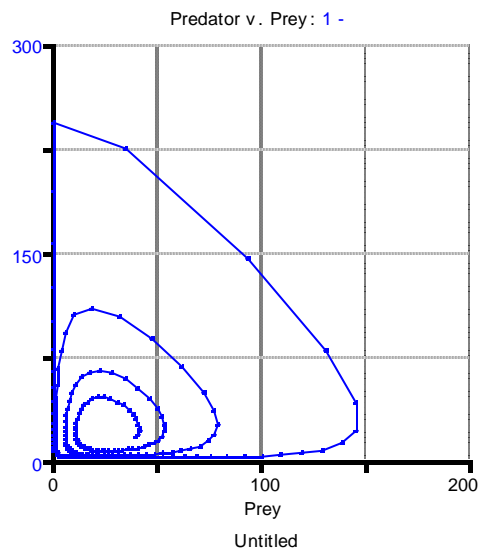


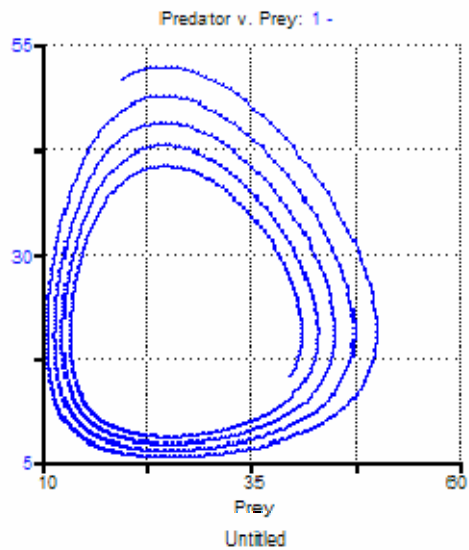
Figure 3 Plot of numerical trajectory with initial conditions prey=40 and predator=15, including field plot with Rosenbrock algorithm in MAPLE.



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Figure 4 Simulated phase plot with Euler $dt=0.5$ for initial conditions prey=40 & pred=15 with Stella software. (Incorrect trajectories).



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Figure 5 Simulated phase plot with Euler $dt=0.1$ for initial conditions prey=40 & pred=15 with Stella software. (Incorrect trajectories).

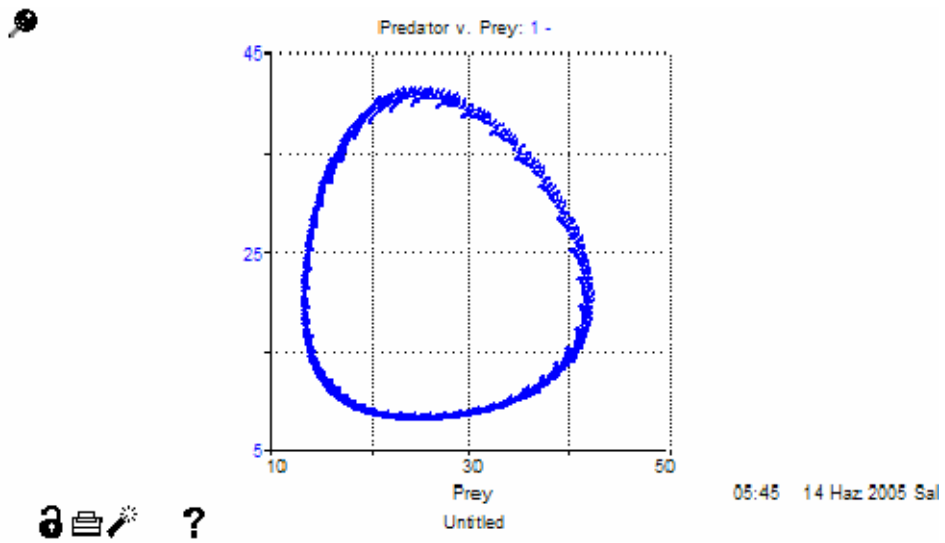


Figure 6 Simulated phase plot with R-K2 $dt = 0.5$ for initial conditions prey=40 & pred=15 with Stella software. (Incorrect trajectories).

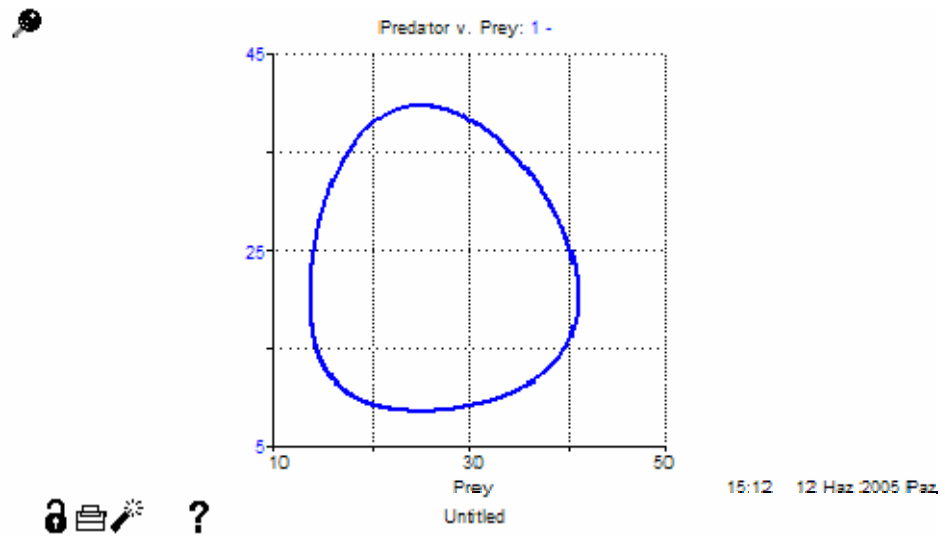


Figure 7 Simulated phase plot with R-K2 $dt=0.1$ for initial conditions prey=40 & pred=15 with Stella software.

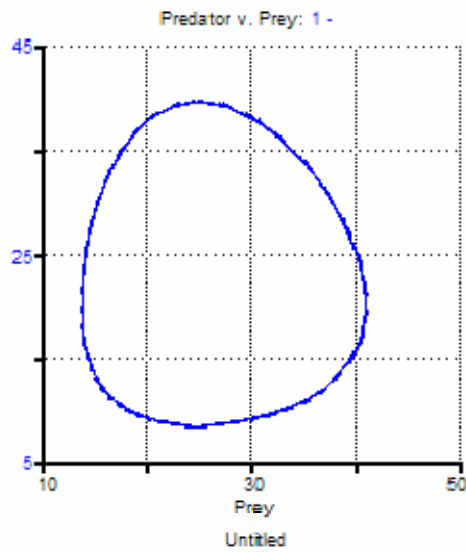


Figure 8 Simulated phase plot with R-K4 dt=0.5 for initial conditions prey=40 & pred=15 with Stella software

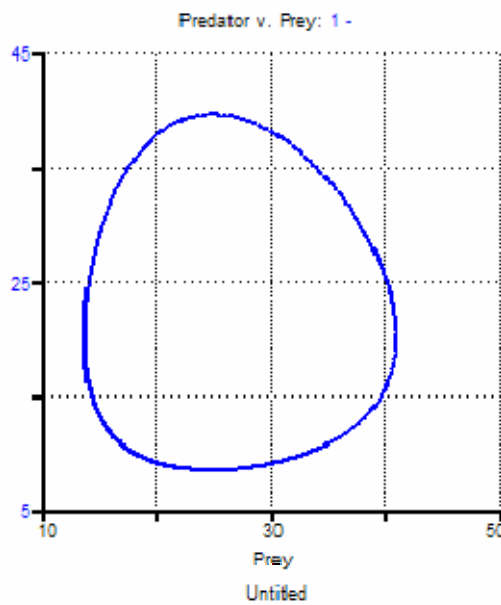


Figure 9 Simulated phase plot with R-K4 dt =0.1 for initial conditions prey=40 & pred=15 with Stella software.

