CHAOTIC BEHAVIOUR IN A SIMPLE MODEL OF URBAN MIGRATION

by

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ABSTRACT

By analysing the dynamics of a simple problem of urban migration, this paper illustrates how chaotic behaviour can be internally generated even in a relatively small (4-level) System Dynamics model.

Two different groups of minority families are considered to move around between three sectors of a city. This migration occurs in response to changes in certain social indicators which we take to be related to the number of families already living in the respective sectors. Type I families, for instance, prefer to live in areas with many households of the same kind and tend to avoid neighbourhoods with many type II families. Type II families, on the other hand, although also they like to live together, are at the same time attracted to areas with many type I families.

For normal parameter values, this system has an unstable equilibrium point. In base case it exhibits a limit cycle behaviour with the non-linear limiting factors associated with a slowing down in the rate of emigration from a certain sector as the number of remaining families approach zero. We show how the system develops through a Feigenbaum cascade of period doubling bifurcations as the inclination of type II families to move into areas with many type I families is reduced by 15%. By calculating the largest Lyapunov exponent for the system we finally show how the chaotic behaviour is quantitatively distinguishable even from the most complicated limit cycle behaviour.

INTRODUCTION

In the field of non-linear dynamics chaos denotes a distinct mode of behaviour in the same way as exponential growth or damped oscillations are characteristic modes of behaviour for linear systems.

Chaos may be described as a behaviour which is bounded in state space, and which seems to have a certain recurrence. Each swing is unique, however, the system never repeats itself, and the true period is infinite. Chaotic behaviour is also characterized by its extreme sensitivity to the initial conditions. Trajectories started at points in state space which are infinitesimally apart will thus generally evolve in entirely different manners. As as result, projection is no longer possible, even for systems which are described by completely deterministic equations of motion.

In a way, the existence of such complicated trajectories even for relatively simple systems should not come as a surprise to us. Tossing a coin, for instance, must be considered an experiment with a simple dynamical system. No one can doubt that the equations of motion governing the flight of a coin are deterministic, and at least in principle, we can certainly specify the forces acting upon the coin as functions of all relevant parameters. Nevertheless, the outcome of such an experiment is usually considered to be completely random, and in many cases coin tosses are even used as idealizations of stochastic processes. Maintaining, however, that the flight of a coin is a deterministic process, the apparently random nature of the outcome can only be explained in terms of an extreme sensitivity to the initial conditions. Even the slightest change in these conditions many reverse the result.

A closer examination of classical mechanics shows (Helleman, 1980) that almost all the knowledge about the behaviour of dynamical systems that we have acquired during the last centuries relates to (i) linear systems, (ii) non-linear systems with one degree of freedom, or (iii) larger systems which decompose into seperate one-dimensional systems. As soon as we move to problems which are a little more complicated, the equations of motion can no longer be integrated, and in <u>most</u> cases the phase space becomes scattered with regions where the trajectories evolve as unpredictably as the toss of a coin. Chaos is therefore not a phenomenon which occurs only in exceptional, pathological systems, but almost all conservative systems with more than a few degrees of freedom exhibit this type of behaviour.

In System Dynamics we deal with macroscopic systems in which large numbers of identical elements (persons, machines, dollars, coffee bags etc.) flow and accumulate, and for which it is impossible to follow the life curve of individual elements in detail. Instead, the description is in terms of aggregate rate and level variables. Such macroscopic systems are <u>dissipative</u> which means that, in the absence of unlimited growth, the trajectories tend to approach (be attracted by) certain points or curves in state space. The simplest case, of course, is that of a stable equilibrium point which is approached by all trajectories starting within a certain region of state space.

Besides in certain degenerate systems in which stable line or plane attractors occur, the stable equilibrium point is the only attractor which can exist in linear systems, and by the laws of thermodynamics it is also the attractor which is bound to control the development of a system which cannot exchange energy or resources with its surroundings. Social and biological systems are neither linear nor thermodynamically isolated, however, and it is therefore quite likely that more complicated attractors will occur.

The next simplest form of an attractor is a limit cycle which is approached by all trajectories starting within a certain region (the basin of attraction) on both sides of the cycle. John Sterman's simple Kondratieff wave model (Sterman, 1983), and Jack Homer's model of worker burn out (Homer, 1984) both show examples of this kind of behaviour. For certain parameter values, one can also obtain limit cycle behaviour in Dennis Meadow's commodity cycle model (Meadows, 1970) as well as in Nathaniel Mass' business cycle model (Mass, 1975). Until recently at least, there has been a clear tendency for System Dynamics practitioners to 'tune' their models for damped oscillations rather than for limit cycle behaviour. This reflects a particular view on the stability of social systems, a view which may not necessarily be correct.

For models with 3 or more state variables, competition between a basic growth tendency and nonlinear limiting factors may cause the attractor to change form. As some parameter B in the system is increased, the simple 1-cycle which closes to itself after a single revolution in phase space may suddenly become unstable to be replaced by an attractor with twice the original period of revolution, a 2-cycle. This phenomenon is referred to as period doubling (Feigenbaum, 1980). The 2-cycle almost closes to itself after one revolution in phase space. It misses a little, however, and precisely returns to its initial point only after two revolutions.

