# The Irregular Disturbance of Control Policy in a Dynamic System

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ABSTRACT

The control policies of a dynamic system are manifest in the rates of change of state variables or levels. Irregular variation in a control will introduce random delay of the flow into the related level. Of particular interest are exogenous rapid disturbances to the rate constants of feedback loops. These disturbances lead to small random delays which build up as the system develops in time. A statistical description of this process can be obtained for a simple feedback loop as a function of the spectrum of the time series of exogenous disturbances. An understanding of this behaviour is useful for modelling stochastic systems.

## INTRODUCTION

In modelling a dynamic system an attempt is made to represent the feedback and control structure of a real system and to simulate possible trajectories of the system's future state. Irregular behaviour is an essential feature of many dynamic systems. Certain systems are known to generate their own chaos even though they are governed by deterministic rate equations (see Lorentz, 1963, for early work and Prigogine, 1980, for a general discussion). These are systems where the rates of change of state variables are functions of interactions between the state variables. Such systems map out pseudo-cyclical trajectories over surfaces of fractional dimension in the system phase space. But random behaviour is important also in more traditionally based causal models. These may be deterministic, in which case the state trajectory can be viewed as an indicator of average behaviour, or noise can be deliberately introduced to represent exogenous or unexplained influences and so to generate samples of random trajectories. The adjustment of the Wharton Model of the US economy in the light of disturbances to the real economy are described by Klein (Klein, 1984) and serve to illustrate the remarks made above.

An area of interest to the present author is the short period fluctuation of time series (Wheeler, 1984) and the present work is concerned with short period disturbances to the control policy of a dynamic system. The control policy dictates the rates of change of state variables or levels. Of particular interest are feedback loops in which exogenous disturbances to the rate constants (delays) will be compounded such that even weak disturbances may build up noticeable random delays. The objective of this work is to obtain a statistical description of the effect, which random variations in the rate constant of a simple feedback loop have, upon the associated state variable or level. A further objective is to suggest a representation for the effect in the context of a computer simulation.

The necessity for such a representation arises because the rate equations of a simulation model are often macro-scale approximations to micro-scale stochastic processes. As an example, the adoption curve of a new product can be modelled by a smooth sigmoid (see Stone, 1980, for details). Whereas the adoption process is actually stochastic and on a micro-timescale the rate of adoption fluctuates about a longer term trend. Examples of such macro-scale approximations are to be found in work on System Dynamics modelling (Randers, 1980). Sanatani (Sanatani, 1981) gives a recent example of a product adoption application. Although the effect of noise has been investigated, for example in industrial models (Forrester, 1961), a simulation can only represent variations with periods which are at least twice the timestep for computation. So it is necessary to investigate the relationship between the spectrum of short period disturbances to a rate constant and the future value of the resulting level.

## STATISTICAL BACKGROUND

The following model represents a non-stationary time series, $\nu(t)$, which reduces to a stationary series after taking first differences of lagged values;

$$\nu(t) = \nu_0 + \dot{\nu}_0 t + \varepsilon(t) \ . \tag{1}$$

In (1), $\varepsilon(t)$ is a random variable whose expected value is zero:

$$< \varepsilon(t) > = 0 \ ,$$

and the derivative $d\varepsilon(t)/dt$ is a stationary random variable. The expected value of $\nu(t)$ is allowed to have a linear trend over a local interval,

$$< \nu(t) > = \nu_0 + \dot{\nu}_0 t \ , \qquad 0 \leqq t \leqq T \ .$$

The notation $<\cdot>$ is used throughout to denote an ensemble average. Many real continuous time series are described by this model. In the next section $\nu(t)$ will represent the rate constant of a feedback loop.