Initial condition set 2:

For initial values prey x=80 and predator y=15 analytic trajectory is:

$$y(x) = -20 \text{ LambertW} \left(- \frac{4800 e^{\left(\frac{2}{25} x - \frac{143}{20} \right)}}{x^2} \right)$$

The solution obtained above is for the principal branch of Lambert function. The principal branch only gives the lower part of the solution. By lower part we mean the part below the isocline $y=20$. For the above portion, Lambert's branch parameter $k=-1$ provides the solution. The complete closed analytic trajectory is union of principal and the upper (-1) branch as given in Figure 10. We should inform that Lambert's function continues to give values out of $\sim (5 \text{ and } 82)$ range, but they consist of imaginary parts.

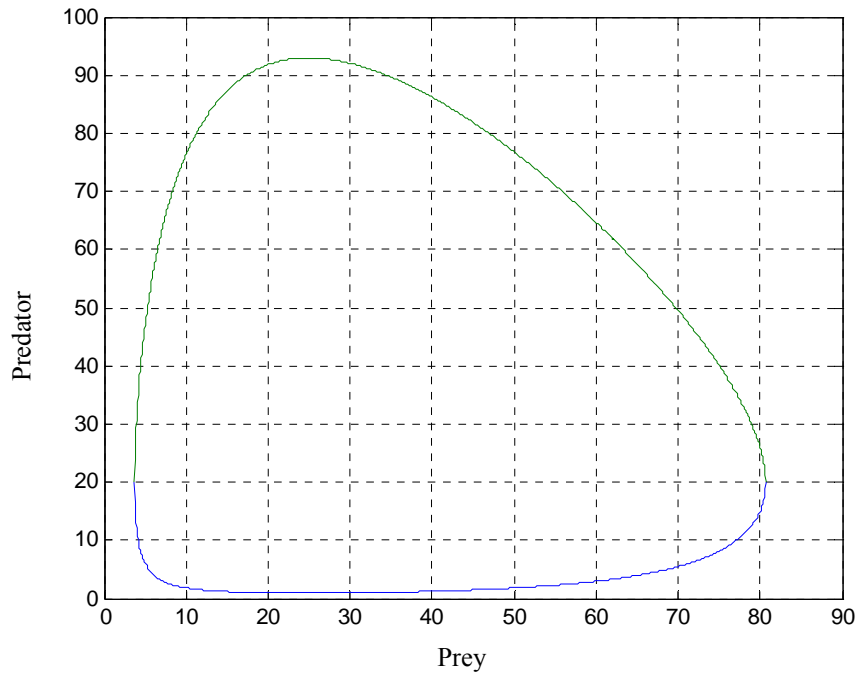


Figure 10 Plot of analytic trajectory with initial conditions prey $x=80$ and predator $y=15$ using MAPLE.

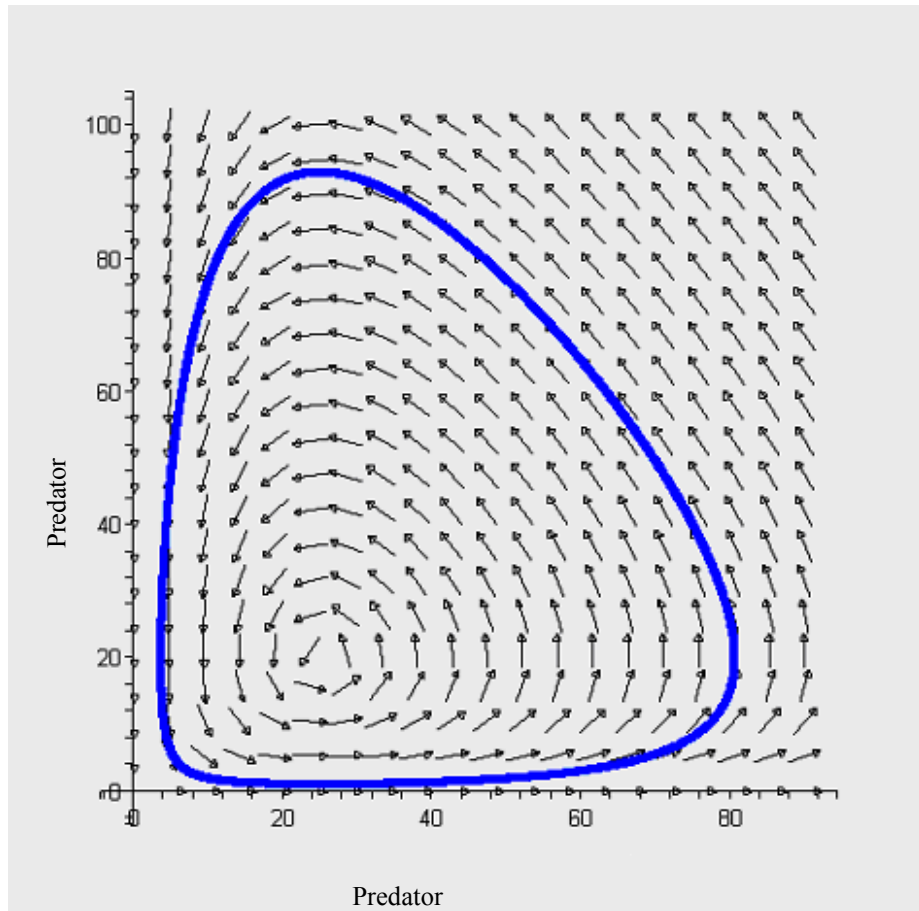
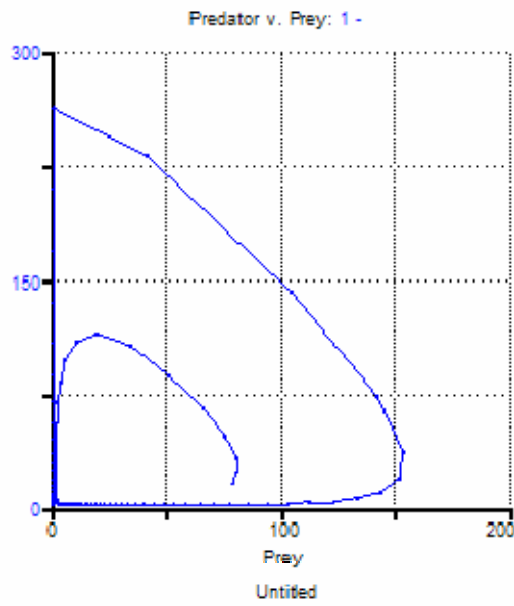


Figure 11 Plot of numerical trajectory with initial conditions prey $x=80$ and predator $y=15$ including field plot with Rosenbrock algorithm in MAPLE.

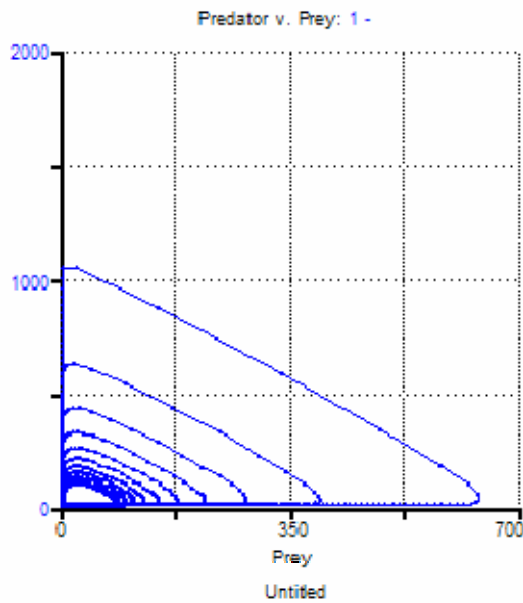
After analytic trajectories we obtain numerical phase and field plots of the differential equation using Rosenbrock algorithm as in Figure 11. Then we obtained Stella phase plots with different dt 's ranging from 0.5 to 0.001 with Euler and R-K methods. For dt value 0.5 Euler is not able to produce a meaningful result, since it hits the computer's numerical storage capacity, after a few iterations (Figure 12). As dt value is decreased, Euler method starts to produce some results, but even with 0.001 it spirals outward (Figures 13 and 14). Other methods like R-K2 have produced a very thick disc for $dt=0.5$ as in Figure 15. As dt gets smaller, the pattern becomes a closed curve like the theoretical one; see Figure 16. R-K4 with dt 0.5 produces a mild error in behavior with very thin disc (Figure 17). But with dt 0.1 and with smaller dt values, R-K4 algorithm yields a correct behavior (see Figure 18).