At a slightly higher value of B, a new period doubling will occur, so that the attractor only returns to its starting point after 4 revolutions in phase space. If the behaviour of the system is spectrally analysed, the first period doubling corresponds to the generation of a component at the half subharmonic frequency of the original limit cycle, with the fundamental and its subharmonic component being phase-locked together. The second period doubling similarly corresponds to the generation of subharmonic signals at 1/4 and 3/4 the original frequency. (In physics and engineering this phenomenon has previously been termed parametric subharmonic generation (Zemon, 1968)).

The remarkable and very powerful result is now that once this route has been initiated, the period doubling bifurcations will continue until at a finite parameter value B_∞, the period becomes infinitely long, and the attractor turns chaotic. Moreover, Feigenbaum (Feigenbaum, 1980) has shown that this route to chaos asymptotically develops in quantitatively the same manner for all dissipatice systems independent of their nature (social, biological, physical or technical) or number of state variables, and independent of the precise form of the equations of motion.

Two universal constants α = 2.5029078... and δ = 4.6692016... exist such that the value of the parameter B at the n'th period doubling is given by

$$B_n = B_\infty - a\delta^{-n} \qquad (1)$$

and such that the split between loops decreases by a factor α from period doubling to period doubling. a is here a (positive or negative) constant, depending on the actual system.

In a recent paper (Rasmussen, 1985) we have demonstrated how John Sterman's simple Kondratieff wave model (Sterman, 1983) through a number of bifurcations finally turns chaotic if the model is driven by a relatively weak sinusoidal signal, representing a business cycle variation in demands for goods. It is the purpose of the present study to illustrate how a similar phenomenon can be endogeneously generated in a relatively simple System Dynamics model.

Preliminary results also indicate that chaotic behaviour can develop if two commodity cycle models are connected via demand cross elasticities. In this case, however, the system follows an alternative route to chaos which involves a competition between the periodicities of the two sub-models. As long as the coupling between the two sub-models is relatively weak, phase locking and quasi periodic behaviour is produced. As the coupling becomes stronger, however, the system suddenly switches into a chaotic state. Alternative routes to chaos have also been identified for physical systems (Eckmann, 1981).

A MODEL OF URBAN MIGRATION

The problem to be studied is related to the migration of two minority populations in the town of Waycross which is situated approximately 400 miles north of Promise Gorge. Figure 1 illustrates the outlay of the downtown area with its three main districts: Richmond (1), Jonesboro (2), and Camden (3). The downtown area houses about 60000 families out of which 3000 are Italian and 3000 Puertorican. The total number of families is assumed to remain constant over time, and so is the distribution between different ethnic groups. For the individual district, however, the number of families of each of the two minority populations may vary as a result of migration from one area to another.

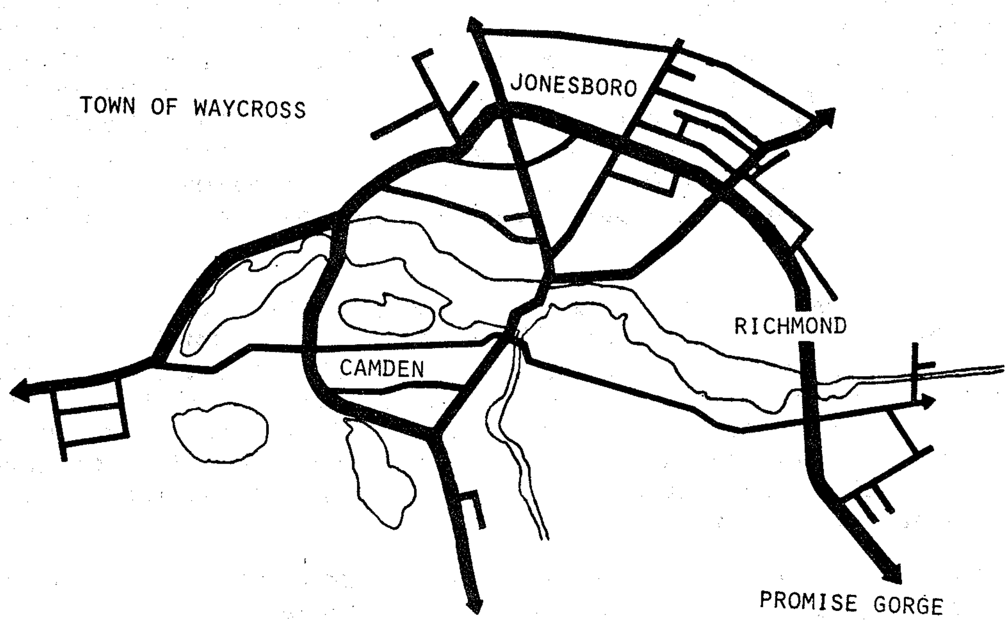


Figure 1. The downtown area of Waycross with its three main districts. For various social reasons, minority families migrate between these districts in response to changes in the relative population sizes.

