Continuous Models and Discrete Time Series Data

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Abstract

We consider cases where reality is best described by a continuous model, and where data are sampled at discrete points in time. Then an exact transformation of the continuous model into a discrete one, or vice versa, is typically very complicated. Simplified transformations might produce great errors if the sampling interval for the time series is approaching natural periods or time constants of the system being modelled. For such problematic cases we discuss implications for system dynamics, traditional discrete model econometrics, and Bayesian statistical methods.

1. Introduction

The transformation of a continuous time model into a discrete one, or the transformation of a discrete time model into a continuous one, is normally very complicated. However, such transformations typically have to be made, somehow, for the purpose of parameter estimation or model testing using either system dynamics or traditional statistical procedures:

- System dynamics (and other methods using continuous simulation models) often make use of parameter estimates that have to be transformed from discrete time econometric models.
- Traditional discrete time econometric models typically rely on continuous time economic theory and restrictions on parameters.
- Bayesian statistical methods using discrete time observations typically rely on continuous time theory and prior information about parameters.

The purpose of this paper is to indicate what problems might arise when using the three methods without making proper transformations. We discuss implications for practical work and ways to reduce the problems. The paper relies on theory developed after the mid 1960's, see Bergstrom (1976) for a collection of central papers. Our main contribution is to supply a business-cycle example, a phenomenon frequently studied in economics and in system dynamics, and to compare three different approximative transformations.

2. When are transformations needed?

The transformations above are only needed if reality is best represented by a continuous model, while the time series data are discrete. Throughout this paper we assume that this is the case; a choice that also seems to represent most cases, see discussions in Koopmans (1950), Forrester (1961) (p.64-65), in Gandolfo (1981) (p.4-7) who categorizes main arguments and give numerous references, and in Richardson (1991).

Transformations are avoided in system dynamics models if the continuous model is based solely on prior information, or if continuous models are used directly in parameter calibration. However, if one wants to use parameter estimates from discrete time models, or want to test relationships in a traditional statistical fashion (make inference), transformations seems to be needed, see Gandolfo (1981) (p.66). Continuous discrete Kalman filtering might offer an alternative, see e.g. Jazwinsky (1970) or Maybeck (1982).

For discrete time econometric models, the continuous time model is the medium through which economic theory and a priori restrictions on parameters are introduced. If discrete time models do not make use of this type of prior information, transformations are not needed. However, the lack of comprehensive time series data in social systems typically imply that a priori restrictions are needed to save degrees of freedom. Furthermore, since numerous models can be fit to historical time series, direct observations of decision making behaviour is needed to discriminate between theories. In this case transformations are needed.

3. The exact transformation of a linear system

Assume that reality can be described by the following linear, continuous differential equation:

$$\frac{dx}{dt} = Ax + Bu \tag{1}$$

where A and B are matrices containing the model parameters, **x** is a vector of state variables, **u** represents exogenous inputs, and we have ignored random disturbances. The solution for **x**(t) is given by:

$$x(t) = e^{At}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{2}$$

We now let x_t represent the observation of **x**(t) at time t, and we use equation 2 to predict x_{t+1} from the previous measurement x_t. To ease the exposition we will not only assume that **u**(t) is integrable but also constant between measurement times. The time interval between measurements is denoted by δ and I denotes the identity matrix.

$$x_{t+1} = e^{A\delta}x_t - A^{-1}(I - e^{A\delta})Bu_t \tag{3}$$

Setting the matrices $\Phi = e^{A\delta}$ and $\Delta = -A^{-1}(I - \Phi)B$, the equation reads:

$$x_{t+1} = \Phi x_t + \Delta u_t \tag{4}$$

To better see the relationship between the continuous model in equation 1 and the discrete model in equations 3 and 4, we write the matrix $e^{A\delta}$ by its Taylor expansion:

$$\Phi = e^{A\delta} = I + A\delta + \frac{1}{2}A^2\delta^2 + \cdots \tag{5}$$

From equation 5 we see that each element ϕ_{ij} of the matrix Φ in principle is a complicated function of all parameters a_{ij} of the matrix A. This is why the transformation between continuous models and discrete ones is so complicated.