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Figure 12 Simulated phase plot with Euler $dt=0.5$ for initial conditions prey=80 & pred=15 with Stella software. (Incorrect trajectories)



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Figure 13 Simulated phase plot with Euler $dt=0.1$ for initial conditions prey=80 & pred=15 with Stella software. (Incorrect trajectories)

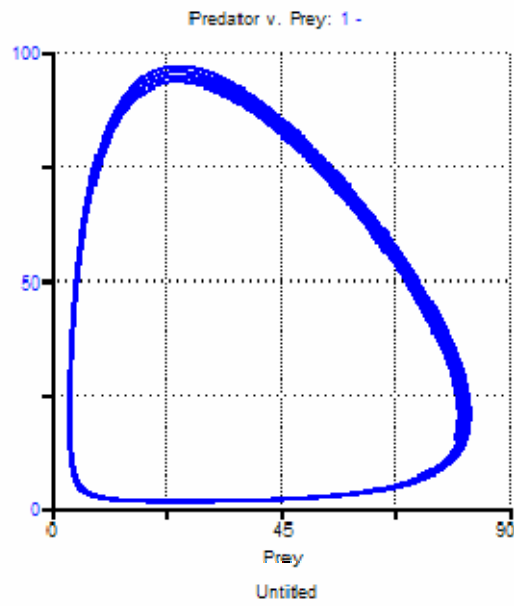


Figure 14 Simulated phase plot with Euler $dt=0.01$ for initial conditions prey=80 & pred=15 with Stella software. (Incorrect trajectories)

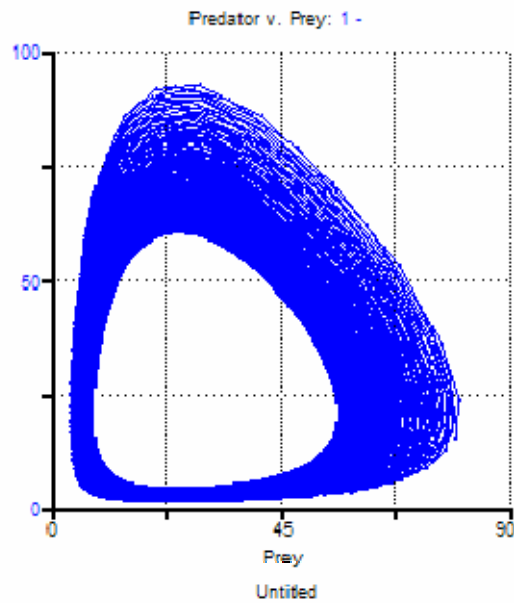


Figure 15 Simulated phase plot with R-K2 $dt=0.5$ for initial conditions prey=80 & pred=15 with Stella software. (Incorrect trajectories)

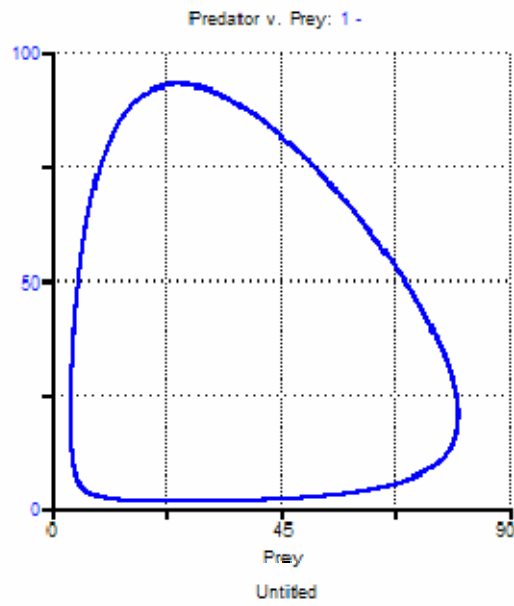


Figure 16 Simulated phase plot with R-K2 $dt=0.1$ for initial conditions prey=80 & pred=15 with Stella software.

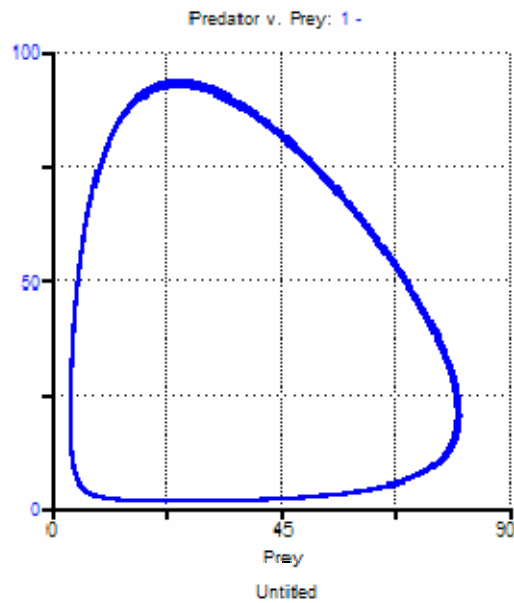


Figure 17 Simulated phase plot with R-K4 $dt=0.5$ for initial conditions prey=80 & pred=15 with Stella software.

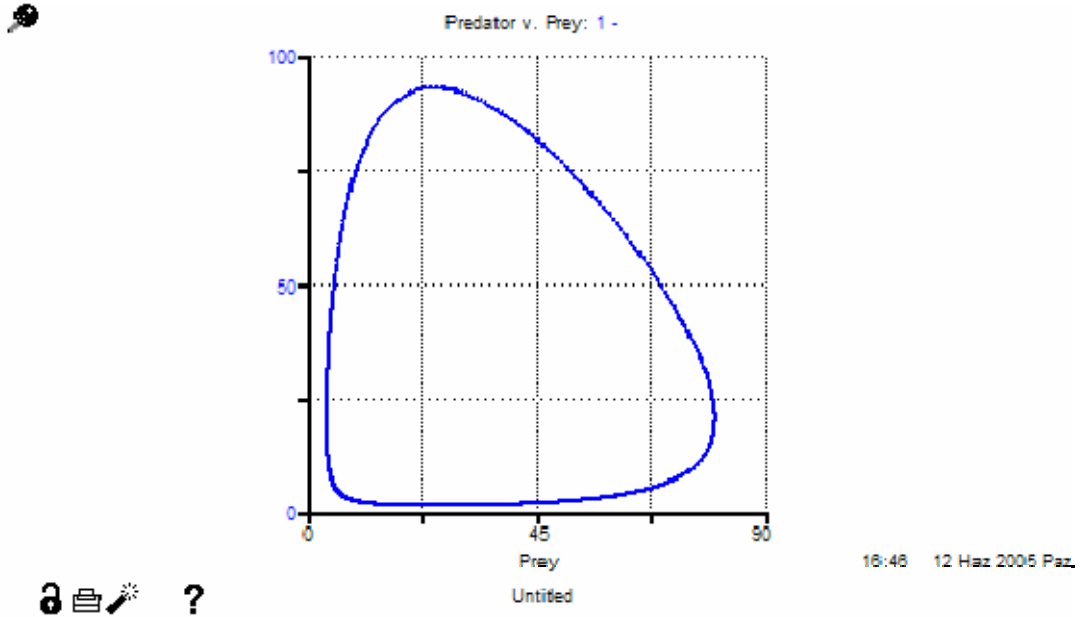


Figure 18 Simulated phase plot with R-K4 dt=0.1 for initial conditions prey=80 & pred=15 with Stella software.