The time derivative $d\nu(t)/dt = \dot{\nu}(t)$ can be expressed as a Fourier-Stieljes integral

$$\dot{\nu}(t) = \dot{\nu}_0 + \int_{-\infty}^{\infty} e^{i\omega t} \dot{Z}(d\omega) \ . \tag{2}$$

Here, $\dot{\nu}(0) = \dot{\nu}_0$ , and the random coefficient $\dot{Z}(d\omega)$ has the properties

$$<\dot{Z}(d\omega) > = 0 \ ; \quad < \dot{Z}(d\omega_1)\dot{Z}^*(d\omega_2)> = \delta(\omega_1 - \omega_2) \ \dot{p}(\omega_1)d\omega_1 \, d\omega_2 \ ; \tag{3}$$

where $\delta(\cdot)$ is a delta function and $\dot{p}(\omega_1)$ is the spectral density of the generating random process. It follows that if

$$Z(d\omega) = (1/i\omega) \ \dot{Z}(d\omega), \qquad \text{and} \qquad p(\omega) = \omega^{-2}\dot{p}(\omega) \text{ , then}$$

$$\nu(t) = \nu_0 + \dot{\nu}_0 t + \int_{-\infty}^{\infty} \{ e^{i\omega t} - 1 \} \, Z(d\omega) \ . \tag{4}$$

In (4) , $\nu(0) = \nu_0$ and similar properties to (3) apply:

$$< Z(d\omega) > = 0; < Z(d\omega_1) \ Z^*(d\omega_2) > = \delta(\omega_1 - \omega_2) \ p(\omega_1)d\omega_1 \, d\omega_2 \ . \tag{5}$$

A comparison of (4) with (5) shows that

$$\varepsilon(t) = \int_{-\infty}^{\infty} \{ e^{i\omega t} - 1 \} \, Z(d\omega) \ . \tag{6}$$

Equations (4) - (6) show the relationship between the realizations of the time series $\nu(t)$ and the spectrum of the generating random process.

These equations can be used as the basis for generating a random sequence with the properties appropriate to a given system by appropriately specifying the spectrum $p(\omega)$, generating random $Z(d\omega)$ and performing the integral (6).

The spectrum $p(\omega)$ determines the autocovariance properties of the $\varepsilon(t)$ time series. Autocovariance for a locally stationary series can be represented in the time domain by the Kolmogorov structure function

$$\Delta(t_1-t_2) = \langle [\varepsilon(t_1) - \varepsilon(t_2)]^2 \rangle = \int_{t_1}^{t_2}\int_{t_1}^{t_2} \langle d\varepsilon(t') \, d\varepsilon(t'') \rangle \quad . \tag{7}$$

The relationship between the structure function $\Delta(t)$ and the spectrum $p(\omega)$ is

$$\Delta(t) = 2 \int_0^\infty d\omega [\cos(\omega t) - 1] \, p(\omega) \quad . \tag{8}$$

A related function which appears in the following analysis is

$$D(t) = \int_0^t t' \, \Delta(t') \, dt' \quad . \tag{9}$$

When $D(t)$ is expressed in terms of the spectrum it is given by

$$D(t) = \int_0^\infty d\omega \, \{\omega t^2 + \frac{2}{\omega}[1 - \cos(\omega t)]\} \frac{dp(\omega)}{d\omega} \tag{10}$$

where it has been assumed that $\omega \, p(\omega) \to 0$ as $\omega \to \infty$ .

## MODELLING CONSIDERATIONS

The purpose of this section is to demonstrate the effect which random disturbances to the rate constant of a feedback loop have upon the associated level. A second purpose is to demonstrate the way to incorporate the effect of short period disturbances in a simulation model.

A simple feedback loop in which a level $\ell(t)$ decays, or grows, with a delay of $\tau$ is described by the equation

$$\ell(t) = \ell(0) + \int_0^t \nu \, \ell(t') \, dt' \quad , \tag{12}$$

where $|\nu| = \tau^{-1}$ and the rate constant $\nu$ is either negative or positive, for decay or growth of the level, respectively. The solution to (12) is

$$\ell(t) = \ell(0) \exp \left[ \int_0^t \nu(t') \, dt' \right] \quad . \tag{13}$$

In the absence of disturbances the system control policy will determine $\nu(t)$. It will be assumed that the linear form

$$\hat{\nu}(t) = \nu_0 + \dot{\nu}_0 t$$

is a good approximation to the chosen policy, represented by $\hat{\nu}(t)$, for some time interval $[0, T]$ and that the system does not adjust the policy within this interval. Then if exogenous disturbances alter the actual $\nu(t)$ the level $\ell(t)$ will be affected but the system's chosen $\hat{\nu}(t)$ will not be altered in response. The appropriate representation of $\nu(t)$ for such a situation is statistical, demonstrating the uncertainty in the true rate:

$$\nu(t) = \nu_0 + \dot{\nu}_0 t + \varepsilon(t) \quad . \tag{14}$$

In (14) $\varepsilon(t)$ is a mean zero random variable. In general $\nu(t)$ will not be statistically stationary but will be reduced to stationarity after sufficient differencing of lagged values. Here it will be assumed that stationarity is achieved after taking first differences. This implies that $\nu(t)$ has the properties of the model (1) introduced in the previous section.