Equation 5 also indicates that the problem vanishes if δ tends towards zero. Then Φ is simply given by $I+A\delta$. For what values of δ this approximation is acceptable, depends on the eigenvalues of A. This is most easily illustrated by the one-dimensional case where $a_{11}=-1/T$; a first order negative feedback loop with a time constant T. For this case equation 5 can be written:

$$\phi_{11} = 1 - \left(\frac{\delta}{T}\right) + \frac{1}{2}\left(\frac{\delta}{T}\right)^2 + \cdots \tag{6}$$

In this case we see that δ must be sufficiently shorter than the time constant T for the simple first order approximation to be acceptable. More advanced approximations will depend on higher orders of δ (or δ/T) than 2, see Gandolfo (1981).

4. A continuous time business cycle model.

To illustrate we use a simple version of a business-cycle model proposed by Metzler (1941) and elaborated on by Mass (1975), Forrester (1976), and Forrester (1982).

Aggregate inventories i increases by production q and are depleted by shipments s of consumer and investment goods. Shipments s are exogenous.

$$\frac{di}{dt} = q - s \tag{7}$$

Desired production is set equal to shipments s plus inventory adjustments $(as-i)/\tau_i$, where desired inventory is the product of shipments s and normal inventory coverage a. The strength of the adjustment is given by the inventory adjustment time τ_i.

$$\frac{dq}{dt} = (s + (as - i)/\tau_i - q)/\tau_q \tag{8}$$

Production follows desired production after an adjustment delay τ_q. This delay combines time used to perceive changes in inventories and shipments, to adjust plans, to hire or fire workers, to train workers, and to organize overtime. Since we are dealing with an aggregate model, this time delay will also capture delays incurred by intra-industry deliveries.

Written in the format of equation 1 the business cycle model can be written:

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\tau_i \tau_q} & -\frac{1}{\tau_q} \end{bmatrix} \begin{bmatrix} i \\ q \end{bmatrix} + \begin{bmatrix} -1 \\ \frac{1}{\tau_q}(1+\frac{a}{\tau_i}) \end{bmatrix} s \tag{9}$$

The eigenvalues[1] for the model shows that it produces cycles when $\tau_i < 4\tau_q$. The periodicity is given by $T=2\pi\sqrt{\tau_i \tau_q}$. Parameter values of $\tau_i = 0.5$ year and $\tau_q = 1.0$ year, give a period of 4.44 years, which is a quite typical length between business cycle peaks.

1 Eigenvalues are: $-\frac{1}{2\tau_q} \pm \frac{1}{2}\sqrt{\frac{1}{\tau_q^2} - \frac{4}{\tau_i \tau_q}}$

5. An exact discrete time version of the b-c model

Equations 3 and 4 shows the general expression for the discrete time model. Below we have derived explicit expressions for the parameters ϕ_{ij} and Δ_j for the business cycle model. :

$$\varphi_{11} = \frac{e^{-\frac{\delta}{2\tau_q}}}{\tau_q\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}}\sin\left(\frac{1}{2}\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}\,\delta\right) + e^{-\frac{\delta}{2\tau_q}}\cos\left(\frac{1}{2}\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}\,\delta\right)$$

$$\varphi_{12} = \frac{2e^{-\frac{\delta}{2\tau_q}}}{\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}}\sin\left(\frac{1}{2}\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}\,\delta\right)$$

$$\varphi_{21} = \frac{-\frac{2}{\tau_i\tau_q}e^{-\frac{\delta}{2\tau_q}}}{\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}}\sin\left(\frac{1}{2}\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}\,\delta\right)$$

$$\varphi_{22} = \frac{-e^{-\frac{\delta}{2\tau_q}}}{\tau_q\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}}\sin\left(\frac{1}{2}\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}\,\delta\right) + e^{-\frac{\delta}{2\tau_q}}\cos\left(\frac{1}{2}\sqrt{-\frac{1}{\tau_q^2}+\frac{4}{\tau_i\tau_q}}\,\delta\right)$$