Initial condition set 3:

For initial values prey x=1 and predator y=1 analytic trajectory is:

$$y(x) = \frac{e^{\left(-\text{LambertW}\left(-\frac{e^{\left(\frac{2}{25}x - \frac{13}{100} \right)}}{20x^2} \right) + \frac{2}{25}x - \frac{13}{100} \right)}}{x^2}$$

The solution obtained above is for the principal branch of Lambert function. The principal branch only gives the lower part of the solution. By lower part we mean the part below the isocline y=20. For the above portion, Lambert's branch parameter k=-1 provides the solution. The complete closed analytic trajectory is union of principal and the upper (-1) branch as given in Figure 19. We should inform that Lamberts function continues to give values out of ~ (0 and 155) range, but they consist of imaginary parts.

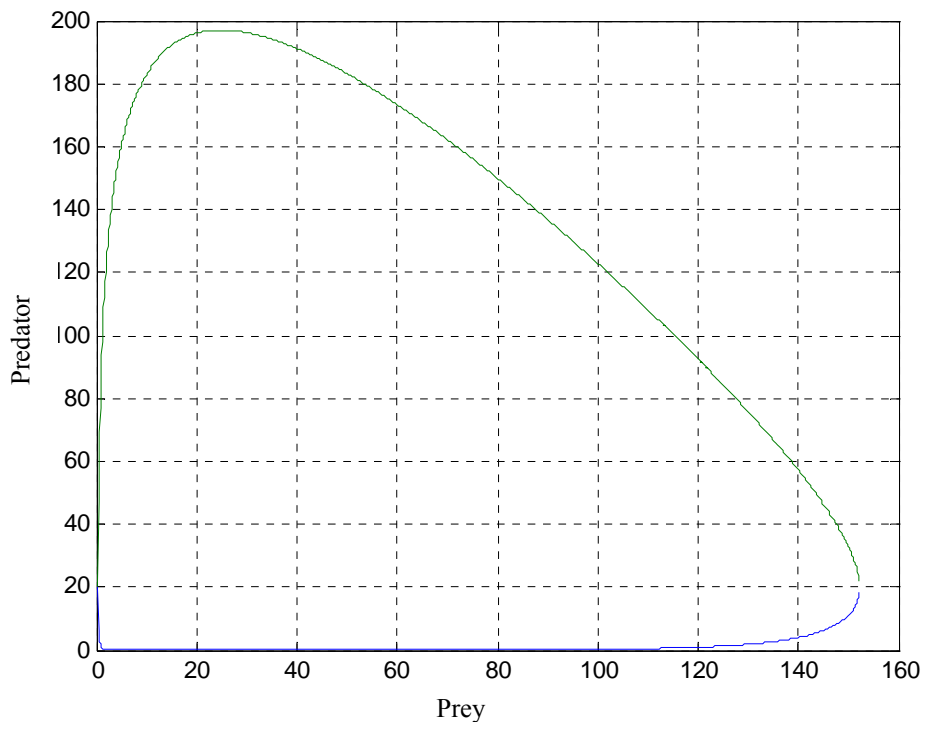


Figure 19 Plot of analytic trajectory with initial condition prey $x=1$ and predator $y=1$ using MAPLE.

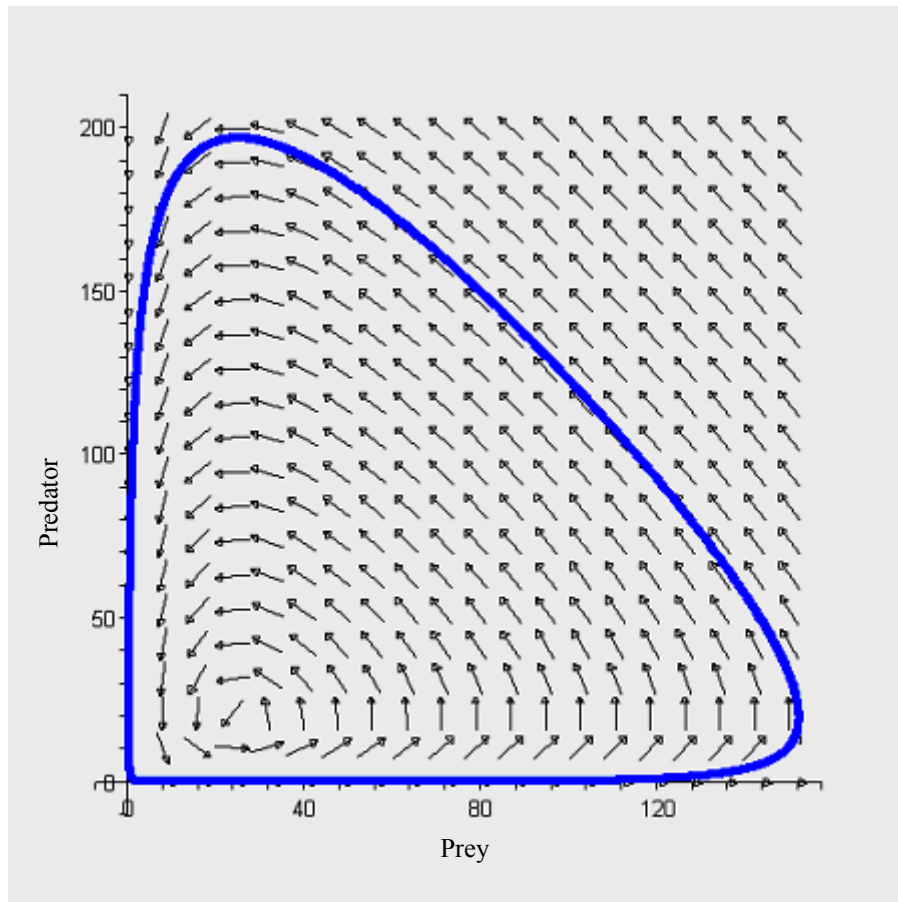


Figure 20 Plot of numerical trajectory with initial conditions prey $x=1$ and predator $y=1$ including field plot with Rosenbrock algorithm in MAPLE

After analytic trajectories we obtain numerical phase and field plots of the differential equation using Rosenbrock algorithm as in Figure 20. Then we obtain Stella phase plots with different dt 's ranging from 0.5 to 0.001 with Euler and R-K methods. (But we only give 0.5 and 0.01 dt valued results Figures 21-26). For dt value 0.5 Euler is not able to produce a meaningful result, since it hits the computer's numerical storage capacity, after a few iterations (Figure 21). As dt value is decreased, Euler method starts to produce some results, but even with 0.001 it spirals outward (Figures 22 and 23). Other methods like R-K2 have produced a very thick disc for $dt=0.5$ as in Figure 24. As dt gets smaller, the pattern becomes a closed curve like the theoretical one; see Figure 25. R-K4 with $dt=0.5$ produces a mild error in behavior with very thin disc (Figure 26). But with $dt=0.1$ and with smaller dt values, R-K4 algorithm yields a correct behavior.

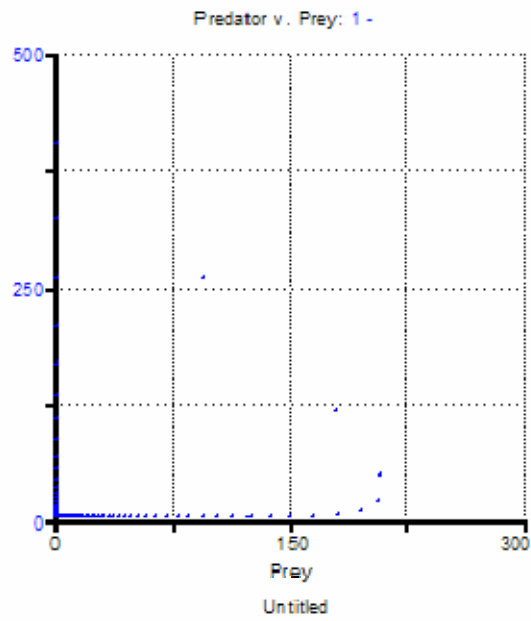


Figure 21 Simulated phase plot with Euler $dt=0.5$ for initial conditions $prey=1$ & $pred=1$ with Stella software. (Incorrect trajectories)