The future value of the level $\ell(t)$ is now given by

$$\ell(t) = \ell(0)\ \exp\left[\ \nu_0 t + \tfrac{1}{2}\dot{\nu}_0 t^2 + \int_0^t \varepsilon(t_1)\ dt_1\right] \tag{15}$$

A statistical description is sought for $\ell(t)$ in order to represent the uncertainty about its future value.

It is reasonable to assume that the integrated random variable $\int_0^t \varepsilon(t')dt'$ is normally distributed. If the disturbances $\varepsilon(t')$ are rapid in comparison with t then the integral will contain a large number of independent terms. The expected value of the integral is zero and its variance is given by

$$D(t) = \int_0^t \int_0^t < \varepsilon(t_1)\ \varepsilon(t_2) > dt_1 dt_2\ . \tag{16}$$

A knowledge of D(t) is necessary if the behaviour of $\ell(t)$ is to be described. Writing $\Delta(t_1 - t_2) = < \left[\varepsilon(t_1) - \varepsilon(t_2)\right]^2 >$, enables (16) to be expressed as

$$D(t) = 2t \int_0^t \Delta(t')\quad dt' - \int_0^t \int_0^t \Delta(t_1 - t_2)\ dt_1 dt_2 \tag{17}$$

The double integral in (17) can be transformed to a single integral since $\Delta(t) = \Delta(-t)$. The result is

$$\int_0^t \int_0^t \Delta(t_1 - t_2)\ dt_1 dt_2 = 2\int_0^t (t - t')\ \Delta(t')\ dt'\quad . \tag{18}$$

Using this result, (17) becomes

$$D(t) = 2 \int_0^t t'\ \Delta(t')\ dt'\quad . \tag{19}$$

If the assumption of normal statistics for the integral of the random variable $\varepsilon(t)$ appearing in (15) is correct, then the level $\ell(t)$ will follow a log-normal distribution. That is, $\ell$ will have the distribution function

$$f(\ \ell\,|t) = \ \ell^{-1}\left[\,2\,\pi\,D(t)\,\right]^{-\frac{1}{2}}\ \exp\{\ -\left[\,\ln(\ell) - <\ln(\ell(t))>\,\right]^2/2D(t)\ \}\ , \tag{20}$$

where

$$< \ln(\ell(t)\ )\ > = \ \ln(\ell(0)) + \nu_0 t + \tfrac{1}{2}\dot{\nu}_0 t^2\quad .$$

It follows from (20) that the moments $<\ell^m(t)>$ are equal to

$$< \ell^m(t)\ > = \ell^m(0)\ \exp\left[\,m\ (\ \nu_0 t + \tfrac{1}{2}\dot{\nu}_0 t^2\ ) + \tfrac{1}{2}m^2\ D(t)\ \right]\quad . \tag{21}$$

The expected value of the level $<\ell(t)>$ is obtained from (21) by setting m = 1. It is seen that the expected value of the level is increased by the factor

$$\exp\ \left[\,\tfrac{1}{2}D(t)\,\right] = \ \exp\left[\,\int_0^t t'\ \Delta(t')\ dt'\ \right] \tag{22}$$

as a result of the uncertainty in $\nu(t')$ over the time interval $[\,0,\ t\,]$.

The next objective of this work is to demonstrate the way in which these effects can be incorporated in a simulation model. To this end, the equation

$$L(t) = L(0) + \int_0^t \left[\,\nu(t_1)\ L(t_1) + F(t_1)\ \right]\ dt_1 \tag{23}$$

will be used to describe the change in a level L(t) as a result of an inflow at a rate F(t) and an outflow at a rate determined by the delay $\tau = -1/\nu$ . As before, $\nu(t)$ is allowed to be disturbed randomly from the value $\hat{\nu}(t)$, which is determined by the system policy for the present interval $[\,0,\ T\,]$. Once again, the model (1) will be adopted for $\nu(t)$ so that the value $\nu(t)$ at a given t will be the sum of a predetermined trend and the accumulation of random movements off the trend. In the situation which is considered now, the simulation is to be advanced through the interval $[\,0,T\,]$ . This represents one time step of the computation. The levels in the model, of which L(t) is representative, change in value as the computation is advanced to the end of the interval.