Italian families naturally prefer to live in Italian neighbourhoods, which implies that these families tend to move from districts with a small to districts with a larger Italian population. Similarly, Puertorican families tend to move to areas where the Puertorican population is already relatively large. For various reasons, however, Puertorican families also like to live in Italian neighbourhoods, and once some Puertoricans start to move into predominantly Italian areas others will follow. The Italians on the other hand do not particularly appreciate Puertoricans, and when Puertorican families start to move into 'their' areas, they usually prefer to move out.

Altogether, the pattern of migration can be represented through the diagram on figure 2. Note, however, that since we have assumed the total number of both Italian and Puertorican families to remain constant, the flowdiagram corresponding to the detailed equations of motion only has two state variables for each type of minority population.

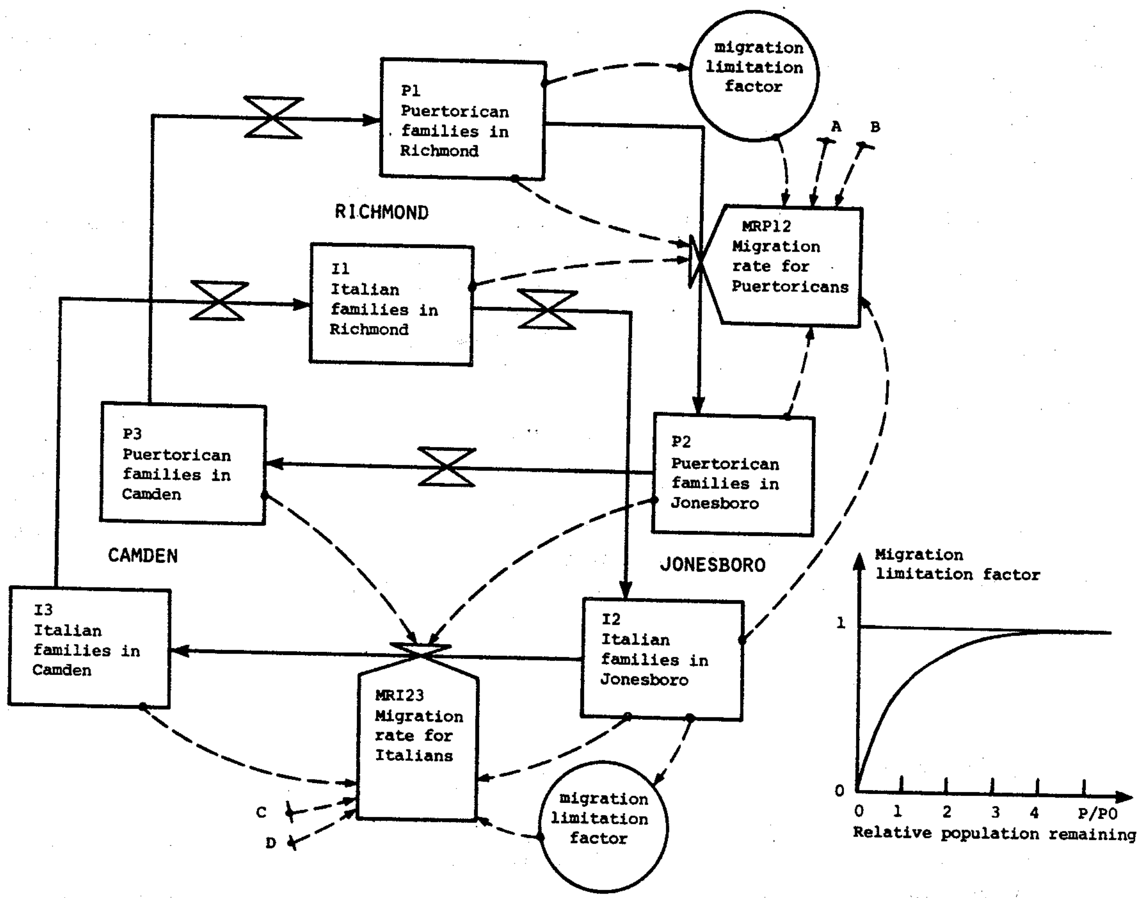


Figure 2. Basic migration pattern in downtown Waycross. The insert shows the factor which slows down emigation from a given district as the number of remaining families approaches zero.

As a first approximation we have taken the inclination of Puertorican families to migrate from Richmond (1) to Jonesboro (2) to be given by

IMP12.K=A(P2.K-P1.K)+B(I2.K-I1.K)

with similar expressions for the inclination of Puertoricans to move from Jonesboro to Camden (3), and from Camden to Richmond.

A and B are here constants. P2 and P1 denote the number of Puertorican families already living in Jonesboro and Richmond, respectively while I2 and I1 represent the number of Italian families in each of the two districts. Note, that the inclination to migrate may change sign, depending upon the relative population sizes in the various districts. In the simulations to be discussed in the next section, we have taken A=1, while B gradually is reduced from 2 in the base case run to 1.1 in the final example. Expressing the relative strength by which Puertorican families are attracted to Italian neighbourhoods, B is the <u>bifurcation parameter</u> of our problem.

Correspondingly, the inclination of Italian families to migrate for instance from Jonesboro to Camden is given by

IMI23.K=C(I3.K-I2.K)+D(P3.K-P2.K)

where the constants C=1 and D=-4.5 remain unchanged in all simulations. I3 and I2 represent the number of Italian families already living in Camden and Jonesboro, while P3 and P2 give the corresponding number of Puertorican families. To obtain the rates of migration we divide the inclinations to migrate by a constant AMD (average migration delay). Rather arbitrarily we have chosen AMD=50 months, the purpose of this constant being primarily to define a scale on the time axis.