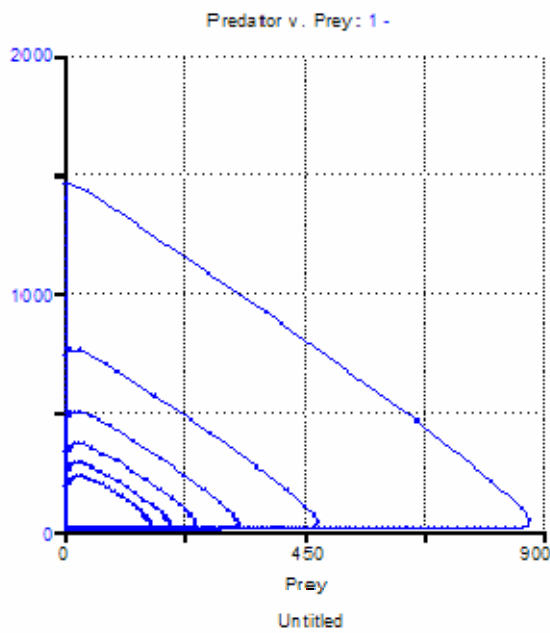


Figure 22 Simulated phase plot with Euler $dt=0.1$ for initial conditions $prey=1$ & $pred=1$ with Stella software. (Incorrect trajectories)

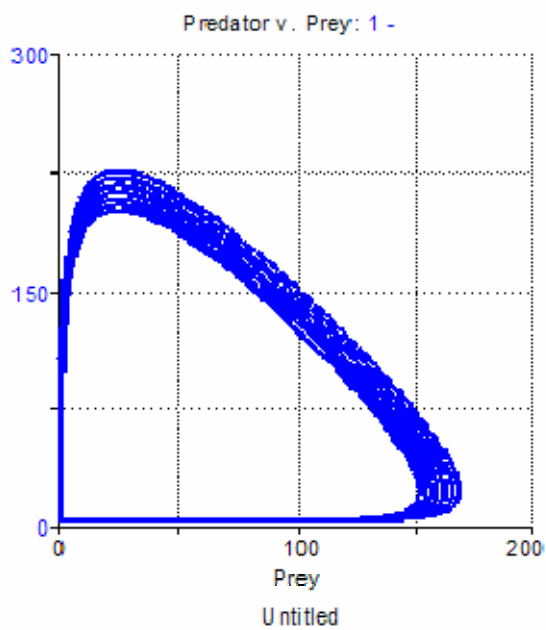


Figure 23 Simulated phase plot with Euler $dt=0.01$ for initial conditions prey=1 & pred=1 with Stella software. (Incorrect trajectories)

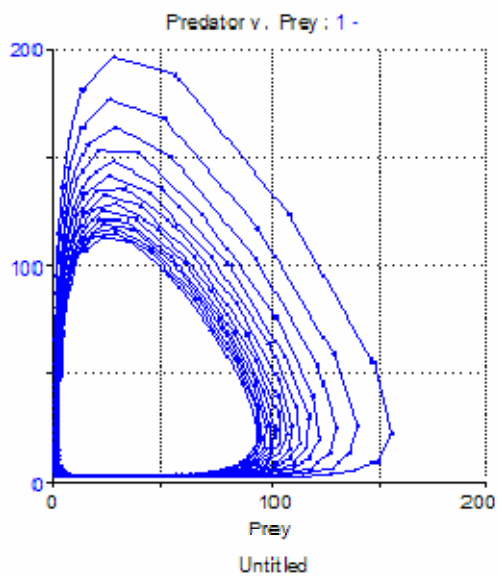


Figure 24 Simulated phase plot with R-K2 $dt=0.5$ for initial conditions prey=1 & pred=1 with Stella software. (Incorrect trajectories)

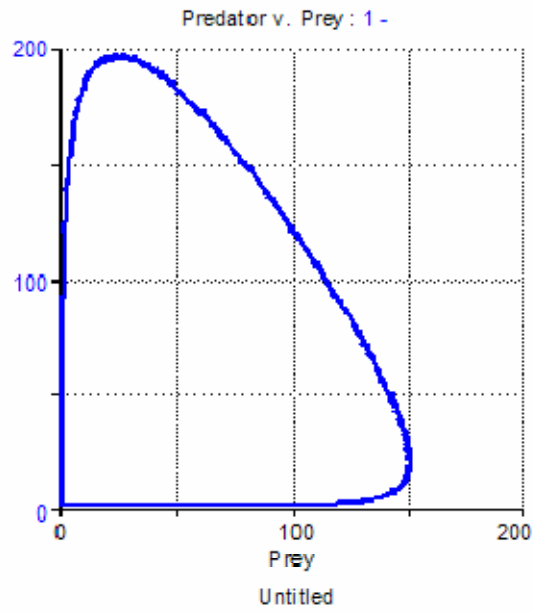


Figure 25 Simulated phase plot with R-K2 $dt=0.1$ for initial conditions prey=1 & pred=1 with Stella software.

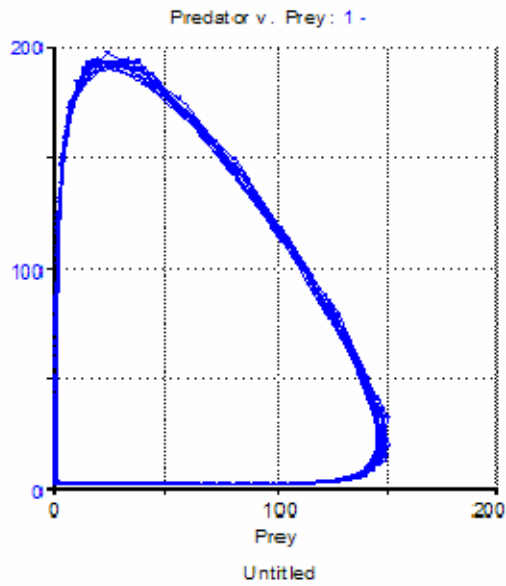


Figure 26 Simulated phase plot with R-K4 $dt=0.5$ for initial conditions prey=1 & pred=1 with Stella software

Competition Model:

The model is:

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = my - nxy$$

where x and y are two competing species and a,b,m,n are (+) constants representing birth, death and interaction rates respectively.

The equilibrium points of this model can be easily obtained to be: (0,0) and (m/n, a/b).

When the stability of these points are analyzed by linearization, it is found that both (0,0) and (m/n, a/b) are unstable. The question that we tackle is the exact shape of phase trajectories of the system, especially around the non-zero equilibrium (m/n, a/b).

Competition model in phase representation becomes:

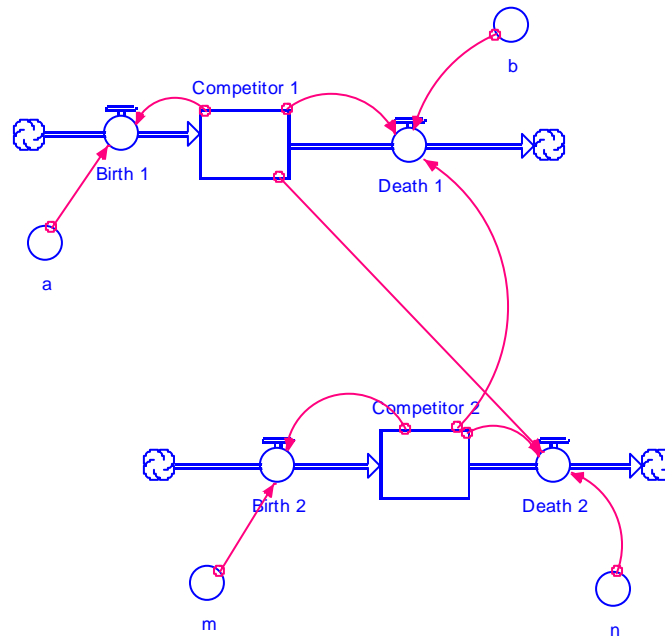
$$\frac{d}{dx}y = \frac{(m-nx)y}{(a-by)x}$$

General solution of phase trajectories obtained using Lambert's W function and Maple software is presented below:

$$y(x) = e^{\left(\frac{-nx + m \ln(x) - a \operatorname{LambertW}\left(-\frac{bx \left(\frac{m}{a}\right) e^{\left(-\frac{nx}{a} - \frac{C}{a}\right)}}{a} \right) - C}{a} \right)}$$

In the numerical solution of we were not able use Stella or other SD software since they can only work for very small ranges makes impossible for comparison needs. Instead we did not make comparison of SD and Maple plots.