$$\Delta_1 = \tau_i(\varphi_{11}-1) + \tau_i\tau_q\varphi_{21} - (\tau_i\varphi_{12} + \tau_i\tau_q(\varphi_{22}-1))\left(\frac{1}{\tau_q}+\frac{\alpha}{\tau_i\tau_q}\right)$$

$$\Delta_2 = -(\varphi_{11}-1) + \varphi_{12}\left(\frac{1}{\tau_q}+\frac{\alpha}{\tau_i\tau_q}\right)$$

Obviously, even the transformation of a second order model is very complicated. Using an exact transformation, it is very difficult to test theory from the continuous model, and to utilize prior information about parameters. The parameters from the continuous model enters the discrete model in a highly non-linear fashion.

6. Simplified discrete time versions of the b-c model.

We consider three simplifications of the transformation to a discrete business cycle model. We begin with a first order Taylor expansion $\Phi = I + A\delta$ and enter the parameters for the business cycle model into equation 4. This is the first order Euler method usually used for simulation in Dynamo and Stella. The general equation for a linear model is:

$$x_{t+1} = x_t + \delta(Ax_t + Bu_t) \tag{10}$$

The business cycle model reads:

$$\begin{bmatrix} i_{t+1} \\ q_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ -\frac{\delta}{\tau_i \tau_q} & 1-\frac{\delta}{\tau_q} \end{bmatrix} \begin{bmatrix} i_t \\ q_t \end{bmatrix} + \begin{bmatrix} -\delta \\ \frac{\delta}{\tau_q}(1+\frac{a}{\tau_i}) \end{bmatrix} s_t \tag{11}$$

A second simplification is to make an exact transformation of the continuous model equation by equation.

$$x_{i,t+1} = e^{a_{ii}\delta} x_{i,t} - a_{ii}^{-1}(1 - e^{a_{ii}\delta})(\sum_{j \neq i} a_{ij} x_{j,t} + \sum_k b_k u_{k,t}) \tag{12}$$

We transform the equation for di/dt treating the variable q as an exogenous variable, and the equation for dq/dt treating i as an exogenous variable. As with the true exogenous variable s, we assume that q and i are constant between the discrete time points.

$$\begin{bmatrix} i_{t+1} \\ q_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & \delta \\ -\frac{(1-e^{-\delta/\tau_q})}{\tau_i} & e^{-\delta/\tau_q} \end{bmatrix} \begin{bmatrix} i_t \\ q_t \end{bmatrix} + \begin{bmatrix} -\delta \\ (1-e^{-\delta/\tau_q})(1+\frac{a}{\tau_i}) \end{bmatrix} s_t \tag{13}$$

The equation by equation transformation is the philosophy usually followed in practical econometrics. The equation for i_t maintains the definition of inventory change (the time step δ is typically set equal to 1 per definition). The equation for q_t represents a linearized Koyck lag, where $e^{-\delta/\tau_q}$ is the lag factor. Actually, the Euler method is also an equation by equation transformation, although simplified with a lag factor equal to $-\delta/\tau_q$.

The third simplification of the transformation takes notice of the difference between rates (flows) and levels (states, stocks or instantaneous variables). The discrete equation is written as:

$$x_{t+1} = x_t + \delta(A\tfrac{1}{2}(x_{t+1} + x_t) + Bu_t) \tag{14}$$

for the case when u(t) is constant between measurement times, see e.g. Gandolfo (1981) p.85. Compared to the first order Euler approximation, $\mathbf{x_t}$ on the right hand side is replaced by the average value $\frac{1}{2}(x_{t+1} + x_t)$ between time points t and t+1. This makes sense because rates are functions of levels, and change as the levels change. The average rate over a time period gives a better prediction of next period's level than the rate at the beginning of the period. Note that the third method introduces simultaneity in that $\mathbf{x}_{t+1}$ appears on the right hand side of the equation. Econometric methods tackle this simultaneity directly[2]. For our comparison of methods we solve for $\mathbf{x}_{t+1}$ and insert parameters from the business cycle model:

$$\mathbf{x}_{t+1}=(I - \frac{\delta}{2}A)^{-1}\{ (I + \frac{\delta}{2}A)\mathbf{x}_t + \mathbf{B}\mathbf{u}_t\} \tag{15}$$

which gives:

$$\begin{bmatrix} i_{t+1} \\ q_{t+1} \end{bmatrix} = \frac{1}{D}\begin{bmatrix} 1+\frac{\delta/2}{\tau_q}-\frac{(\delta/2)^2}{\tau_i\tau_q} & \delta \\ -\frac{\delta}{\tau_i\tau_q} & 1-\frac{\delta/2}{\tau_q}-\frac{(\delta/2)^2}{\tau_i\tau_q} \end{bmatrix}\begin{bmatrix} i_t \\ q_t \end{bmatrix} + \frac{1}{D}\begin{bmatrix} -1-\frac{a\delta/2}{\tau_i\tau_q} \\ \frac{(\delta/2)^2}{\tau_i\tau_q}+\frac{1}{\tau_q}(1+\frac{a}{\tau_i}) \end{bmatrix} s_t$$

$$\text{where } D=1+\frac{\delta/2}{\tau_q}+\frac{(\delta/2)^2}{\tau_i\tau_q} \tag{16}$$

To summarize, in all four transformations we have assumed that exogenous variables $\mathbf{u}(t)$ are constant between discrete time points. For the Euler and the Equation by equation methods, we have assumed that also endogenous rates are constant between discrete time point; they are treated as exogenous variables. The Exact transformation fully accounts for rates being functions of levels, while the Rate approximation picks up the first order effect of levels on rates.

Note that with the exception of exogenous variables $\mathbf{u}(t)$, we have assumed that all level variables are measured at the discrete time points, e.g. inventories in our business cycle model could be reported at year-ends. We will stick to this assumption throughout the paper. However, we acknowledge that there are important exceptions. Level variables might be reported by their average values over the measurement interval. In particular this is likely to be the case for level variables that appear directly as rates in other parts of the model. Production in the business cycle model is an example (in this paper we assume that it is measured instantaneously). Gandolfo (1981) shows how the mixed case with both level and rate variables on the left-hand side (instantaneous and flow measurements) can be treated by integrating the variables over the measurement interval. This treatment complicates the Rate approximation method because error terms will no longer be serially uncorrelated. Gandolfo also describes a transformation to avoid this correlation.

7. Numerical assessment of the simplifications

Before we enter numbers into the equations above, we repeat the main purposes of the transformations. Either theory and prior information about parameters are transformed from a continuous model into a discrete one, or parameter estimates from discrete models are transformed to be used in continuous models. If we assume that the continuous model is an exact representation of reality, we note that unbiased estimators yield the same discrete model parameters as the Exact transformation.

We use the same parameter values as earlier in the paper: τ_i=0.5, τ_q=1.0, and a=0.2, which imply business-cycles with a period of 4.44 years. The correct discrete model parameters, ϕ_{ij} and Δ_j, using the Exact transformation are compared to the corresponding parameters for the simplified methods.

2 E.g. by using a three-stage least squares. The error terms will be serially uncorrelated only if all the variables are measured at the same point in time, see Gandolfo (1981).

First we compare parameter values for the case when yearly data on inventories and production are available, δ=1, see table 1. Deviations from the correct parameters are considerable. Of the three alternatives, the Rate adjustment is clearly the best. Rate adjustment yields parameters which on average differ from the correct parameters by 34 percent (ϕ_{22} is not included due to wrong sign). Correspondingly, the Euler method gives an average deviation of 129 percent and the Equation by equation method leads to a 115 percent average deviation.

Table 1: Parameter values of discrete models, δ=1.