At this stage, the model is still linear. The equilibrium point is unstable, however, and the model will therefore produce a growing oscillatory behaviour. At a certain amplitude this will lead to the population sizes becoming negative, which is clearly a meaningless result. It is therefore necessary to introduce nonlinearities which slow down the rate of emigration out of a certain district as the number of remaining families approaches zero. The rate of emigration of Puertorican families out of Richmond is therefore multiplied by a limitation factor

DP1.K = 1-EXP(-P1.K/PO)

with the scaling population PO=400 families.

As long as the number of Puertorican families in Richmond P1 is well above PO, the limitation factor is only slightly less than unity. For P1=PO, DP1 is reduced to 0.63, and as P1 becomes much smaller than PO, DP1 approaches zero. The detailed relation is sketched in the insert of figure 2. It is necessary to stress, however, that any limitation factor which will serve a similar function as DP1, be it a DYNAMO-table function or an alternative analytical expression, can be expected to give fundamentally the same simulation results. In accordance with our usual paradigm, the basic behaviour of a dynamical system is little sensitive to the precise form of the table functions. When bifurcations can occur, we only have to give the term 'basic behaviour' a somewhat more general meaning than we are otherwise used to. Introducing an alternative limitation factor may change the dynamics from say limit cycle behaviour to chaos. The ability, however, to generate both periodic and chaotic behaviour is conserved. The threshold for chaotic behaviour (B_∞) is just moved to another parameter value.

Together with the limitation factors we have also introduced a number of shift functions which, depending on the direction of migration, apply limitation factors corresponding to the populations under reduction. The total DYNAMO-program hereafter reads:

```
*   MIGRATION OF MINORITY POPULATIONS IN WAYCROSS
NOTE
NOTE PUERTORICAN POPULATION
NOTE
L  P1.K=P1.J+(DT)(MRP31.JK-MRP12.JK)
N  P1=P1I
L  P2.K=P2.J+(DT)(MRP12.JK-MRP23.JK)
N  P2=P2I
A  P3.K=3000-P1.K-P2.K
A  IMP12.K=A(P2.K-P1.K)+B(I2.K-I1.K)
R  MRP12.KL=IMP12.K*(B12.K*DP1.K+(1-B12.K)*DP2.K)/AMD
A  IMP23.K=A(P3.K-P2.K)+B(I3.K-I2.K)
R  MRP23.KL=IMP23.K*(B23.K*DP2.K+(1-B23.K)*DP3.K)/AMD
A  IMP31.K=A(P1.K-P3.K)+B(I1.K-I3.K)
R  MRP31.KL=IMP31.K*(B31.K*DP3.K+(1-B31.K)*DP1.K)/AMD
A  DP1.K=1-EXP(-P1.K/PO)
A  DP2.K=1-EXP(-P2.K/PO)
A  DP3.K=1-EXP(-P3.K/PO)
A  B12.K=TABHL(SHIFT,IMP12.K,-15,15,5)
A  B23.K=TABHL(SHIFT,IMP23.K,-15,15,5)
A  B31.K=TABHL(SHIFT,IMP31.K,-15,15,5)
C  A=1
C  B=2.0
C  AMD=50    months
T  SHIFT=0/.05/.15/.5/.85/.95/1
C  PO=400    families

NOTE
NOTE ITALIAN POPULATON
NOTE
L  I1.K=I1.J+(DT)(MRI31.JK-MRI12.JK)
N  I1=I1I
L  I2.K=I2.J+(DT)(MRI12.JK-MRI23.JK)
N  I2=I2I
A  I3.K=3000-I1.K-I2.K
A  IMI12.K=C(I2.K-I1.K)+D(P2.K-P1.K)
R  MRI12.KL=IMI12.K*(A12.K*DI1.K+(1-A12.K)*DI2.K)/AMD
A  IMI23.K=C(I3.K-I2.K)+D(P3.K-P2.K)
R  MRI23.KL=IMI23.K*(A23.K*DI2.K+(1-A23.K)*DI3.K)/AMD
A  IMI31.K=C(I1.K-I3.K)+D(P1.K-P3.K)
R  MRI31.KL=IMI31.K*(A31.K*DI3.K+(1-A31.K)*DI1.K)/AMD
A  DI1.K=1-EXP(-I1.K/PO)
A  DI2.K=1-EXP(-I2.K/PO)
A  DI3.K=1-EXP(-I3.K/PO)
A  A12.K=TABHL(SHIFT,IMI12.K,-15,15,5)
A  A23.K=TABHL(SHIFT,IMI23.K,-15,15,5)
A  A31.K=TABHL(SHIFT,IMI31.K,-15,15,5)
C  C=1
C  D=-4.5
```

SIMULATION RESULTS

The simulation results to be presented in this section were obtained with constant values of the parameters A, C and D, and only the parameter B has been gradually reduced from run to run. For most simulations we show a plot of the Puertorican population in Richmond as function of time together with a phase-plot in which the Puertorican population in Jonesboro is depicted vs. the Puertorican population in Richmond.