The corresponding system dynamics stock-flow model is given below:



$$\begin{aligned} \text{Competitor}_1(t) &= \text{Competitor}_1(t - dt) + (\text{Birth}_1 - \text{Death}_1) * dt \\ \text{INIT Competitor}_1 &= 1 \\ \text{INFLOWS:} \\ \text{Birth}_1 &= \text{Competitor}_1 * a \\ \text{OUTFLOWS:} \\ \text{Death}_1 &= \text{Competitor}_2 * \text{Competitor}_1 * b \\ \text{Competitor}_2(t) &= \text{Competitor}_2(t - dt) + (\text{Birth}_2 - \text{Death}_2) * dt \end{aligned}$$

$$\begin{aligned} \text{INIT Competitor}_2 &= 1 \\ \text{INFLOWS:} \\ \text{Birth}_2 &= \text{Competitor}_2 * m \\ \text{OUTFLOWS:} \\ \text{Death}_2 &= \text{Competitor}_1 * \text{Competitor}_2 * n \\ a &= 0.2 \\ b &= 0.01 \\ m &= 0.4 \\ n &= 0.016 \end{aligned}$$

Figure 27 System Dynamics model and equations of the competition model

For initial values competitor1 $x=10$ and competitor2 $y=10$ analytic trajectory is obtained as:

$$y(x) = -20 \text{ LambertW} \left(-\frac{1}{200} x^2 e^{\left(-\frac{2}{25} x + \frac{3}{10} \right)} \right)$$

The solution obtained above is for the principal branch of Lambert function. The principal branch only gives the lower part of the solution. By lower part we mean the part below the isocline $y=20$. For the above portion, Lambert's branch parameter $k=-1$ provides the solution. The complete closed analytic trajectory is union of principal and the upper (-1) branch as given in Figure 28. We should inform that Lambert's function continues to give values out of $\sim (0 \text{ and } 13)$ range, but they consist of imaginary parts.

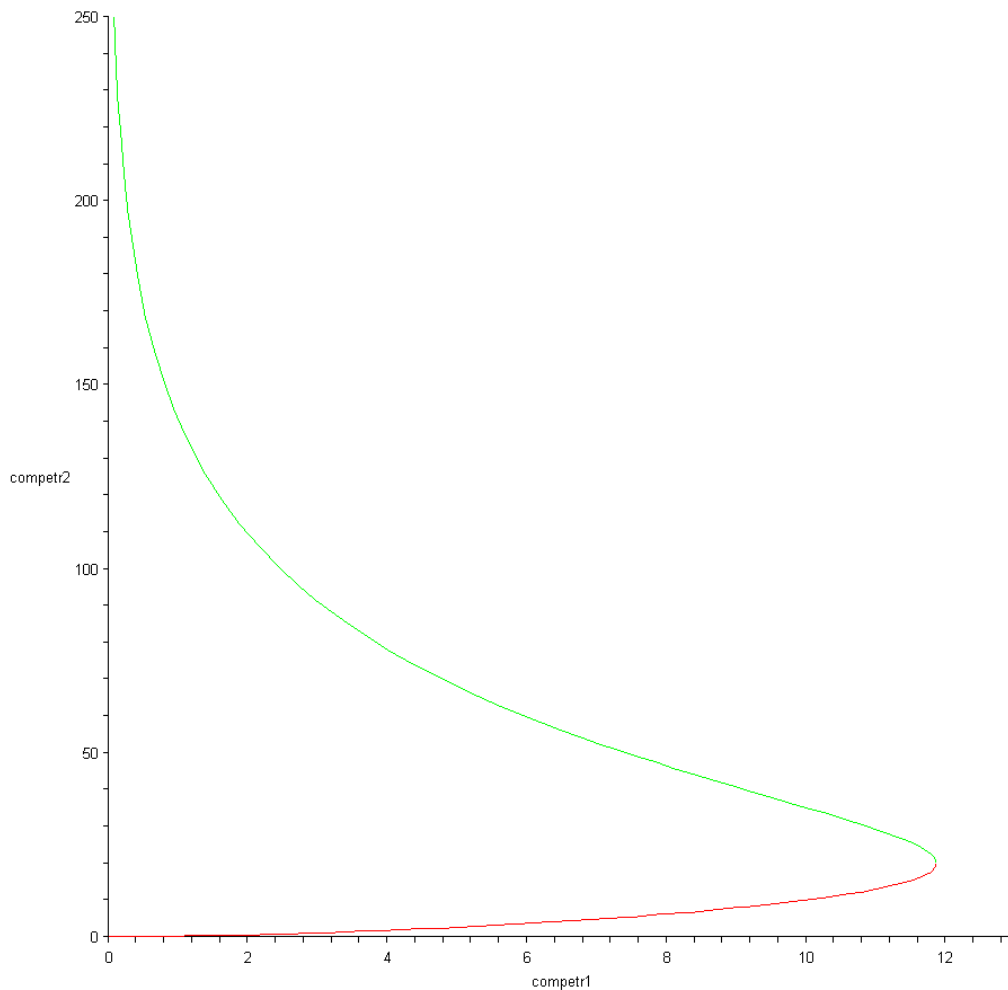


Figure 28 Plot of analytic trajectory, with initial conditions competitor1 $x=10$ and competitor2 $y=10$, using MAPLE.

For initial values competitor1 $x=30$ and competitor2 $y=10$ analytic trajectory is:

$$y(x) = -20 \operatorname{LambertW} \left(-\frac{1}{1800} x^2 e^{\left(-\frac{2}{25} x + \frac{19}{10} \right)} \right)$$

The solution obtained above is for the principal branch of Lambert function. Unlike other cases principal branch solution covers solution space of this case. The plot of analytical solution is given below in Figure 29.

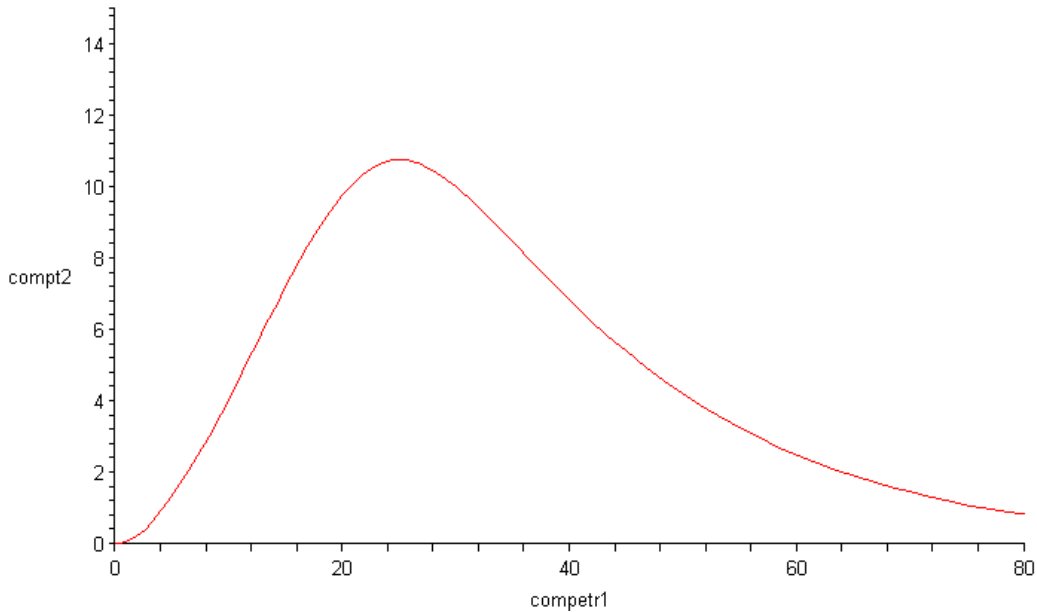


Figure 29 Plot of analytic trajectory with initial conditions competitor1 x=30 and competitor2 y=10 using MAPLE

For initial values competitor1=45 and competitor2=30 analytic trajectory is:

$$y(x) = -20 \operatorname{LambertW} \left(-1, -\frac{1}{1350} x^2 e^{\left(-\frac{2}{25} x + \frac{21}{10} \right)} \right)$$

The solution obtained above is for the principal branch of Lambert function. The principal branch only gives the lower part of the solution. By lower part we mean the part below the isocline $y=20$. For the above portion, Lambert's branch parameter $k=-1$ provides the solution. The complete closed analytic trajectory is union of principal and the upper (-1) branch as given in Figure 30. We should inform that Lamberts function continues to give values out of $\sim (45 \text{ and } \infty)$ range, but they consist of imaginary parts.