	ϕ11	ϕ12	ϕ21	ϕ22	Δ1	Δ2
Exact	0,371	0,444	-0,889	-0,073	-0,319	1,251
Euler	1,000	1,000	-2,000	0,000	-1,000	1,400
Equation by eqn.	1,000	1,000	-1,264	0,368	-1,000	0,885
Rate adjustment	0,500	0,500	-1,000	0,000	-0,600	0,950

Next, we compare parameters for the case with quarterly data, δ=0.25, see table 2. Deviations from correct values are much smaller than in the case with yearly data. Still the Rate adjustment gives the best results with an average deviation of 4.3 percent. Deviations for the Euler and the Equation by equation methods are respectively 10.8 and 11.1 percent.

Table 2: Parameter values of discrete models, δ=0.25.

	ϕ11	ϕ12	ϕ21	ϕ22	Δ1	Δ2
Exact	0,943	0,217	-0,433	0,726	-0,205	0,360
Euler	1,000	0,250	-0,500	0,750	-0,250	0,350
Equation by eqn.	1,000	0,250	-0,442	0,779	-0,250	0,310
Rate adjustment	0,946	0,216	-0,432	0,730	-0,227	0,309

For monthly data, δ=1/12, average deviations become 1.5 percent for the Rate adjustment, and 3.0 percent for the other two. Finally, we note that as δ tends towards zero, all parameters converge towards the correct values, as expected.

8. Discussion and conclusions

The starting point for this paper has been cases where reality is best described by a continuous model, and where data are sampled at discrete points in time. Then an exact transformation of the continuous model into a discrete one, or vice versa, is typically very complicated. Simplified procedures produce great errors if the period between data points is approaching natural periods or time constants of the system being modelled. For such problematic cases we discuss implications for system dynamics, traditional discrete model econometrics, and Bayesian statistical methods.

It has been argued that system dynamics models would benefit from more extensive use of formal statistical methods. The above results introduce an important qualification to this claim. Parameter estimates obtained by a discrete version of a continuous simulation model, could introduce serious biases in the simulation model if an approximative transformation is used. Take the estimation of τ_q from the parameter ϕ_{22} as an example. Assume first that ϕ_{22} has been estimated correctly with no bias. For the case with yearly data, the Equation by equation method deems the negative parameter estimate implausible, while the Euler method gives an estimate of τ_q equal to 0.93, only 7 percent below the correct value. For the case with quarterly data, the Equation by equation method give an estimate of τ_q of 0.78, while the Euler method gives 0.91. Corresponding biases are found for estimates of τ_i based on ϕ_{21} and estimates of τ_q. For yearly data, the Euler method gives an estimate of τ_i of 1.21, 142 percent above the correct value. For quarterly data, both methods gives an estimate of τ_i of 0.63, 26 percent above the correct value.

To conclude: The transformation biases, even for quarterly data, together with likely estimation biases (e.g. short sample biases), are likely to be of comparable size to typical mistakes made when dealing with prior data. Thus, using simplified transformations and ordinary statistical methods, it is not obvious that greater reliance on formal econometric methods would benefit system dynamics models. Tables 1 and 2 indicate that the Rate adjustment will do better.

Traditional discrete time econometric models are argued to benefit from economic theory and restrictions on parameters formulated in continuous models. The above results also introduce an important qualification to this claim.

An example is provided by the inventory equation. Both the Euler and the Equation by equation methods use the inventory expression as it is in the continuous model, with the exception that the rates of change (q_t and s_t) are multiplied by the time interval δ. No parameter is left to be estimated. This introduces serious biases in the discrete model. In the case with yearly data, the three parameters are over-estimated by on average 163 percent. With quarterly data, the three parameters are over-estimated by on average 14 percent. Tables 1 and 2 indicate that the Rate adjustment will reduce the biases.

One obvious response to the transformation problem would be to play down the importance of theory, to ignore restrictions on parameters, and "let the data speak for themselves". This can be done by the use of black-box models, e.g. ARMA models, or by for instance a methodology developed at the London School of Economics, see e.g. Henry and Richard (1982) and Henry and Richard (1983). Maybe transformation biases explain why black-box models at times out-perform structural models.