Figure 3 shows the results of the base case run in which B=2. For reasons of

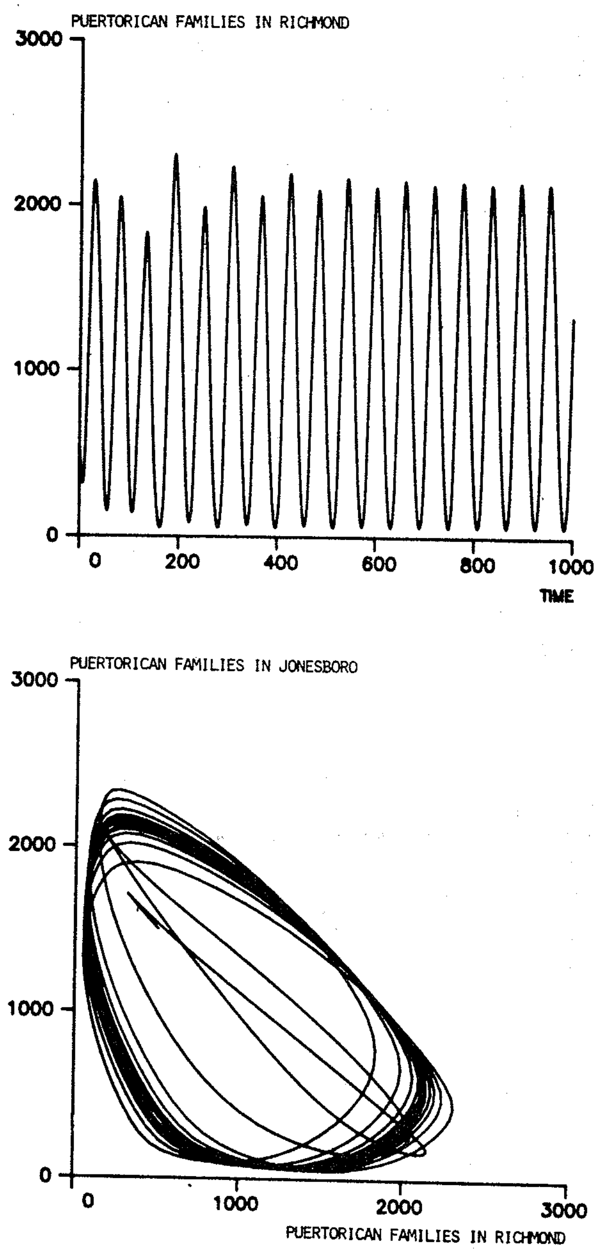


Figure 3. Time- and phase-plots obtained with the migration model for B=2 (base case). The figure shows the approach of a trajectory to the limit cycle attractor.

symmetry, our model has an equilibrium point in which both the Puertorican and the Italian populations are equally distributed between the three districts. As in the linear case, this equilibrium point is unstable. Even if the model is initiated in equilibrium, the slightest disturbance triggers an expanding oscillation. Due to the migration limitation factors, however, the amplification now stops at a certain wave amplitude - before any of the populations become negative. The non-linear restrictions thus cause the model to have a self-sustained oscillation with finite amplitude, a limit cycle.

The phase-plot in figure 3 shows how the limit cycle is approached by a trajectory starting close to the equilibrium point. The rate at which this approach occurs is a measure of the dissipation (loss) in the system. In the present case, the approach is rather slow, and because our problem has 4 state variables, the projection into a two-dimensional state space looks a little complicated.

To accentuate the form of the stable attractor rather than of the initial transient, in the following simulations we have started the plotting routine only after the transient has died out. Figure 4 thus shows a phase-plot of the limit cycle attractor for B=2.

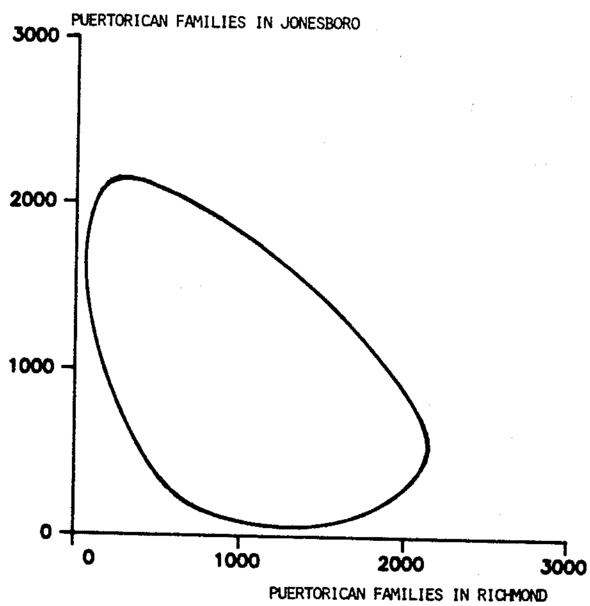
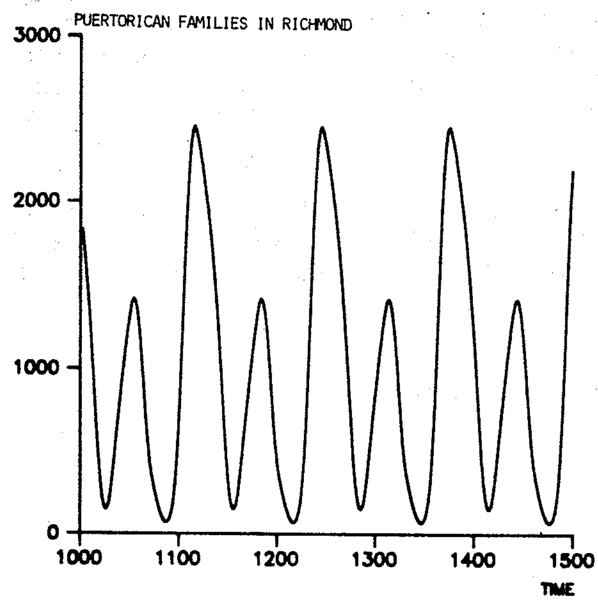