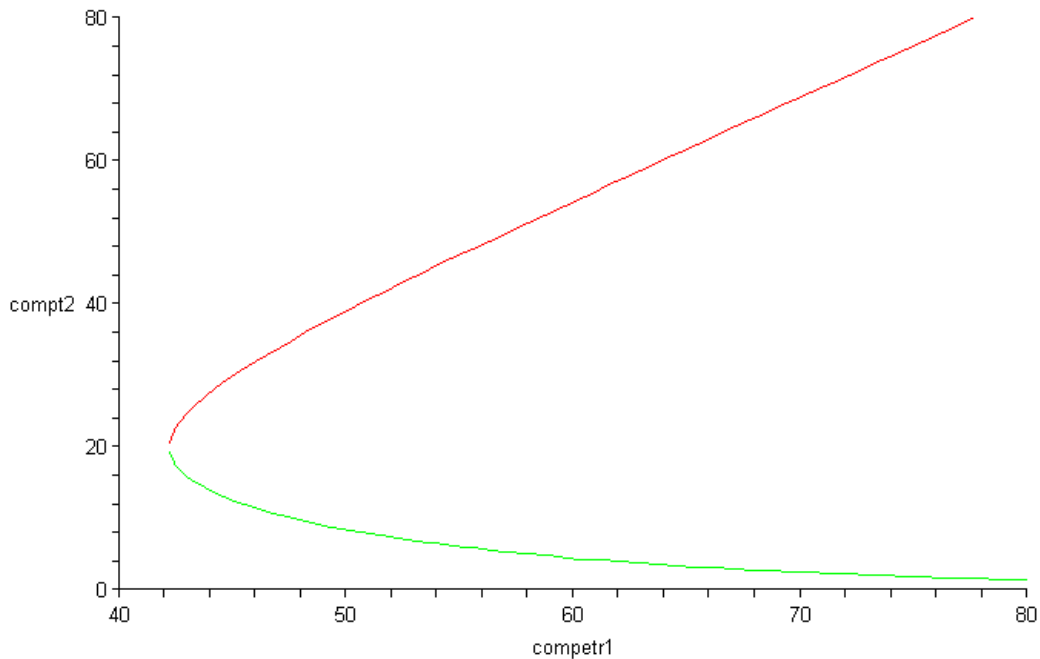


Figure 30 Plot of analytic trajectory with initial conditions competitor1 x=45 and competitor2 y=30 using MAPLE

For initial values competitor1 x=45 and competitor2 y=50 analytic trajectory is:

$$y(x) := -20 \operatorname{LambertW} \left(-1, -\frac{1}{200} x^2 e^{\left(-\frac{2}{25} x + \frac{3}{10} \right)} \right)$$

The solution obtained above is completely given by (-1) branch of Lambert function. The reader can see from above that the solution of the initial values competitor1=30 and competitor2=10 was a mirror image of this case (compare Figure 29 and 31); except that the principal (0) branch of the Lambert function was active in the previous case. A problem was that MAPLE was not able find the analytical solution with initial values competitor1=45 and competitor2=50, for some rason. But thanks to similarity between these two cases, we were able to deduce the analytic behavior in this last case with small number calculation and tests.. The resulting trajectory can be seen in Figure 31.

Finally, with this competition model, we were unable use effectively Stella or other SD software for numerical phase trajectory plotting. Such SD simulation software can only plot very short trajectory branches in each simulation, in very small ranges. This makes it near impossible to construct the complete phase trajectories out of these very small pieces. So, to test simulation results by comparison with analytical ones, we only used Maple's numerical solution results. These numerical trajectories are shown in Figure 32. Considering Figures 28, 29, 30 and 31 altogether, we observe that Maple's numerical solutions are in well agreement with analytical trajectories.

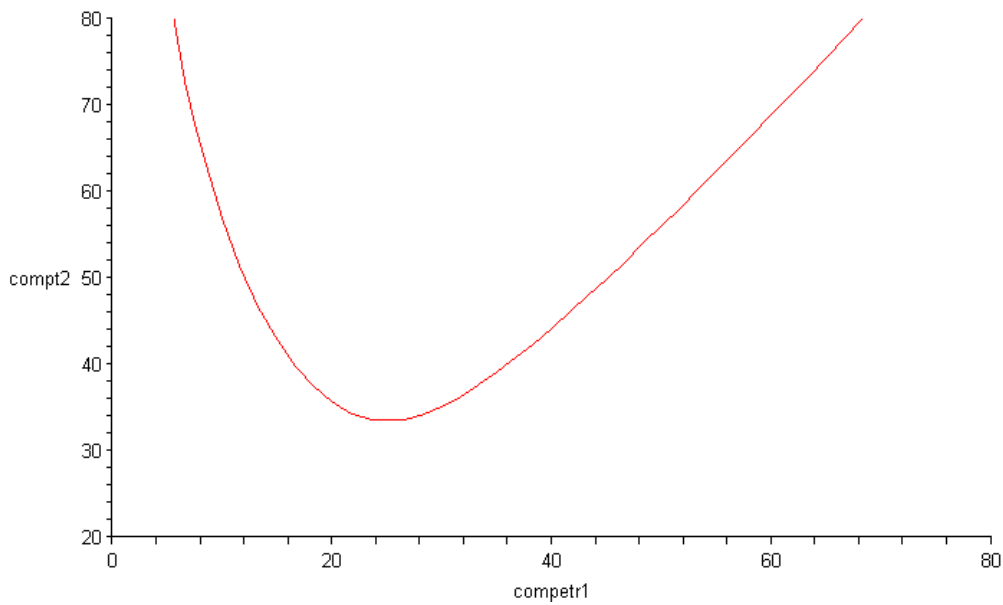


Figure 31 Plot of analytic trajectory with initial conditions competitor1 $x=45$ and competitor2 $y=50$ using MAPLE

