# Local Forecasting of Chaotic Time Series

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## Prologue

Measured time series are usually the basis for characterizing a dynamical system. In practical cases, however, it is not possible to observe all relevant dynamical variables pertaining the system. The most common case is limited to a scalar time evolution of a variable for a finite duration of time. One of the most challenging endeavor is to predict the continuation of the time evolution. A finite-dimensional linear system produces a signal which is characterized by finite number of frequencies. Based upon this fact, either in frequency or in time domain there are methods for time series prediction. For nonlinear processes, however, these methods become inappropriate since a global model cannot be applied to the entire state space where the signal lives. Eckmann and Ruelle (Eckmann and Ruelle 1985) suggested first the idea of finding the relation between the delay coordinates of a point and the points appear some time later in the state space. A competition was also arranged to test the success of prediction algorithms proposed until 1993 (Gershenfeld and Weigend 1993). Among those registered for the competition, two methods prove to be the most successful (Sauer 1993; Wan 1993). One of which uses a connectionist neural network (Wan 1993) and the other utilizes the delay coordinate embedding based methodology (Sauer 1993) based upon the Eckmann-Ruelle proposition. More recently, wavelets (Parlitz and Mayer-Kress 1995) and genetic algorithms (Szpiro 1997) have also been suggested for nonlinear predictions.

Suppose that the time evolution of the system behavior is reconstructed in the state space (Abarbanel, Brown et al. 1993). The forecasting problem may then be formulated as follows: n different points on the attractor located in the state space are known. These points are P(1), P(2), ..., P(n). With respect to a fixed reference frame a point is represented by m numbers, m being the dimension of the state space. There are two questions: (i) Can one determine the point P(n+1), and (ii) if (i) is achieved, how far the consecutive points P(n+1), P(n+2), ..., P(n+n*) can be found, that is what is the maximum value for n*?

## Model

We first postulate that the point P(n+1) can be found by using the time evolution information of spatial neighbors of P(n) which are located within a certain cut off distance. We next presume that the succeeding point in time $P(k_i + 1)$ of each neighboring point $P(k_i)$ can be determined by a linear combination of the preceding points of $P(k_i + 1)$. These points are $P(k_i)$, $P(k_i - 1)$, $P(k_i - 2)$, ..., $P(k_i - d + 1)$, where d is referred to as the model dimension. Here i can take values from 1 to p where p is the number of neighbors that falls into the sphere whose center is at P(n) and radius is

the cut off distance r. This scheme introduces a set of unknown interpolation coefficients $C_1$, $C_2$, ..., $C_d$. Therefore, the same interpolation scheme can be constructed for each neighboring point $P(k_i)$ of P(n) with the same interpolation coefficients $C_1$, $C_2$, ..., $C_d$. This statement may be formulated as

$$C_1 P(k_i) + C_2 P(k_i - 1) + ... + C_d P(k_i - d + 1) = P(k_i + 1) \quad (1)$$

where $C_j = [\, c_{j1}\, c_{j2} \ldots c_{jm} \,]$. Eq. 1 contains p number of equations and d x m number of unknown coefficients $c_{jm}$, since j can take values from 1 to p and the dimension of the state space is m. However, m different sets of d x m coefficients and m different equation sets are needed for each component. Therefore, one can form a system of linear algebraic equations in the form of A u = b, where u denotes the column vector consisting of the interpolation coefficients $c_{jm}$. Here A and b contains the component values of the neighboring points. The dimension of A is (m x p) x (m x d x m) and the dimensions of u and b are (m x d x m) x 1 and (mxp) x 1, respectively. The linear system of interest usually turns out to be overdetermined. A solution can be found by the singular value decomposition technique. Were the system to be underdeterminate due to a few spatial neighbors satisfying (m x d x m) > (m x p), the minimum number of neighbors that makes the system determinate would be collected by automatically enlarging the cut off distance. The point P(n + 1) following the last point P(n) can then be calculated by the following equation

$$P(n + 1) = C_1 P(n) + C_2 P(n - 1) + ... + C_d P(n - d + 1) \quad (2)$$

The same scheme is followed to calculate P(n+2) that follows the last point which is now P(n+1).

We need to define an error between the predicted and the actual trajectory of the system so as to monitor the accuracy of the prediction model within the range of locality assumption. Root-mean-square (RMS) error, which is calculating the root mean square of the differences between the predicted and the actual data at all points, gives a single number about the error. Instead, we need the evolution of the error along the prediction horizon. This is referred to as moving RMS error. The error of this nature may be expressed as

$$e(i) = \left(1/\sqrt{l_w}\right) \sqrt{\sum_{k = i - l_w/2 + 1}^{k = i + l_w/2} [x(k) - \bar{x}(k)]^2} \quad (3)$$

where N is the number of predictions and $l_w$ is the window of the RMS error. Herein x(k) and $\bar{x}(k)$ are the actual and predicted data, respectively.

**A benchmark case**

Complex signals was commonly assumed to be the output of a complicated system with a large number of active degrees of freedom. However, realization of nonlinear systems with relatively

small number of degrees of freedom, while deterministic in principle, can create output signals that look complex and mimic stochastic signals, such as the Lorenz model (Lorenz 1963) which is described by the following set of ordinary differential equations

$$\dot{x} = -\sigma x + \sigma y \qquad \dot{y} = -xz + rx - y \qquad \dot{z} = -xy - bz \tag{4}$$

The parameters in Eq. 4 is most commonly selected to be $\sigma = 10$, $r = 28$, and $b = 8/3$. For generating the data, the Lorenz equations displayed in Eq. 4 are integrated for $3 \times 10^4$ time steps with the step size of $5 \times 10^{-3}$. The Bulirsch-Stoer method (Press, Teukolsky et al. 1992) is used to obtain the time evolution of the parameters x, y, and z whose starting values are $x(0) = y(0) = z(0) = 10$. The time-delayed embedding technique together with the false nearest neighbor technique is used to reconstruct the state space using the convection amplitude, x.