Figure 4. Phase-plot of the limit cycle attractor for B=2 (1-cycle).

Figure 5 shows the simulation results obtained for B=1.8. In the time-plot we now observe alternating high and low maxima for the Puertorican population in Richmond, and the period of the stable attractor has now doubled relative to the 1-cycle in figure 4. In the phase-plot we see how the attractor has folded itself, and it now closes only after two revolutions.

If B is reduced to 1.765 we obtain the simulation results of figure 6. There are now 4 different maxima in the time plot, two high maxima and two low maxima. The difference between the two low maxima is approximately 17%, while the difference between the two high maxima only amouns to about 3%. Compared with a difference of about 45% between the high and low maxima generated by the first period doubling, this agrees with the theoretical prediction that the split between the loops must decrease from period doubling to period doubling. The universal constant α which describes this reduction in loop split only applies after the first few period doublings have removed those aspects of the process which depend on the particular problem. From the phase-plot of figure 6 we see how the attractor now has folded twice, a 4-cycle.



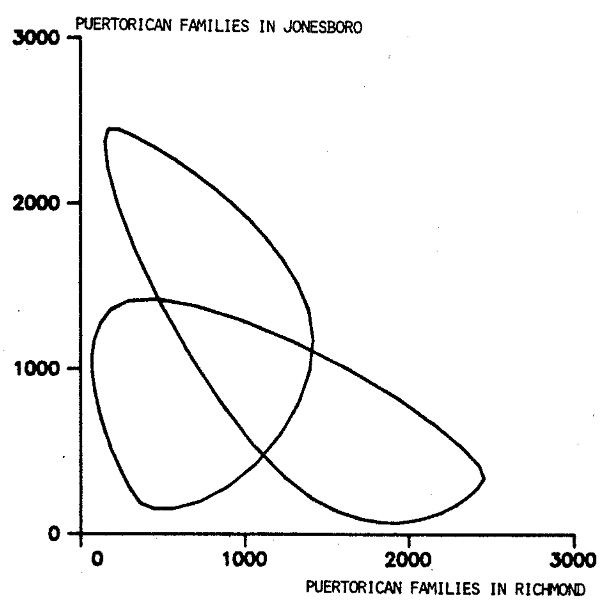


Figure 5. Time- and phase-plots for the stable 2-cycle attractor obtained with B=1.8.

As B is further reduced, the bifurcation process continues, although it becomes harder and harder to follow with the tools that we are applying in this study. For B=1.760 we obtain the 8-cycle shown in figure 7. At B≅1.7 the threshold to chaos is exeeded, and from then on the system in general behaves in an aperiodic and random manner. This is illustrated in figures 8 and 9 for B=1.3 and B=1.1, respectively. Now, the phase-plots for the stable attractor no longer close, and the variation of the Puertorican population with time appears completely stochastic with random succession of high and low, broad and narrow maxima. When looking at figure 9 it should be recalled that the curves describe a stationary behaviour for a simple, purely deterministic system.