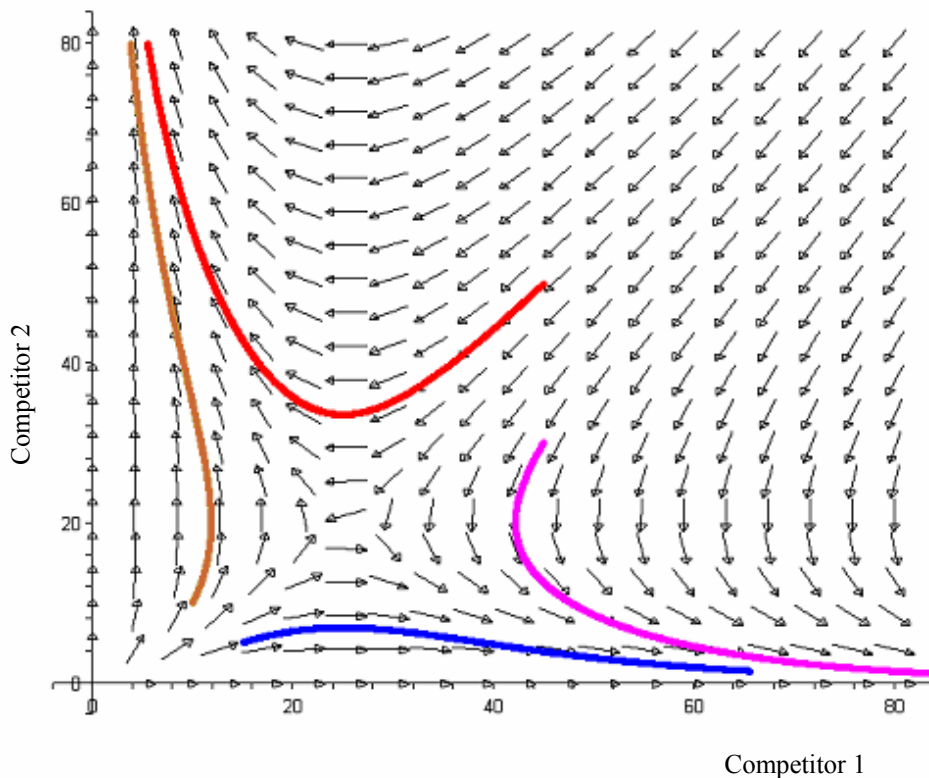


Figure 32 Plot of numeric trajectory with initial conditions competitor1 $x=45$ and competitor2 $y=50$; competitor1 $x=10$ and competitor2 $y=10$; competitor1 $x=45$ and competitor2 $y=10$; and competitor1 $x=45$ and competitor2 $y=3$, with Rosenbrock algorithm in MAPLE.

Conclusion

This study compares the analytical & numerical trajectories in phase plane so as to verify the correctness of simulated solutions of system dynamics models. The research is carried out on two example system dynamics models. The first one is a basic predator-prey model, a second-order non-linear stiff system with one stable and one unstable equilibrium point and closed-orbit phase trajectories. The second example is a basic model of competition between two species, also a second order nonlinear model that has two unstable equilibrium points. The phase plots in this case constitute unstable saddle trajectories. An important result is that for both of these non-linear models, it was possible to obtain exact phase trajectories analytically, by making use of Lambert's W functions, as suggested and implemented by Maple software. Numerical trajectories are obtained by Stella and Vensim simulation software. Phase trajectories are plotted with different representative initial conditions and different integration methods and in some cases, significant errors are discovered in numerical solutions, as compared to the exact analytical results. Simulations are also repeated with different time step (dt) values in order to see if and when the resulting phase trajectories start deviating qualitatively from the analytical ones. It is also shown that for stiff systems MAPLE software is highly reliable in numerical solution as well, thanks to the robust numerical algorithms included in the package. Another advantage of the MAPLE is the ease of creating phase plots. Thus, although analytical time trajectories are typically impossible to obtain for nonlinear models, analytical phase trajectories obtainable by Lambert's W function, Maple software and related tools offer promising research areas for system dynamics method.

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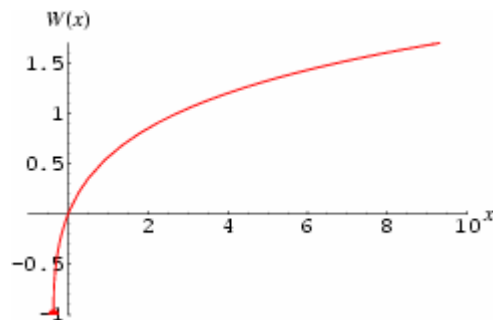
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Appendix -LambertW Function:



The LambertW function, also called the omega function, is the inverse function of

$$f(W) = We^W.$$

The plots above show the function along the real axis. The principal value of the LambertW-function is LambertW[z]. Different branches of the function are available as LambertW [k, z], where k is any integer and k = 0 corresponds to the principal value.

Lambert (1758) considered the solution to

$$x^a - x^b = (a - b)vx^{a+b},$$

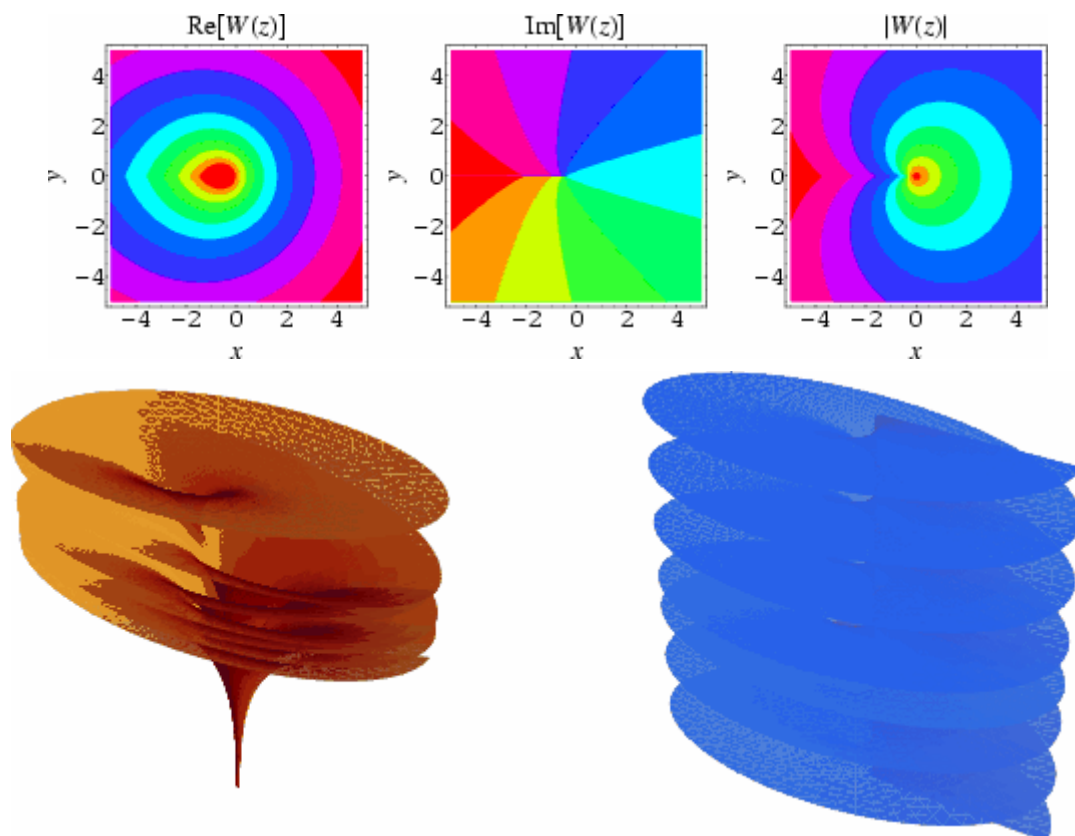
now known as Lambert's transcendental equation. Euler (1783) wrote a paper about Lambert's transcendental equation in which he introduced a special case which reduces to $wa^w = lx$, which is nearly the definition of $W(x)$, although Euler proposed defining a function more like $-W(-x)$. Euler considered series solutions in this paper and, in the first paragraph, explicitly quotes Lambert as the one who first considered this equation. Eisenstein (1844) considered the series of the infinite power tower

$$h(z) = z^{z^{\dots}},$$

which can be expressed in closed form as

$$h(z) = -\frac{W(-\ln z)}{\ln z}.$$

Applications have been discovered in statistical mechanics, quantum chemistry, combinatorics, enzyme kinetics, the physiology of vision, the engineering of thin films, hydrology, and the analysis of algorithms (Hayes 2005).



The LambertW-function is illustrated above in the complex plane.

$W(x)$ is real for $x \geq -1/e$. It has the special values

$$\begin{aligned} W(-\frac{1}{2} \pi) &= \frac{1}{2} i \pi \\ W(-e^{-1}) &= -1. \end{aligned}$$

$W(1) = 0.567143 \dots$ is called the omega constant and can be considered a sort of "golden ratio" of exponentials since

$$\exp[-W(1)] = W(1), \quad \ln \left[\frac{1}{W(1)} \right] = W(1).$$

The LambertW-Function has the series expansion

$$\begin{aligned} W(x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-2}}{(n-1)!} x^n \\ &= x - x^2 + \frac{3}{2} x^3 - \frac{8}{3} x^4 + \frac{125}{24} x^5 - \frac{54}{5} x^6 + \frac{16807}{720} x^7 + \dots \end{aligned}$$