For prediction, different initial points are collected as input. Five different time series for x with different initial points, 16, 18, 20, 22, and $24 \times 10^3$ are prepared for prediction. Continuation of the amplitude data is predicted for $10^3$ time steps for each initial condition. Results are then compared with the actual continuation with each initial condition which is calculated by integrating the Lorenz equations with the Bulirsch-Stoer method further in time. The parameter set of the algorithm are identical for all five runs. The model dimension d, which represents the number of neighbors taken along the same trajectory of the point, is set to 3 and the radius of the sphere r, which controls the number of neighbors taken on the adjacent trajectories is set to 0.2. A representative comparison study is placed in Figure 1a. The moving RMS error obtained by averaging over five sets is displayed in Figure 1b. The results indicate that different starting points contribute to the accumulation of error differently. However, the averaged error demonstrates that the error start growing considerably after around 600 time steps, then fluctuates within an interval about 200 steps. The later steps however, diverge considerably from the actual continuation. The algorithm is coded in C. The calculations are performed on a SG Power Indigo workstation. A CPU time of 22s is sufficient for each data set.

**Epilogue**

Predictions follow the actual trajectory for all of the components for a certain time. The predicted results then starts diverging from the actual one. This is observed whenever an orbit approaches to an unstable fixed point of the system. Therein the sensitivity to the initial conditions is so critical that the error accumulated by the prediction algorithm also grows exponentially. The predicted trajectory gets away slightly from the true continuation first, and as it comes nearby the unstable fixed point once more, this slight difference results in a shift to a completely different trajectory. However, the predicted trajectory catches the actual continuation with a time lag, as the method utilizes the state space neighbors for predictions. Though the success of the forecasting is

comparable with those of Sauer and Wan (Sauer 1993; Wan 1993), the method presented herein is much simpler and faster.

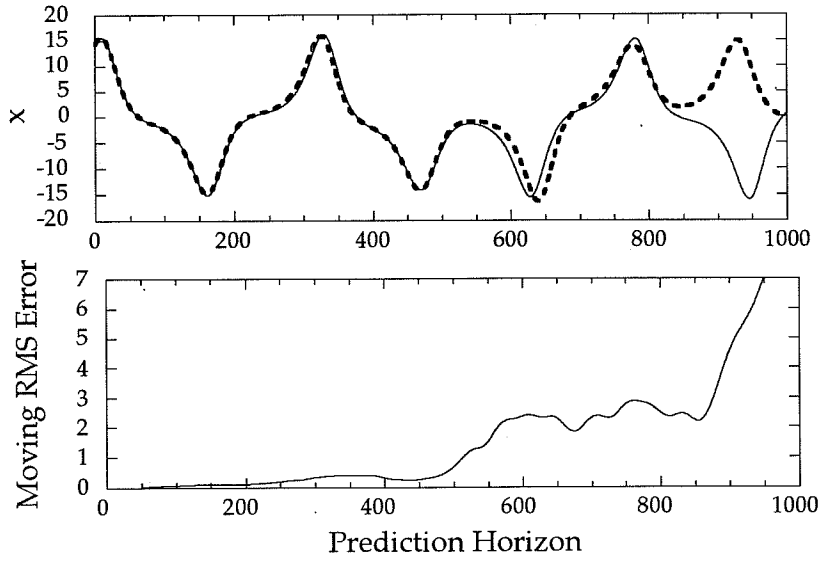


Figure 1. Time series prediction for the displacement x from the Lorenz attractor. (a) In the top figure, solid curve and dashed curve are the actual and the predicted time series, respectively. (b) In the bottom figure, the moving RMS error of the prediction is displayed.

**References**


Abarbanel, H. D. I., R. Brown, et al. 1993. The Analysis of Observed Chaotic Data in Physical Systems. *Reviews of Modern Physics* **65**(4): 1331-1392.

Eckmann, J.-P. and D. Ruelle. 1985. Ergodic Theory of Chaos and Strange Attractors. *Reviews of Modern Physics* **57**: 617-656.

Gershenfeld, N. A. and A. S. Weigend. 1993. The Future of Time Series: Learning and Understanding. *Time Series Prediction: Forecasting the Future and Understanding the Past.* Addison-Wesley. 1-70.

Lorenz, E. 1963. Deterministic Non-periodic Flow. *Journal of Atmospherical Sciences* **20**: 130-141.

Parlitz, U. and G. Mayer-Kress. 1995. *Physical Review E* **51**: 2709.

Press, W. H., S. A. Teukolsky, et al. 1992. *Numerical Recipes.* Cambridge University Press.

Sauer, T. 1993. Time Series Prediction by Using Delay Coordinate Embedding. *Time Series Prediction: Forecasting the Future and Understanding the Past.* Addison-Wesley. 241-260.

Szpiro, G. G. 1997. Forecasting Chaotic Time Series with Genetic Algorithms. *Physical Review E* **55**: 2557-2568.

Wan, E. A. 1993. Time Series Prediction by Using a Connectionist Network with Internal Delay Lines. *Time Series Prediction: Forecasting the Future and Understanding the Past.* Addison-Wesley. 195-217.