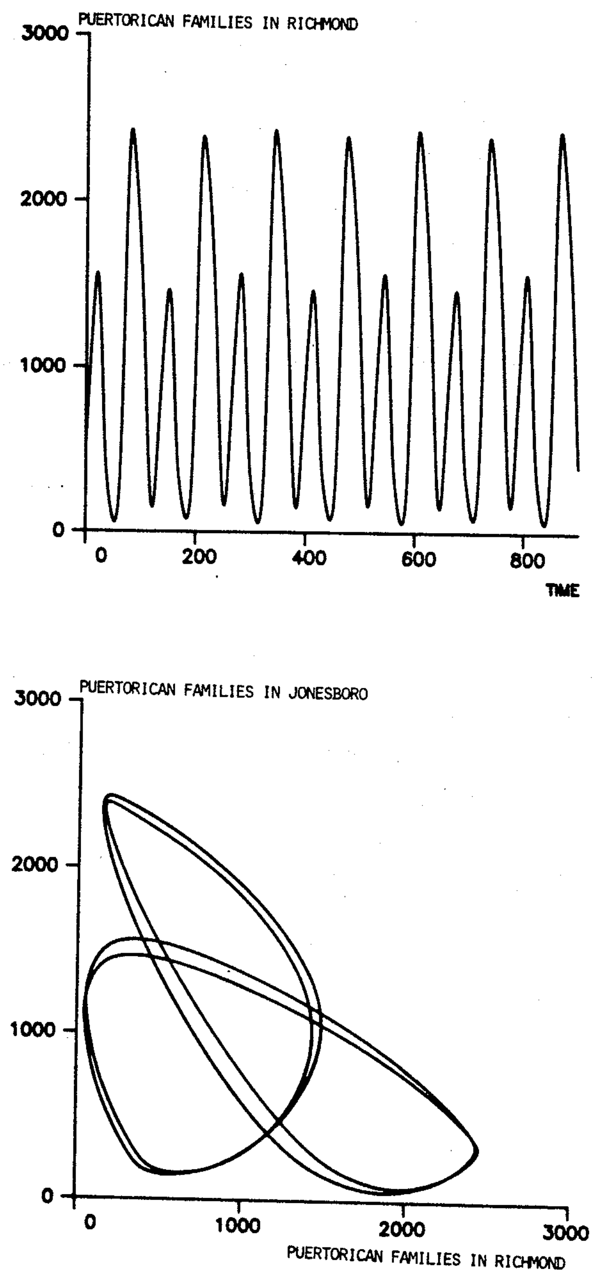


Figure 6. Time- and phase-plots for the stable 4-cycle attractor obtained with B=1.765.

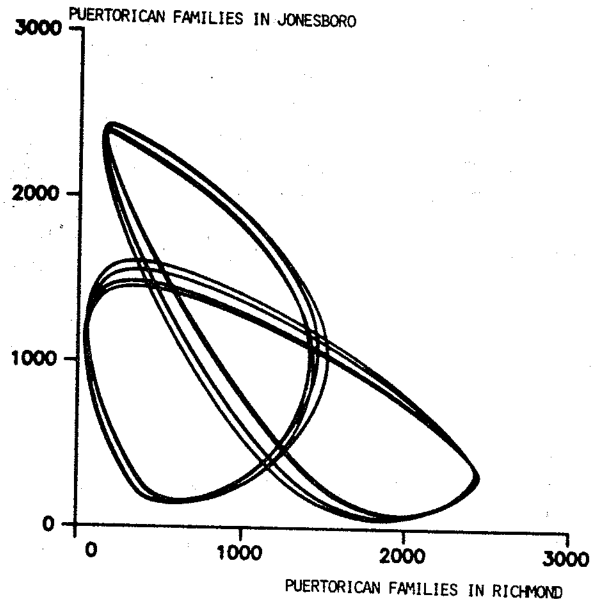


Figure 7. Phase-plot for the stable 8-cycle attractor obtained with B=1.760.

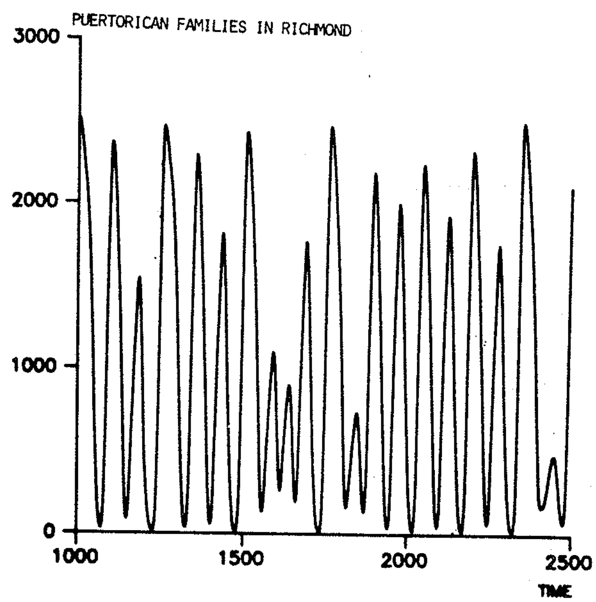


Figure 8. Time-plot for the chaotic attractor obtained with B=1.3.