Any change in L(t) due to variations in $\nu(t)$ with period greater than 2T can be computed by representing such variations explicitly, whether they be the result of control policy changes or exogenous disturbances. But variations with period less than 2T will be ignored unless their effect on the level L(t) is specifically introduced. To do this it is necessary to know the expected behaviour of L(t) as a function of the disturbance time series.

It will be assumed that the change in level is small so that (23) can be approximated by

$$L(t) = L(0)\left[1 + \nu_0 t + \tfrac{1}{2}\dot{\nu}_0 t^2\right] + \int_0^t F(t_1)\,dt_1 + \int_0^t L(t_1)\varepsilon(t_1)\,dt_1 \quad . \qquad (24)$$

In order to evaluate the final term (24) is used to give

$$L(t_2)\varepsilon(t_2) = \left\{L(0)\left[1 + \nu_0 t_2 + \tfrac{1}{2}\dot{\nu}_0 t_2^2\right] + \int_0^{t_2} F(t_1)dt_1\right\}\varepsilon(t_2) + \int_0^{t_2} L(t_1)\varepsilon(t_1)\varepsilon(t_2)\,dt_1 \quad . \qquad (25)$$

In (25) it is seen that the value of $\varepsilon(t_2)$ is uncorrelated with the expression in brackets which relates to values at times $t_1 < t_2$ so that if in (25) the expectation value is considered,

$$< L(t_2)\varepsilon(t_2) > = <\int_0^{t_2} L(t_1)\varepsilon(t_1)\varepsilon(t_2)\,dt_1> \simeq L(0)\int_0^{t_2} <\varepsilon(t_1)\varepsilon(t_2)>dt_1 \qquad (26)$$

The integral appearing in (26) results in

$$\int_0^{t_2} <\varepsilon(t_1)\varepsilon(t_2)>\,dt_1 = t_2\,\Delta(t_2) \quad . \qquad (27)$$

Now equations (26) and (27) can be used with (24) to give the expected value of the level

$$<L(t)> = L(0)\left[1 + \nu_0 t + \tfrac{1}{2}\dot{\nu}_0 t^2\right] + \int_0^t F(t_1)dt_1 + L(0)\int_0^t t_2\Delta(t_2)dt_2. \qquad (28)$$

It follows from (19) that (28) can be written in the form

$$<L(t)> = L(0)\left[1 + \nu_0 t + \tfrac{1}{2}\dot{\nu}_0 t^2 + \tfrac{1}{2}D(t)\right] + \int_0^t F(t_1)\,dt_1 \quad . \qquad (29)$$

This result is in agreement with the earlier results (21) and (22), as can be seen if, in (21), m is set equal to unity and the exponential is expanded as a series such that terms of second order are ignored. Equation (29) shows that the feedback component of the expected level <L(t)> is increased as a result of the disturbance to the rate. This suggests that the effect of the disturbances can be modelled by introducing a multiplicative factor

$$\exp\left[\psi(t)\right]$$

where $\psi(t)$ is a mean zero, normally distributed, integrated random variable which acts on the feedback component in the rate equation. This is to say that the disturbed level can be simulated by modifying (23) to read

$$L(t) = L(0)\left[\int_0^t \hat{\nu}(t_1)\,L(t_1)dt_1\right]\exp\left[\psi(t)\right] + \int_0^t F(t_1)dt_1 \qquad (30)$$

where the statistical properties of $\psi(t)$ are arranged to satisfy

$$< \psi(t) > = 0 \quad \text{and} < \psi(t_1)\psi(t_2) > = D(t_1 - t_2) \quad . \qquad (31)$$

The autocovariance function of the perturbation $\psi(t)$ is seen to equal D(t) whose relationship to the spectrum of short period disturbances was given earlier in (10).

## CONCLUSION

The problem which has been considered is that of representing the behaviour of a system level when the rate constant, or equivalently the delay, in the corresponding feedback loop is subject to disturbances at periods which are shorter than can be represented in a simulation time step. A solution to this problem has been demonstrated together with details of the means of its implementation.

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