According to Simon (1984), it would be a step in the wrong direction to play down the importance of prior information: "It is not likely that important new facts can be obtained by applying sophisticated statistical techniques to aggregate time series. The residual fluctuations in the data are mostly below the level of random noise". (p.51). "The - - strategy for economics is obvious: to secure new kinds of data at the micro level, data that will provide direct evidence about the behavior of economic agents and the ways in which they go about making their decisions." (p.40). This leads to Bayesian methods where not only restrictions on parameters are used, but also prior estimates.

The above results also introduce an important qualification to the use of Bayesian statistics. Using simplified transformations to enter prior information in discrete time models introduces biases. The above example with biases in τ_q and τ_i serves to examplify. Again, the Rate adjustment will serve to reduce biases.

The transformation biases could also be used to argue for stronger reliance on prior estimates and for formal statistical testing of prior information. Better prior information would reduce the need for formal testing of theories against time series data (one might know a priori that a parameter is significantly different from zero)[3]. Similarly the transformation biases could be used to argue for continuous model calibration or possibly continuous-discrete Kalman filtering, and structual and behavioural tests of the types proposed by Forrester and Senge (1980) and Zellner (1981).

3 In fact there exist studies where prior information by itself explains historical behaviour. Meadows (1970) has developed a continuous commodity cycle model. When he enters prior information about lifetimes, gestation periods, offspring per litter etc. for chicken, pigs, and cattle, the model produces cycles with periods very close to historical observations (2.5, 4 and 15 years).

References

Bergstrom, A.R. (1976). *Statistical Inference in Continuous Time Economic Models.* Contributions to Economic Analysis. Amsterdam: North-Holland.

Forrester, J. and Senge, P. (1980). "Tests for building confidence in system dynamics models." *TIMS Studies in the Management Sciences.* **14**: 209-28.

Forrester, J.W. (1961). *Industrial Dynamics.* Cambridge: MIT Press.

Forrester, J.W. (1976). "Business Structure, Economic Cycles, and National Policy." *Futures.* (June 1976): 195-214.

Forrester, N.B. (1982). "A Dynamic Synthesis of Basic Macroeconomic Theory: Implications for Stabilization Policy Analysis." Paper No. D-3384, System Dynamics Group, Sloan School of Management, MIT, Cambridge.

Gandolfo, G. (1981). *Qualitative Analysis and Econometric Estimation of Continuous Time Dynamic Models.* Contributions to Economic Analysis. Amsterdam: North-Holland.

Henry, D.F. and Richard, J.F. (1982). "On the Formulation of Empirical Models in Dynamic Econometrics." *Journal of Econometrics* **20**: 3-33.

Henry, D.F. and Richard, J.F. (1983). "The Econometric Analysis of Economic Time Series." *International Statistical Review* **51**: 111-163.

Jazwinsky, A.H. (1970). *Stochastic Processes and Filtering Theories.* Academic Press.

Koopmans, T.C. (1950). "Models involving a continuous time variable." In Koopmans, T.C., ed. *Statistical inference in dynamic economic models,*. New York: Wiley.

Mass, N.J. (1975). *Economic cycles: An analysis of underlying causes.* Cambridge, Mass.: Wright-Allen Press.

Maybeck, P.S. (1982). *Stochastic Models Estimation and Control.* Academic Press.

Meadows, D.L. (1970). *Dynamics of commodity production cycles.* Cambridge, Mass.: Wright- Allen Press.

Metzler, L.A. (1941). "The nature and stability of inventory cycles." *Review of Economics and Statistics.* **23**(3): 113-29.

Richardson, G. (1991). *Feedback Thought in Social Science and Systems Theory.* Philadelphia: University of Pensylvania Press.

Simon, H.A. (1984). "On the Behavioral and Rational Foundations of Economic Dynamics." *Journal of Economic Behavior and Organization.* **5**: 35-55.

Zellner, A. (1981). "Philosopy and Objectives of Econometrics." In Currie, D., et al., ed. *Macroeconomic Analysis: Essays in Macroeconomics and Econometrics.* England: Croom Helm.