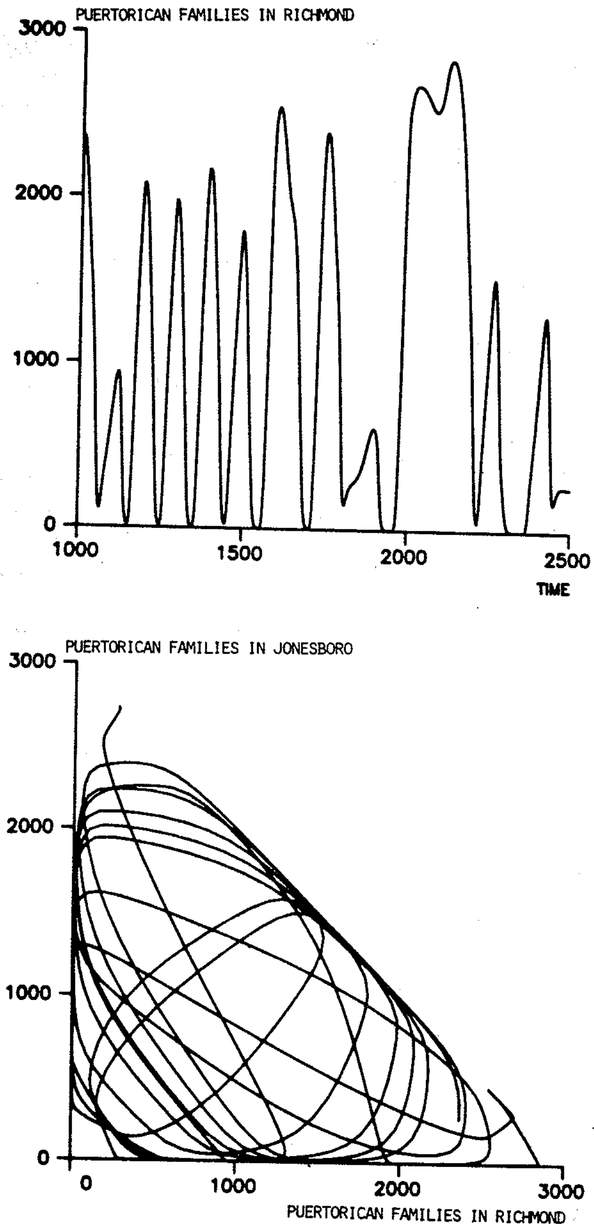


Figure 9. Time- and phase-plots for the chaotic attractor obtained with B=1.1. Note that the deterministic process now appears completely random.

DISCUSSION

From a stricter point of view, simulations of time-behaviour can of course not be taken as proof for chaos. The system in figure 9 may still be periodic, just with a period of more than 1500 months. During the last decade, a number of methods have therefore been developed (or adopted) which can give more convincing evidence for chaotic behaviour. On such method is to determine the largest Lyapunov exponent L_1 (Young, 1983 and Froeyland, 1983). If this exponent is positive, the attactor is chaotic. (For a limit cycle $L_1=0$, and in the case of a stable equilibrium point $L_1<0$).

The Lyaponov exponents L_i are limiting values for time approaching infinity of certain functions $\lambda_i(t)$ which are derived from the eigenvalues of the locally linearized system as one moves along a trajectory. The number of Lyapunov exponents and the number of functins $\lambda_i(t)$ equal the number of state variables in the system. In our problem there are thus 4 Lyapunov exponents. It is a relatively simple matter (Rasmussen, 1985) to construct a DYNAMO-program which will calculate the function $\lambda_1(t)$ associated with the largest Luapunov exponent as well as the sum $\Sigma\lambda_i(t)$ over the four λ-functions. $\lambda_1(t)$ and $\Sigma\lambda_i(t)$ control the fundamental stability properties of the system.

Unfortunately, the Euler integration procedure of DYNAMO II is not good enough for this purpose. (It gives $L_1>0$ for a limit cycle). We have therefore performed our simulations by means of COLTS using a Gear Predictor Corrector integration procedure. COLTS (Behrens,1980) is a simulation language which accepts DYNAMO statements, but has a number of facilities beyond those of DYNAMO II.

Figure 10 shows the Puertorican population in Richmond as a function of time plotted together with a curve representing $t\cdot\lambda_1(t)$. As time approaches infinity, the (average) slope of this last curve determines the value of the largest Lyaponov exponent. It is seen that $L_1>0$. This agrees with the requirements for a chaotic attractor. A more detailed exposition of the use of Lyapunov exponents and other non-local stability measures will be presented in a forthcoming paper.

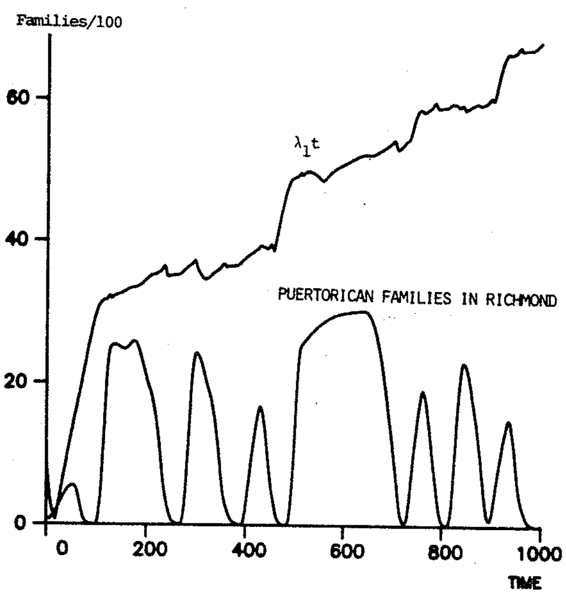


Figure 10. The Puertorican population in Richmond as a function of time plotted together with the function $t\cdot\lambda_1(t)$. In this simulation B=1.3, and the attractor is chaotic.

ACKNOWLEDGMENTS

We are indebted to Prof. W. Weidlich, Institute of Theoretical Physics, University of Stuttgart, West Germany, who during a meeting in Oslo in the fall of 1984 suggested that chaos could occur in a simple migration system. We have corresponded with Prof. Weidlich during the development of our work. Considering, however, our differences in methodology as well as in detailed model structure, Prof. Weidlich has suggested that we publish our results independently.

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